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Harmonic curvatures and generalized helices in $\mathbb{E}^n$

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Abstract

In n-dimensional Euclidean space $\mathbb{E}^n$, harmonic curvatures of a non-degenerate curve defined by Özdamar and Hacisalihoğlu [Özdamar E, Hacısalihoglu HH. A characterization of Inclined curves in Euclidean n-space. Comm Fac Sci Univ Ankara, Ser A1 1975;24:15–23]. In this paper, we give some characterizations for a non-degenerate curve α to be a generalized helix by using its harmonic curvatures. Also we define the generalized Darboux vector D of a non-degenerate curve α in n-dimensional Euclidean space $\mathbb{E}^n$ and we show that the generalized Darboux vector D lies in the kernel of Frenet matrix $M(s)$ if and only if the curve α is a generalized helix in the sense of Hayden.
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1. Introduction

Natural scientists have long held a fascination, sometimes bordering on mystical obsession for helical structures in Nature. Helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. This fact was published for the first time by Watson and Crick in 1953 (see [17]). They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiester bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals forces forming a double helix, lipid bilayers, bacterial flagella in *Salmonella* and *E. coli*, aerial hyphae in actynomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures) (see [2,3]).

A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$, is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [15]) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant.

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For a given couple of one variable functions (eventually curvature and torsion parametrized by arclength) one might like to get an arclength parametrized curve for which the couple works as the curvature and torsion functions. This problem is usually referred as "the solving natural equations problem". The natural equations for general helices can be integrated, nor only in $\mathbb{R}^3$, but also in the 3-sphere $\mathbb{S}^3$ (the hyperbolic space is poor in this kind of curves and only helices are general helices). Indeed one uses the fact that general helices are geodesics either of right general cylinders or of Hopf cylinders, according to the curve lies in $\mathbb{R}^3$ or $\mathbb{S}^3$, respectively (see [1]). If both of $k_1(s) \neq 0$, and $k_2(s)$ are constant it is, of course, a general helix. We call it a circular helix. Its known that straight line and circle are degenerate-helix examples ($k_1(s) = 0$, if the curve is straight line and $k_2(s) = 0$, if the curve is a circle (see [10,11])).

In fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral example is k-Fibonacci spirals. These curves appear naturally from studying the k-Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden section appear very often in theoretical physics and physics of the high energy particles (see [4,5]). Three-dimensional, k-Fibonacci spirals was studied from a geometric point of view in [6].

Recall that a curve α is called a W-curve, if it has constant Frenet curvatures. W-curves in the Euclidean space $\mathbb{E}^n$ have been studied intensively. The simplest examples are circles as a planar W-curves and helices (circular helix) as non-planar W-curves in $\mathbb{E}^3$.

The notion of a generalized helix can be generalized to higher dimension in two way: In [14], the same definition is proposed but in $\mathbb{E}^n$, i.e., a generalized helix as a curve $\alpha : \mathbb{R} \to \mathbb{E}^n$ such that its tangent vector forms a constant angle with a given direction X at $\mathbb{E}^n$. Other way, in [8] (see also [12]), a curve called a "generalized helix" which was defined by Hayden as a curve $\alpha : \mathbb{R} \to \mathbb{E}^n$ such that all the vectors of the Frenet frame makes a constant angle with the fixed direction in $\mathbb{E}^n$. In this case the generalized helix has the following properties:

$$\frac{k_{n-1}}{k_{n-2}} = \text{const.}, \frac{k_{n-3}}{k_{n-4}} = \text{const.}, \ \ldots, \frac{k_2}{k_1} = \text{const.}, \quad \text{if } n \text{ is odd},$$

$$\frac{k_{n-1}}{k_{n-2}} = \text{const.}, \frac{k_{n-3}}{k_{n-4}} = \text{const.}, \ \ldots, \frac{k_3}{k_2} = \text{const.}, \quad \text{if } n \text{ is even}.$$

We call such helix, generalized helix in the sense of Hayden. In the case n is even, Hayden gave the following important result in [9]: "If the dimension n is even, there not exist a non-degenerate curve whose vectors $V_1, V_2, \ldots, V_n$ make constant angles with a parallel vector field along it".

