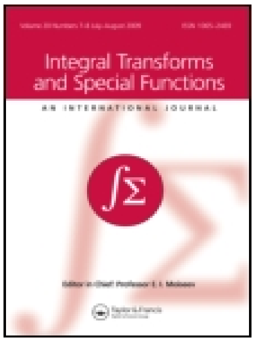

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V. S. Guliyev & Y. Y. Mammadov

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(L_p, L_q) Boundedness of the fractional maximal operator associated with the Dunkl operator on the real line

V.S. Guliyev[a,b]* and Y.Y. Mammadov[c,d]

[a] *Department of Mathematics, Ahi Evran University, Kirsehir, Turkey;* [b] *Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan;* [c] *Department of Mathematics, Nakhcivan State University, Nakhchivan, Azerbaijan;* [d] *Nakhchivan Teacher-Training Institute, Nakhchivan, Azerbaijan*

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On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In this paper, we obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional maximal operator associated with the Dunkl operator on $\mathbb{R}$ from the spaces $L_{p,\alpha}(\mathbb{R})$ to the spaces $L_{q,\alpha}(\mathbb{R})$ and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$.

Keywords: Dunkl operator; generalized translation operator; Dunkl transform; fractional maximal operator; fractional integral operator

2000 Mathematics Subject Classification: Primary: 42B20, 42B25, 42B35; Secondary: 47G10

1. Introduction

On the real line, the Dunkl operators are differential-difference operators introduced in 1989 by Dunkl [8] and are denoted by Λ_α, where $\alpha > -1/2$ is a real parameter (see (4)). These operators are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The Dunkl kernel E_α is used to define the Dunkl transform $\mathcal{F}_\alpha$, which was introduced by Dunkl [9]. Rosler [18] showed that the Dunkl kernels verify a product formula. This allows us to define the Dunkl translation τ_x, $x \in \mathbb{R}$. As a result, we have the Dunkl convolution.

The Hardy–Littlewood maximal function is an important tool of harmonic analysis. It was first introduced by Hardy and Littlewood in 1930 [15] for functions defined on the circle and later it was extended to the Euclidean spaces, various Lie groups, symmetric spaces, and some weighted measure spaces [4,10,22,24,25]. In the setting of hypergroups, the versions of Hardy–Littlewood maximal functions were given in [5] for the Jacobi hypergroups of compact type, in [6] for the Jacobi-type hypergroups, in [22] for the one-dimensional Bessel–Kingman hypergroups, in [2] for the one-dimensional Chebli–Trimeche hypergroups, in [11] (see also [12]) for the n-dimensional

*Corresponding author. Email: vagif@guliyev.com

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Bessel–Kingman hypergroups ($n \geq 1$), in [13] for the Laguerre hypergroups, and in [1,14,16,21] for the Dunkl operator on the real line.

In the paper, we define and study the fractional maximal function using harmonic analysis associated with the Dunkl operator on $\mathbb{R}$. We obtain the necessary and sufficient conditions for the boundedness of the fractional maximal operator associated with the Dunkl operator on $\mathbb{R}$ from the spaces $L_{p,\alpha}(\mathbb{R})$ to the spaces $L_{q,\alpha}(\mathbb{R})$ and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$.

The paper is organized as follows. In Section 2, we give the main result on the boundness of the fractional maximal function associated with the Dunkl operator on $\mathbb{R}$. In Section 3, we present some definitions and auxiliary results. In Section 4, we give some lemmas needed to facilitate the proofs of our theorems. The main result of the paper is the boundedness of the fractional maximal operator associated with the Dunkl operator on $\mathbb{R}$, established in Section 5. We prove the boundedness of the fractional maximal operator from the spaces $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$, $1 < p < (2\alpha+2)/\beta$, $1/p - 1/q = \beta/(2\alpha+2)$, from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak Lebesgue spaces $WL_{q,\alpha}(\mathbb{R})$, $1 - 1/q = \beta/(2\alpha+2)$ and from the spaces $L_{(2\alpha+2)/\beta}(\mathbb{R})$ to $L_\infty(\mathbb{R})$. We show that the conditions on the parameters ensuring the boundedness cannot be weakened.