In this paper, we give some characterizations for a non-degenerate curve α to be a generalized helix by using harmonic curvatures of the curve in n-dimensional Euclidean space $\mathbb{E}^n$. Also, we obtain a vector D for a non-degenerate curve α and we called it a *generalized Darboux vector*, then we study the relationship between the generalized Darboux vector D and the Darboux vector d in the same space.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be arbitrary curve in the Euclidean n- space $\mathbb{E}^n$. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{E}^n$ given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{E}^n$. In particular, the norm of a vector $X \in \mathbb{E}^n$ is given by $\|X\|^2 = \langle X, X \rangle$. Let $\{V_1, V_2, \ldots, V_n\}$ be the moving Frenet frame along the unit speed curve α, where V_i $(i = 1, 2, \ldots, n)$ denote ith Frenet vector fields. Then the Frenet formulas are given by

$$\begin{cases} V_1'(s) = k_1(s) V_2(s), \\ V_i'(s) = -k_{i-1}(s) V_{i-1}(s) + k_i(s) V_{i+1}(s), \quad i = 2, 3, \ldots, n-1, \\ V_n'(s) = -k_{n-1}(s) V_{n-1}(s), \end{cases}$$

where k_i $(i = 1, 2, \ldots, n-1)$ denote ith curvature functions of the curve [7,10]. If all curvatures $k_i (i = 1, 2, \ldots, n-1)$ of the curve nowhere vanish in $I \subset \mathbb{R}$, then the curve is called a non-degenerate curve.

Definition 2.1. [13]Let α be a unit curve in $\mathbb{E}^n$. Harmonic curvatures of α is defined by

$$H_i : I \subset \mathbb{R} \to \mathbb{R}, \quad i = 0, 1, 2, \ldots, n-2,$$

$$H_i = \begin{cases} 0, & i = 0 \\ \frac{k_1}{k_2} & i = 1 \\ \{V_1[H_{i-1}] + H_{i-2}k_i\}\frac{1}{ki+1}, & i = 2, 3, \ldots, n-2. \end{cases}$$

3. Harmonic curvatures and generalized helices

In this section, we give some characterizations for generalized helices by using the harmonic curvatures of the curve.

Theorem 3.1. [13]. *Let $\alpha(s)$ be a unit speed generalized helix in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{H_1, H_2, \ldots, H_{n-2}\}$ be denote the Frenet frame and the higher ordered harmonic curvatures of the curve, respectively. Then the following equations is holds*

$$\langle V_{i+2}, X\rangle = H_i\langle V_1, X\rangle, \quad 1 \leqslant i \leqslant n-1, \tag{1}$$

where X is axis of a helix α.

By using Theorem 3.1, we have the following corollary:

Corollary 3.1. *If X is axis of a helix α, then we can write*

$$X = \lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_n V_n.$$

From the Theorem 3.1, we get

$$\lambda_i = \langle V_i, X\rangle = H_{i-2}\langle V_1, X\rangle,$$

where $\langle V_1, X\rangle = \cos\theta =$ constant.

By the definition of the harmonic curvature, we obtain

$$X = \cos\theta(V_1 + H_1V_3 + \cdots + H_{n-2}V_n). \tag{2}$$

Also

$$D = V_1 + H_1V_3 + \cdots + H_{n-2}V_n$$

is axis of the helix α.

Definition 3.1. Let $\alpha(s)$ be a unit speed non-degenerate curve in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{H_1, H_2, \ldots, H_{n-2}\}$ be denote the Frenet frame and the higher ordered harmonic curvatures of the curve, respectively. The vector

$$D = V_1 + H_1V_3 + \cdots + H_{n-2}V_n \tag{3}$$

is called the *generalized Darboux vector* of the curve α.

Theorem 3.2. *Let $\alpha(s)$ be a unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{H_1, H_2, \ldots, H_{n-2}\}$ be denote the Frenet frame and the higher ordered harmonic curvatures of the curve, respectively. Then α is a generalized helix if and only if D is a constant vector.*

Proof. Let α be a generalized helix in $\mathbb{E}^n$ and X be axis of α. From the Corollary 3.1, we get

$$X = \cos\theta(V_1 + H_2V_3 + \cdots + H_{n-2}V_n), \tag{4}$$

where $\cos\theta$ is a constant and so we can easily see that D is constant.