2. Main result

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$\mathrm{d}\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\,\mathrm{d}x.$$

For every $1 \leq p \leq \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(\mathrm{d}\mu_\alpha)(\mathbb{R})$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}\mu_\alpha(x)\right)^{1/p} < \infty, \quad \text{if } p \in [1,\infty)$$

and

$$\|f\|_{\infty,\alpha} \equiv \|f\|_{L_{\infty,\alpha}} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|, \quad \text{if } p = \infty.$$

For $1 \leq p < \infty$, we denote by $WL_{p,\alpha}(\mathbb{R})$ the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r\left(\mu_\alpha\{x \in \mathbb{R} : |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \leq \|f\|_{p,\alpha}, \quad \text{for all } f \in L_{p,\alpha}(\mathbb{R}).$$

For all $x, y, z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x,y,z),$$

where

$$\sigma_{x,y,z} = \begin{cases} \dfrac{x^2+y^2-z^2}{2xy}, & \text{if } x, y \in \mathbb{R}\setminus 0,\\ 0, & \text{otherwise,} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) = \begin{cases} d_\alpha \dfrac{([(|x|+|y|)^2 - z^2][z^2 - (|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha+(1/2)))$ and $A_{x,y} = [||x|-|y||, |x|+|y|]$.

PROPOSITION 2.1 [24] *The signed kernel W_α is even and satisfies the following properties*:

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$
$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z),$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\,\mathrm{d}\mu_\alpha(z) \leq 4.$$

In the sequel, we consider the signed measure $\nu_{x,y}$ on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\,\mathrm{d}\mu_\alpha(z), & \text{if } x,y \in \mathbb{R}\setminus 0, \\ \mathrm{d}\delta_x(z), & \text{if } y = 0, \\ \mathrm{d}\delta_y(z), & \text{if } x = 0. \end{cases}$$

DEFINITION 2.2 *For $x, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\,\mathrm{d}\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form [18]:

$$\tau_x f(y) = C_\alpha \int_0^\pi f_{\mathrm{e}}\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_1(x,y,\theta)(\sin\theta)^{2\alpha}\,\mathrm{d}\theta$$
$$+ C_\alpha \int_0^\pi f_{\mathrm{o}}\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_2(x,y,\theta)(\sin\theta)^{2\alpha}\,\mathrm{d}\theta,$$

where $f = f_{\mathrm{e}} + f_{\mathrm{o}}$, f_{o} and f_{e} being the odd and the even parts of f, respectively, with $C_\alpha = \Gamma(\alpha+1)/(\sqrt{\pi}\,\Gamma(\alpha+1/2))$,

$$h_1(x,y,\theta) = 1 - \mathrm{sgn}(xy)\cos\theta \quad \text{and}$$
$$h_2(x,y,\theta) = \begin{cases} \dfrac{(x+y)[1-\mathrm{sgn}(xy)\cos\theta]}{\sqrt{x^2+y^2-2|xy|\cos\theta}}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0. \end{cases}$$

PROPOSITION 2.3 [17]

(i) *The operator τ_x, $x \in \mathbb{R}$, is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ onto itself.*
(ii) *For all $f \in \mathcal{E}(\mathbb{R})$ and $x, y \in \mathbb{R}$, we have*

$$\tau_x f(y) = \tau_y f(x), \quad \tau_0 f(x) = f(x),$$
$$\tau_x \circ \tau_y = \tau_y \circ \tau_x, \quad \Lambda_\alpha \circ \tau_x = \tau_x \circ \Lambda_\alpha.$$

Let $B(x,t)=\{y\in\mathbb{R}\colon |y|\in]\max\{0,|x|-t\},|x|+t[\}$ and $t>0$. Then $B(0,t)=]-t,t[$ and

$$\mu_\alpha(]-t,t[)=(2^{\alpha+1}(\alpha+1)\,\Gamma(\alpha+1))^{-1}\,t^{2\alpha+2}.$$

Now we define the fractional maximal function associated with the Dunkl operator by

$$M_\beta f(x)=\sup_{r>0}(\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1}\int_{B(0,r)}\tau_x|f|(y)\,\mathrm{d}\mu_\alpha(y),\quad 0\le\beta<2\alpha+2.$$

If $\beta=0$, then $M\equiv M_0$ is the Hardy–Littlewood maximal operator associated with the Dunkl operator [16].

In [1,16,21] the following theorem was proved (see also [14]).

THEOREM 2.4

(1) *If* $f\in L_{1,\alpha}(\mathbb{R})$, *then for every* $\beta>0$,

$$\mu_\alpha\{x\in\mathbb{R}\colon Mf(x)>\beta\}\le\frac{C_1}{\beta}\int_{\mathbb{R}}|f(x)|\,\mathrm{d}\mu_\alpha(x),$$

where $C_1>0$ *is independent of* f.

(2) *If* $f\in L_{p,\alpha}(\mathbb{R})$, $1<p\le\infty$, *then* $Mf\in L_{p,\alpha}(\mathbb{R})$ *and*

$$\|Mf\|_{p,\alpha}\le C_2\|f\|_{p,\alpha},$$

where $C_2>0$ *is independent of* f.

COROLLARY 2.5 *If* $f\in L^{\mathrm{loc}}_{1,\alpha}(\mathbb{R})$, *then*

$$\lim_{r\to0}\frac{1}{\mu_\alpha(B(0,r))}\int_{B(0,r)}|\tau_x f(y)-f(x)|\,\mathrm{d}\mu_\alpha(y)=0,$$

for a.e. $x\in\mathbb{R}$.

COROLLARY 2.6 *If* $f\in L^{\mathrm{loc}}_{1,\alpha}(\mathbb{R})$, *then*

$$\lim_{r\to0}\frac{1}{\mu_\alpha(B(0,r))}\int_{B(0,r)}\tau_x f(y)\,\mathrm{d}\mu_\alpha(y)=f(x),$$

for a.e. $x\in\mathbb{R}$.

For the fractional maximal operator associated with the Dunkl operator M_β, the following theorem is valid.

THEOREM 2.7 *Let* $0\le\beta<2\alpha+2,\ 1/p-1/q=\beta/(2\alpha+2),\ 1\le p\le(2\alpha+2)/\beta$.

(1) *If* $p=1,\ f\in L_{1,\alpha}(\mathbb{R})$, *then for all* $\theta>0$,

$$\int_{\{x\in\mathbb{R}:M_\beta f(x)>\theta\}}\mathrm{d}\mu_\alpha(x)\le\left(\frac{C_3}{\theta}\int_{\mathbb{R}}|f(x)|\,\mathrm{d}\mu_\alpha(x)\right)^q,\tag{1}$$

where C_3 *is independent of* f.

(2) *Let* $1 < p < (2\alpha+2)/\beta,\ f \in L_{p,\alpha}(\mathbb{R})$, *then* $M_\beta f \in L_{q,\alpha}(\mathbb{R})$ *and*

$$\left(\int_{\mathbb{R}} (M_\beta f(x))^q \,\mathrm{d}\mu_\alpha(x)\right)^{1/q} \leq C_4 \left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}\mu_\alpha(x)\right)^{1/p}. \tag{2}$$

where C_4 *is independent of* f.

(3) *Let* $p = (2\alpha+2)/\beta,\ f \in L_{p,\alpha}(\mathbb{R})$, *then* $M_\beta f \in L_\infty(\mathbb{R})$ *and*

$$\sup_{x\in\mathbb{R}} M_\beta f(x) \leq 4\left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}\mu_\alpha(x)\right)^{1/p}. \tag{3}$$

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the fractional maximal operator M_β to be bounded from the spaces $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$, $1 < p < q < \infty$, and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$, $1 < q < \infty$.