Conversely, if D is constant vector, then we can see that

$$\langle D, V_1\rangle = 1.$$

Thus we get $\cos\theta = \frac{1}{\|D\|}$, where θ is constant angle between D and V_1. In this case, we can define unique axis of the helix as follows:

$$X = (\cos\theta)D.$$

where $\langle X, V_1\rangle = \cos\theta =$ constant. Therefore X is a constant. So, this complete the proof. □

Corollary 3.2. *In three-dimensional Euclidean space, from Eq.* (3), *we can write the axis of a non-degenerate curve as*;

$$D = V_1 + \frac{k_1}{k_2} V_3,$$

where k_1 and k_2 are curvatures of the curve. If we take derivative of D along the curve, we get

$$\nabla_{V_1} D = \left(\frac{k_1}{k_2}\right)' V_3. \tag{5}$$

Thus, from the above equation, if the curve is a generalized helix, then from Theorem 3.2, we have $\nabla_{V_1} D = 0$, then we get $\frac{k_1}{k_2}$ is constant. If $\frac{k_1}{k_2}$ is constant, from Eq. (5), we obtain $\nabla_{V_1} D = 0$ and D is a constant vector. From Theorem 3.2, the curve is a generalized helix. This is a new proof of the Lancret theorem.

Corollary 3.3. *In four-dimensional Euclidean space, from Eq.* (3), *we get axis of a non-degenerate curve as*

$$D = V_1 + \frac{k_1}{k_2} V_3 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' V_4,$$

where k_1, k_2 and k_3 are curvatures of the curve. If all curvatures of the curve are constants, i.e., the curve is a W-curve, then we get

$$D = V_1 + \frac{k_1}{k_2} V_3. \tag{6}$$

If we take derivative of Eq. (6) *along the curve, we obtain*

$$D' = \nabla_{V_1} D = \left(\frac{k_1 k_3}{k_2}\right) V_4. \tag{7}$$

So, we can easily see that D' is not equal to zero, then D is not constant vector. In this case, according to Theorem 3.2, the curve is not helix.

Remark. In [13], Özdamar and Hacısalihoğlu gave the following characterization for general helices by using harmonic curvatures of the curve.

Theorem . *Let $\alpha(s)$ be a unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$ with Frenet vectors $\{V_1, V_2, \ldots, V_n\}$, and harmonic curvatures $\{H_1, H_2, \ldots, H_{n-2}\}$. Then α is a general helix, if and only if $\sum_{i=1}^{n-2} H_i^2 = constant$.*

The above theorem is true for the case necessity but not true for the case sufficiency. The following example show us why the case sufficiency is not true?

Example. $\alpha(s) = \left(a\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), a\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), b\cos\left(\frac{1}{\sqrt{a^2r^2+b^2}}s\right), b\sin\left(\frac{1}{\sqrt{a^2r^2+b^2}}s\right)\right)$ is a unit speed curve in $\mathbb{E}^4$. It is easily obtain the Frenet vectors and curvatures as follows:

$$V_1 = \begin{pmatrix} \frac{-ar}{\sqrt{a^2r^2+b^2}}\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{ar}{\sqrt{a^2r^2+b^2}}\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \\ \frac{-b}{\sqrt{a^2r^2+b^2}}\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{b}{\sqrt{a^2r^2+b^2}}\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right) \end{pmatrix},$$

$$k_1 = \frac{\sqrt{a^2r^4+b^2}}{a^2r^2+b^2},$$

$$V_2 = \begin{pmatrix} \frac{-ar^2}{\sqrt{a^2r^4+b^2}}\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{-ar^2}{\sqrt{a^2r^4+b^2}}\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \\ \frac{-b}{\sqrt{a^2r^4+b^2}}\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{-b}{\sqrt{a^2r^4+b^2}}\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right) \end{pmatrix},$$

$$k_2 = \frac{abr(r^2-1)}{(a^2r^2+b^2)\sqrt{a^2r^4+b^2}},$$

$$V_3 = \begin{pmatrix} \frac{b}{\sqrt{a^2r^2+b^2}} \sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{-b}{\sqrt{a^2r^2+b^2}} \cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right) \\ \frac{-ar}{\sqrt{a^2r^2+b^2}} \sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{ar}{\sqrt{a^2r^2+b^2}} \cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right) \end{pmatrix},$$

$$k_3 = \frac{r}{\sqrt{a^2r^4+b^2}},$$

$$V_4 = \begin{pmatrix} \frac{b}{\sqrt{a^2r^4+b^2}} \cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{b}{\sqrt{a^2r^4+b^2}} \sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right) \\ \frac{-ar^2}{\sqrt{a^2r^4+b^2}} \cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \frac{-ar^2}{\sqrt{a^2r^4+b^2}} \sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right) \end{pmatrix}.$$