THEOREM 2.8 *Let* $0 \leq \beta < 2\alpha + 2$ *and* $1 \leq p \leq (2\alpha+2)/\beta$.

(1) *If* $p = 1$, *then the condition* $1 - 1/q = \beta/(2\alpha+2)$ *is necessary and sufficient for the boundedness of* M_β *from* $L_{1,\alpha}(\mathbb{R})$ *to* $WL_{q,\alpha}(\mathbb{R})$.
(2) *If* $1 < p < (2\alpha+2)/\beta$, *then the condition* $1/p - 1/q = \beta/(2\alpha+2)$ *is necessary and sufficient for the boundedness of* M_β *from* $L_{p,\alpha}(\mathbb{R})$ *to* $L_{q,\alpha}(\mathbb{R})$.
(3) *If* $p = (2\alpha+2)/\beta$, *then* M_β *is bounded from* $L_{p,\alpha}(\mathbb{R})$ *to* $L_\infty(\mathbb{R})$.

3. Preliminaries

For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x) - f(-x)}{2}\right). \tag{4}$$

Note that $\Lambda_{-1/2} = \mathrm{d}/\mathrm{d}x$.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \ \ x \in \mathbb{R},$$

has a unique solution $E_\alpha(\lambda x)$ called the Dunkl kernel [8,19,26] and is given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where j_α is the normalized Bessel function of the first kind and order α [27], defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

We can write for $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ [18, p. 295],

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_{-1}^{1} (1-t^2)^{\alpha-1/2}(1-t)\,\mathrm{e}^{i\lambda x t}\,\mathrm{d}t.$$

Note that $E_{-1/2}(\lambda x) = \mathrm{e}^{\lambda x}$.

The Dunkl transform $\mathcal{F}_\alpha$ of a function $f \in L_{1,\alpha}(\mathbb{R})$ is given by

$$\mathcal{F}_\alpha f(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x) f(x)\, \mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}.$$

Here, the integral makes sense since $|E_\alpha(ix)| \le 1$ for every $x \in \mathbb{R}$ [18, p. 295].

Note that $\mathcal{F}_{-1/2}$ agrees with the Fourier transform $\mathcal{F}$, given by

$$\mathcal{F} f(\lambda) := (2\pi)^{-1/2} \int_{\mathbb{R}} \mathrm{e}^{-i\lambda x} f(x)\, \mathrm{d}x, \quad \lambda \in \mathbb{R}.$$

PROPOSITION 3.1 [7,20]

(i) *For all* $f \in L_{1,\alpha}(\mathbb{R})$, *we have* $\|\mathcal{F}_\alpha f\|_{\infty,\alpha} \le \|f\|_{1,\alpha}$.
(ii) *For all* $f \in \mathcal{S}(\mathbb{R})$, *we have*

$$\mathcal{F}_\alpha(\Lambda_\alpha f)(\lambda) = i\lambda \mathcal{F}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R},$$

where Λ_α *is the Dunkl operator given by* (4).
(iii) $\mathcal{F}_\alpha$ *is a topological automorphism on* $\mathcal{S}(\mathbb{R})$, *which extends to a topological automorphism on* $\mathcal{S}'(\mathbb{R})$.

THEOREM 3.2 [7,20]

(i) *Plancherel theorem*: *The Dunkl transform* $\mathcal{F}_\alpha$ *is an isometric automorphism of* $L_{2,\alpha}(\mathbb{R})$. *In particular*, $\|\mathcal{F}_\alpha f\|_{2,\alpha} = \|f\|_{2,\alpha}$.
(ii) *Inversion formula*: *Let* f *be a function in* $L_{1,\alpha}(\mathbb{R})$, *such that* $\mathcal{F}_\alpha f \in L_{1,\alpha}(\mathbb{R})$, *then*

$$\mathcal{F}_\alpha^{-1} f(x) = \mathcal{F}_\alpha f(-x), \quad a.e.\ x \in \mathbb{R}.$$

THEOREM 3.3 [18]

(i) *Let* $\alpha > -1/2$ *and* $\lambda \in \mathbb{C}$. *The Dunkl kernel* E_α *satisfies the following product formula*:

$$E_\alpha(\lambda x) E_\alpha(\lambda y) = \int_{\mathbb{R}} E_\alpha(\lambda z)\, \mathrm{d}\nu_{x,y}(z), \quad x, y \in \mathbb{R}.$$

(ii) *The measures* $\nu_{x,y}$ *have the following properties*:

$$\operatorname{supp}(\nu_{x,y}) = A_{x,y} \cup (-A_{x,y}), \quad \|\nu_{x,y}\| := \int_{\mathbb{R}} \mathrm{d}|\nu_{x,y}|(z) \le 4.$$

PROPOSITION 3.4 [20]

(i) *If* f *is an even positive continuous function*, *then* $\tau_x f$ *is positive.*
(ii) *For all* $x \in \mathbb{R}$, *the operator* τ_x *extends to* $L_{p,\alpha}(\mathbb{R})$, $p \ge 1$, *and we have for* $f \in L_{p,\alpha}(\mathbb{R})$,

$$\|\tau_x f\|_{p,\alpha} \le 4\|f\|_{p,\alpha}. \tag{5}$$

(iii) *For all* $x, \lambda \in \mathbb{R}$, *and* $f \in L_{1,\alpha}(\mathbb{R})$, *we have*

$$\mathcal{F}_\alpha(\tau_x f)(\lambda) = E_\alpha(i\lambda x)\mathcal{F}_\alpha f(\lambda).$$

Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_\alpha$ of f and g by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y)\, \mathrm{d}\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The generalized convolution $*_\alpha$ is associative and commutative [24]. Note that $*_{-1/2}$ agrees with the standard convolution $*$.

PROPOSITION 3.5 [20]

(i) *If f is an even positive function and g a positive function with compact support, then $f *_\alpha g$ is positive.*

(ii) *Assume that $p, q, r \in [1, +\infty[$ satisfying $1/p + 1/q = 1 + 1/r$ (the Young condition). Then the map $(f, g) \mapsto f *_\alpha g$, defined on $\mathcal{E}_c \times \mathcal{E}_c$, extends to a continuous map from $L_{p,\alpha}(\mathbb{R}) \times L_{q,\alpha}(\mathbb{R})$ to $L_{r,\alpha}(\mathbb{R})$, and we have*

$$\|f *_\alpha g\|_{r,\alpha} \le 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(iii) *For all $f \in L_{1,\alpha}(\mathbb{R})$ and $g \in L_{2,\alpha}(\mathbb{R})$, we have*

$$\mathcal{F}_\alpha(f *_\alpha g) = (\mathcal{F}_\alpha f)(\mathcal{F}_\alpha g).$$

4. Proof of Theorems 2.7 and 2.8

Proof of Theorem 2.7 The fractional maximal function $M_\beta f(x)$ may be interpreted as a fractional maximal function defined on a space of homogeneous type. By this, we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure μ satisfying

$$\mu(E(x, 2r)) \le C_0 \mu(E(x, r)) \tag{6}$$

with a constant C_0 independent of x and $r > 0$. Here, $E(x, r) = \{y \in X\colon \rho(x, y) < r\}$ and $\rho(x, y) = |x - y|$. Let (X, ρ, μ) be a space of homogeneous type. Define

$$M_{\mu,\beta} f(x) = \sup_{r>0} \mu(E(x, r))^{-1+\beta/(2\alpha+2)} \int_{E(x,r)} |f(y)|\, \mathrm{d}\mu(y), \quad 0 \le \beta < 2\alpha + 2.$$