The generalized Darboux vector of the above curve is $D = V_1 + \frac{k_1}{k_2}V_3 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' V_4$. Since $\left(\frac{k_1}{k_2}\right)' = 0$, we get $D = V_1 + \frac{k_1}{k_2}V_3$. Thus we obtain $D' = \frac{k_1k_3}{k_2}V_4 \neq 0$. Although $H_1^2 + H_2^2 =$ constant, D is not constant vector. According to the Theorem 3.2, α is not a generalized helix. Also, the above example is a good example for Corollary 3.3.

The following theorem is a new characterization of helices.

Theorem 3.3. *Let $\alpha(s)$ be a unit speed non-degenerate curve in n-dimensional Euclidean space $\mathbb{E}^n$ with Frenet vectors $\{V_1, V_2, \ldots, V_n\}$, and harmonic curvatures $\{H_1, H_2, \ldots, H_{n-2}\}$. If α is a generalized helix, then $\sum_{i=1}^{n-2} H_i^2 =$ constant.*

The following theorem is a explicit characterization for a non-degenerate curve to be a generalized helix.

Theorem 3.4. *Let $\alpha(s)$ be a unit speed non-degenerate curve in n-dimensional Euclidean space $\mathbb{E}^n$ with Frenet vectors $\{V_1, V_2, \ldots, V_n\}$, and harmonic curvatures $\{H_1, H_2, \ldots, H_{n-2}\}$. Then α is a generalized helix if and only if $V_1[H_{n-2}] + k_{n-1}H_{n-3} = 0$.*

Proof. If we take derivative of D a long the curve α, we get

$\nabla_{V_1} D = (V_1[H_{n-2}] + k_{n-1}H_{n-3})V_n$, where ∇ denotes the Levi–Civita connection in $\mathbb{E}^n$. Since α is a generalized helix, D is a constant vector. Thus we obtain $\nabla_{V_1} D = 0$ or $V_1[H_{n-2}] + k_{n-1}H_{n-3} = 0$.

Conversely, we assume that the equation $V_1[H_{n-2}] + k_{n-1}H_{n-3} = 0$ holds, we easily obtain that D is a constant vector, then from Theorem 3.2, we have α is generalized helix in $\mathbb{E}^n$, which completes the proof. □

4. Geometrical means of the generalized Darboux vector D

It is well known that, when a point moves a long a curve α in Euclidean space $\mathbb{E}^3$, its Frenet trihedral (T, N, B), parallely translated to the origin, defines a rigid motion around the origin called *Frenet motion*. The instantaneous axis of rotation of Frenet motion that we call *Darboux axis*, is determined by the *Darboux vector* $d = k_2T + k_1B$, where k_1 and k_2 are the curvature and the torsion of the curve, respectively.

A rigid motion in Euclidean space has an instantaneous axis of rotation (*Darboux axis*) only if the space is of odd dimension.

Lemma 4.1. [16].*The Darboux axis of the time s is determined by the kernel of the Frenet matrix given with respect to the basis T, N, B,*

$$M(s) = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{pmatrix}.$$

Proposition 4.1. [16]. *The Darboux axis of α at the time s is determined by the kernel of the Frenet matrix $M(s)$ given with respect to the basis $T, N_1, N_2, \ldots, N_{2k}$ in $\mathbb{E}^{2k+1}$, $k > 2$. The Darboux vector is given by as a curve, in $\mathbb{E}^{2k+1}$, $k > 2$*

$$d = a_0T + a_1N_2 + \cdots + a_kN_{2k},$$

where $a_0 = k_2k_4 \ldots, k_{2k}$, $a_1 = \frac{k_1}{k_2}a_0$, $a_2 = \frac{k_3}{k_4}a_1, \ldots, a_i = \frac{k_{2i-1}}{k_{2i}}a_{i-1}$.

Proposition 4.2. [16]. *The Darboux vector* $d = a_0T + a_1N_2 + \cdots + a_{kN2k}$ *lies in the kernel of the Frenet matrix* $M(s)$ *in* $\mathbb{E}^{2k+1}$, $k > 2$.

Now, we study the relationship between the vectors; the generalized Darboux vector D and the Darboux vector d in n-dimensional Euclidean space $\mathbb{E}^n$.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a non-degenerate unit speed curve with Frenet vectors $V_1, V_2, \ldots, V_n$ and curvatures $k_1, k_2, \ldots, k_{n-1}$, then the Frenet matrix, $M(s)$ given by

$$M(s) = \begin{pmatrix} 0 & k_1 & 0 & \cdots & 0 & 0 \\ -k_1 & 0 & k_2 & \cdots & 0 & 0 \\ 0 & -k_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_{n-1} \\ 0 & 0 & 0 & \cdots & -k_{n-1} & 0 \end{pmatrix}.$$

Then we get,

$$M(s)D = \begin{bmatrix} 0 \\ 0 \\ H'_1 \\ H'_2 \\ \vdots \\ H'_{n-2} \end{bmatrix} - (H'_{n-2} + k_{n-1}H_{n-3}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{8}$$

Now we can ask the question: When does the generalized Darboux vector D lies in the kernel of $M(s)$? To obtain the answer of this question, from (7), we have the following cases:

Case I. For $n = 3$, we have $D = V_1 + H_1V_3$, then we easily obtain that $M(s)D = 0$.

Corollary 4.1. *In three-dimensional Euclidean space* $\mathbb{E}^3$, *for any non-degenerate unit speed curve* α, *the generalized Darboux vector D of the curve* α *lies in the kernel of* $M(s)$. *Then we get* $D = \frac{1}{k_2}d$.

Case II. For $n > 3$, and n even, then $M(s)$ is a regular matrix and there is only zero vector its kernel. Since $D \neq 0$, then, it can not be in the kernel of $M(s)$.

Case III. For $n > 3$, and n odd, then from (7) we easily obtain that:

$$H_2 = H_4 = \cdots = H_{n-3} = 0$$

and

$$H_1 = \frac{k_1}{k_2} = \text{const.}, H_3 = \frac{k_1}{k_2}\frac{k_3}{k_4} = \text{const.}, H_5 = \frac{k_1}{k_2}\frac{k_3}{k_4}\frac{k_5}{k_6} = \text{const.}, \ldots,$$
$$H_{n-2} = \frac{k_1}{k_2}\frac{k_3}{k_4}\frac{k_5}{k_6}\cdots\frac{k_{n-2}}{k_{n-1}} = \text{const.}$$

Thus, we get,

$$\frac{k_1}{k_2} = \text{const.}, \frac{k_3}{k_4} = \text{const.}, \ \cdots, \ \frac{k_{n-2}}{k_{n-1}} = \text{const.}$$

which means that the curve is a generalized helix in the sense of Hayden.

Corollary 4.2. *In n-dimensional Euclidean space* $\mathbb{E}^n$*(n-odd), for any non-degenerate unit speed curve* α, *the generalized Darboux vector D of the curve* α *lies in the kernel of* $M(s)$ *if and only if the curve* α *is a generalized helix in sense of Hayden.*

In this case, we have $D = \frac{1}{a_0}d$, *where* $a_0 = k_2k_4k_6\cdots k_{n-1}$.

Corollary 4.3. *W-curves in n-dimensional Euclidean space* $\mathbb{E}^n$*(n-odd), are generalized helix in the sense of Hayden.*

5. Conclusion

Helix one of the most fascinating curve in science and nature. Scientist have long held a fascination, sometimes bordering on mystical obsession, for helical structures in nature. Indeed a helix (also known as circular helix) is a easiest example of spirals. A curve of constant slope or generalized helix in Euclidean space $\mathbb{E}^3$ is defined by the property that its tangent indicatrix is a planar curve. The straight line perpendicular to this plane is called the axis of the generalized helix.

In this paper, we give some characterizations for a non-degenerate curve α to be a generalized helix by using harmonic curvatures of the curve in n-dimensional Euclidean space $\mathbb{E}^n$. Also, we obtain a vector D for a non-degenerate curve α and we called it a generalized Darboux vector, then we study the relationship between the generalized Darboux vector D and the Darboux vector d in the same space.

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