It is well known that the fractional maximal operator $M_{\mu,\beta}$ is of weak type $(1, q)$, $1 - 1/q = \beta/(2\alpha + 2)$ and is bounded from $L_p(X, \mu)$ to $L_q(X, \mu)$, $1/p - 1/q = \beta/(2\alpha + 2)$ for $1 < p < (2\alpha + 2)/\beta$ [3]. We shall use this result in the case in which $X = \mathbb{R}$, $\rho(x, y) = |x - y|$, and $\mathrm{d}\mu(x) = \mathrm{d}\mu_\alpha(x)$. It is clear that this measure satisfies the doubling condition (6).

We will show that

$$M_\beta f(x) \le C_5 M_{\mu,\beta} f(x), \tag{7}$$

where $C_5 > 0$ is independent of f.

From the definition of the generalized shift operator, it follows that $\tau_x \chi_{B(0,r)}(y)$ is supported in $B(x, r)$.

Moreover,

$$0 \le \tau_x \chi_{B(0,r)}(y) \le \min\left\{1, \frac{2C_\alpha}{2\alpha + 1}\left(\frac{r}{x}\right)^{2\alpha+1}\right\}, \quad \forall y \in B(x, r). \tag{8}$$

In the case $|x| \le r$, this follows from the simple inequality $0 \le \tau_x \chi_{B(0,r)}(y) \le 1$.

To prove (8) in the case $|x| > r$, we proceed as follows:

$$\begin{aligned}\tau_x \chi_{B(0,r)}(y) &= C_\alpha \int_{\{\theta\in(0,\pi):(x^2+y^2-r^2)/2|xy|\le\cos\theta\}} (\sin\theta)^{2\alpha}\,\mathrm{d}\theta\\ &= C_\alpha \int_{(x^2+y^2-r^2)/2|xy|}^{1} (1-t^2)^{\alpha-1/2}\,\mathrm{d}t\\ &\le 2_+^{\alpha-1/2} C_\alpha \int_{(x^2+y^2-r^2)/2|xy|}^{1} (1-t)^{\alpha-1/2}\,\mathrm{d}t\\ &= \frac{2_+^{\alpha-1/2}C_\alpha}{2\alpha+1}\left(1-\frac{x^2+y^2-r^2}{2|xy|}\right)^{\alpha+1/2}\\ &\le \frac{2C_\alpha}{2\alpha+1}\left(\frac{r}{|x|}\right)^{\alpha+1/2}\left(\frac{r-|x-y|}{|y|}\right)^{\alpha+1/2},\end{aligned}$$

where $a_+ = a$ if $a \ge 0$ and $a_+ = 0$ if $a < 0$.

In the case $|y| > |x|$,

$$\tau_x \chi_{B(0,r)}(y) \le \frac{2C_\alpha}{2\alpha+1}\left(\frac{r}{|x|}\right)^{2\alpha}$$

and in the case $|y| < |x|$, the inequality $(r-|x-y|)/|y| < (r/|x|)$ is equivalent to $r < |x|$. Therefore, we have

$$\tau_x \chi_{B(0,r)}(y) \le \frac{2C_\alpha}{2\alpha+1}\left(\frac{r}{|x|}\right)^{2\alpha},$$

which proves (8) in the case $|y| < |x|$ as well.

Also,

$$\begin{aligned}\mu_\alpha B(x,r) &= (2^{\alpha+1}\Gamma(\alpha+1))^{-1}\int_{B(x,r)} |y|^{2\alpha+1}\,\mathrm{d}y\\ &\le (2^{\alpha+1}\Gamma(\alpha+1))^{-1}\begin{cases} 2\int_{|x|-r}^{|x|+r} y^{2\alpha+1}\,\mathrm{d}y, & r<|x|\\ 2\int_0^{|x|+r} y^{2\alpha+1}\,\mathrm{d}y, & r\ge|x|\end{cases}\\ &\le \frac{2^{\alpha+1}}{\Gamma(\alpha+1)}\begin{cases} r|x|^{2\alpha+1}, & r<|x|\\ r^{2\alpha+2}, & r\ge|x|\end{cases}\\ &= \frac{2^{\alpha+1}}{\Gamma(\alpha+1)} r^{2\alpha+2}\begin{cases} (|x|/r)^{2\alpha+2}, & r<|x|\\ 1, & r\ge|x|.\end{cases}\end{aligned}$$

Then,

$$\begin{aligned}M_\beta f(x) &= \sup_{r>0}(\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1}\int_{\mathbb{R}} \tau_x|f(y)|\chi_{B(0,r)}(y)\,\mathrm{d}\mu_\alpha(y)\\ &= \sup_{r>0}(\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1}\int_{\mathbb{R}} |f(y)|\tau_x\chi_{B(0,r)}(y)\,\mathrm{d}\mu_\alpha(y)\\ &= \sup_{r>0}(\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1}\int_{B(x,r)} |f(y)|\tau_x\chi_{B(0,r)}(y)\,\mathrm{d}\mu_\alpha(y).\end{aligned}$$

Thus,

$$M_\beta f(x) \le M_{1,\beta} f(x) + M_{2,\beta} f(x),$$

where

$$M_{1,\beta} f(x) = \sup_{r \geq |x|} (\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1} \int_{B(0,r)} \tau_x |f(y)| \, \mathrm{d}\mu_\alpha(y),$$

$$M_{2,\beta} f(x) = \sup_{r < |x|} (\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1} \int_{B(0,r)} \tau_x |f(y)| \, \mathrm{d}\mu_\alpha(y).$$

If $r \geq |x|$, then $\mu_\alpha B(x,r) \leq (2^{\alpha+1}/\Gamma(\alpha+1)) r^{2\alpha+2}$, also

$$\mu_\alpha B(0,r) = (2^{\alpha+1} (\alpha+1) \Gamma(\alpha+1))^{-1} r^{2\alpha+2}$$

and $\tau_x \chi_{B(0,r)}(y) \leq 1$ for all $y \in B(x,r)$. This yields

$$\begin{aligned} M_{1,\beta} f(x) &= \sup_{r \geq |x|} (\mu_\alpha B(0,r))^{(\beta/(2\alpha+2))-1} \int_{B(x,r)} |f(y)| \tau_x \chi_{B(0,r)}(y) \, \mathrm{d}\mu_\alpha(y) \\ &\leq (2^{2\alpha+2}(\alpha+1))^{1-(\beta/(2\alpha+2))} \sup_{r>0} (\mu B(x,r))^{(\beta/(2\alpha+2))-1} \int_{B(x,r)} |f(y)| \, \mathrm{d}\mu(y) \\ &\leq C_6 M_{\mu,\beta} f(x). \end{aligned}$$

If $r < |x|$, then by (8) $\mu_\alpha B(x,r) \leq (2^{\alpha+1}/\Gamma(\alpha+1)) r |x|^{2\alpha+1}$ and

$$\tau_x \chi_{B(0,r)}(y) \leq \frac{2C_\alpha}{2\alpha+1} \left(\frac{r}{x}\right)^{2\alpha+1}$$

for all $y \in B(x,r)$. Thus, we have

$$\begin{aligned} M_{2,\beta} f(x) &= \sup_{r<|x|} (\mu_\alpha B(0,r))^{(\beta/(2\alpha+2))-1} \int_{B(x,r)} |f(y)| \tau_x \chi_{B(0,r)}(y) \, \mathrm{d}\mu_\alpha(y) \\ &\leq C_7 \sup_{r>0} (\mu B(x,r))^{(\beta/(2\alpha+2))-1} \int_{B(x,r)} |f(y)| \mathrm{d}\mu(y) \leq C_8 M_{\mu,\beta} f(x). \end{aligned}$$

Therefore we get (7), which completes the proof (1) and (2).

(3) Let $p = (2\alpha+2)/\beta$, $f \in L_{p,\alpha}(\mathbb{R})$, then applying the Hölders inequality and inequality (5) we have

$$\begin{aligned} &(\mu_\alpha B(0,r))^{(\beta/(2\alpha+2))-1} \int_{B(0,r)} \tau_x |f(y)| \mathrm{d}\mu_\alpha(y) \\ &\leq (\mu_\alpha B(0,r))^{(\beta/(2\alpha+2))-1+(1/p')} \left(\int_{B(0,r)} (\tau_x |f(y)|)^p \, \mathrm{d}\mu_\alpha(y) \right)^{1/p} \\ &= \|\tau_x f\|_{p,\alpha} \leq 4 \|f\|_{p,\alpha}. \end{aligned}$$

Theorem 2.7 has been proved. ■

Proof of Theorem 2.8 Sufficiency part of the proof follows from Theorem 3.3.

Necessity (1) Let M_β be bounded from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$.

Define $f_r(x) := f(rx)$, then

$$\|f_r\|_{p,\alpha} = r^{-((2\alpha+2)/p)}\|f\|_{p,\alpha}$$

and

$$\|M_\beta f_r\|_{WL_{q,\alpha}} = r^{-\beta-((2\alpha+2)q)}\|M_\beta f\|_{WL_{q,\alpha}}.$$

By the boundedness M_β from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$,

$$\begin{aligned}\|M_\beta f\|_{WL_{q,\alpha}} &= r^{\beta+((2\alpha+2)/q)}\|M_\beta f_r\|_{WL_{q,\alpha}}\\ &\le Cr^{\beta+((2\alpha+2)/q)}\|f_r\|_{1,\alpha} = Cr^{\beta+((2\alpha+2)/q)-(2\alpha+2)}\|f\|_{1,\alpha},\end{aligned}$$

where C depends only on q and α.

If $1 < 1/q + (\beta/(2\alpha+2))$, then for all $f \in L_{1,\alpha}(\mathbb{R})$ we have $\|M_\beta f\|_{WL_{q,\alpha}} = 0$ as $r \to 0$. Similarly, if $1 > 1/q + (\beta/(2\alpha+2))$, then for all $f \in L_{1,\alpha}(\mathbb{R})$ we obtain $\|M_\beta f\|_{WL_{q,\alpha}} = 0$ as $r \to \infty$.

Hence, we get $1 = (1/q) + (\beta/(2\alpha+2))$.

(2) Let $1 < p < (2\alpha+2/\beta)$, $f \in L_{p,\alpha}(\mathbb{R})$ and assume that the inequality

$$\|M_\beta f\|_{q,\alpha} \le C\|f\|_{p,\alpha} \tag{9}$$

holds, where C depends only on p, q, and β.

We have

$$\|M_\beta f_r\|_{q,\alpha} = r^{-\beta-((2\alpha+4)/q)}\|M_\beta f\|_{q,\alpha}.$$

By inequality (9),

$$\|M_\beta f\|_{q,\alpha} \le Cr^{\beta+((2\alpha+2)/q)-((2\alpha+2)/p)}\|f\|_{p,\alpha}.$$

If $1/p > 1/q + (\beta/(2\alpha+2))$, then for all $f \in L_{p,\alpha}(\mathbb{R})$ we have $\|M_\beta f\|_{q,\alpha} = 0$ as $r \to 0$, which is impossible. Similarly, if $1/p < 1/q + (\beta/(2\alpha+2))$, then for all $f \in L_{p,\alpha}(\mathbb{R})$ we obtain $\|M_\beta f\|_{q,\alpha} = 0$ as $r \to \infty$, which is also impossible.

Therefore, $1/p = 1/q + (\beta/2\alpha+2)$.

Thus, the proof of Theorem 2.8 is completed. ■

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