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A symbolic algorithm for exact power series solutions of *n*th order linear homogeneous differential equations with polynomial coefficients near an ordinary point

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Abstract

We developed an algorithm in Kıymaz and Mirasyedioğlu [O. Kıymaz and Ş. Mirasyedioğlu, An algorithmic approach to exact power series solutions of second order linear homogeneous differential equations with polynomial coefficients, Appl. Math. Comp. 139 (1) (2003) 165–178] for computing exact power series solutions of second order linear homogeneous differential equations with polynomial coefficients, near a point $x = x_0$. In this paper we present a symbolic algorithm to compute the exact power series solutions of *n*th order linear homogeneous differential equations with polynomial coefficients, near an ordinary point.

Keywords: Power series solutions; Generalized hypergeometric series; Symbolic computation

1. Introduction

In 1992, Koepf [2] used generalized hypergeometric series and the definition of a recurrence equation (RE) of hypergeometric type for calculating the power series of a given function. Koepf showed that each formal Laurent series (FLS) of hypergeometric type satisfies linear differential equations (DE) with polynomial coefficients [2]. In [1], we developed a new algorithm to find the exact power series solutions of the second order linear DE with polynomial coefficients by using some transformations and the coefficient formula in [2].

The main purpose of this paper is to give a symbolic algorithm that computes the exact power series solutions of *n*th order linear homogeneous DE with polynomial coefficients near an ordinary point.

Without loss of generality we assume that, all solutions will be computed at the point $x_0 = 0$.

2. Preliminaries

This section introduces some basic definitions and notations which will be used through the paper.

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Definition 2.1. A (generalized) hypergeometric function has a series representation $\sum_{n=0}^{\infty} c_n$ with c_{n+1}/c_n a rational function of n. The ratio c_{n+1}/c_n can be factored and it's usually written as

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)x}{(n+b_1)(n+b_2)\cdots(n+b_q)(n+1)}. \tag{2.1}$$

Then if $c_0 = 1$, the equation can be solved for c_n as

$$c_n = \frac{(a_1)_n(a_2)_n\cdots(a_p)_n x^n}{(b_1)_n(b_2)_n\cdots(b_p)_n n!},$$

where $(a)_n$ denotes the Pochammer symbol (or shifted factorial)

$$(a)_n = \begin{cases} 1 & n = 0, \\ a(a+1)\cdots(a+n-1) & n \in IN, \end{cases}$$

and

$$pFq\begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{pmatrix}, x\Bigg) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n n!} x^n \tag{2.2}$$

is the usual notation [3].

Definition 2.2. Let $F = \sum_{n=n_0}^{\infty} a_n x^n (a_{n_0} \neq 0)$ be a formal Laurent series (FLS) for some $n_0 \in \boldsymbol{Z}$. Then F is called to be hypergeometric type if it has a positive radius of convergence, and if its coefficients satisfy a RE of the form

$$\begin{aligned} a_{n+m} &= R(n)a_n, \quad n \geqslant n_0, \\ a_n &= A_n, \quad n = n_0, n_0+1, \ldots, n_0+m-1 \end{aligned} \tag{2.3}$$

for some $m \in \boldsymbol{IN}$, $A_n \in \boldsymbol{C}(n = n_0+1, \ldots, n_0+m-1)$, $A_{n_0} \in \boldsymbol{C} \setminus \{0\}$, and some rational function R. The number m is then called the symmetry number of F. A RE of type (2.3) is also called to be of hypergeometric type [2]. Note that a function of the form $f(x^m)$ and so as the formal power series (FPS) of the form $F(x^m)$ is called m-fold symmetric. It is clear that, each FLS with symmetry number m can be represented as the sum of m shifted m-fold symmetric functions [2].

Lemma 2.3. *Let $x_0 = 0$ be an ordinary point of a nth order linear DE with polynomial coefficients. The RE of the DE, can be computed applying the transformation*

$$x^m y^{(k)} \to (n+1-m)_k a_{n+k-m} \tag{2.4}$$

to the all terms in DE. Here $(n + 1 - m)_k$ denotes the Pochammer of $n + 1 - m$ [1].

To get the exact power series solutions of a given DE, we use the fact that; the coefficients c_n of the hypergeometric series $\sum_{n=0}^{\infty} c_n x^n$ are the unique solution of the special RE (2.1) with the initial condition $c_0 = 1$ [2].

Theorem 2.4. *If the corresponding RE of a nth order linear DE with polynomial coefficient is hypergeometric type, then the exact power series solutions of DE can be computed.*

Proof. If the RE is hypergeometric type then it is of the form (2.3). So the symmetry number is equal to m. Applying the transformation $n \to mn$ to RE we get

$$a_{m(n+1)} = R(mn)a_{mn}.$$

Applying a second transformation $a_{mn} \to c_n$, we get the following equality:

$$c_{n+1} = R(mn)c_n.$$

The last equality is of the form (2.1) with multiplying the variable x. Making similar calculations, with the transformations $n \to mn + 1, \ldots, mn + (m-1)$, we get m equations of the form (2.1). For each equation, we can find the coefficients of corresponding power series. So, we can represent the solutions as the sum of m shifted m-fold symmetric functions. □

3. The algorithm

Here we present an algorithm that compute the exact power series solutions of a given nth order linear DE with polynomial coefficients at the ordinary point $x_0 = 0$.

Algorithm 3.1

(1) Apply the transformation (2.4) to the all terms at the lhs of the input. Write the results in the equation. Then construct the RE.
(2) In the rhs of the RE, apply a substitution which makes the index of coefficient c, n. Determine the symmetry number m. Represent the solutions as the sum of m shifted m-fold symmetric functions.
(3) From these, choose the series, which are of the form (2.1). Find the explicit formula for the coefficients using some calculations and the hypergeometric coefficient formula (2.2). Then construct the results.

Theorem 3.2. *Let $x = x_0$ be the ordinary point of the nth order linear homogeneous DE with polynomial coefficients. Then Algorithm 3.1 computes the exact power series solutions correctly if the RE of the DE is hypergeometric type.*

Proof. The proof of the step (1) is given by Koepf in [2]. In step (2), we transformed the RE to the form (2.3) by a substitution that makes the index of coefficient c, n. Then applying the transformations described in proof of Theorem 2.4 we get m shifted m-fold symmetric functions. After choosing the series which are of the form (2.1) in step (3), the proof of the calculations for finding the explicit formula is also given by Koepf in [2]. □

Example 3.3. Assume that $y''' + xy' + 2y = 0$ is given. Then $x = 0$ is an ordinary point. So we can apply the Algorithm 3.1. With transformation (2.4) we get the RE as

$$a_{n+3} = -\frac{(n+2)a_n}{(n+1)(n+2)(n+3)}.$$

The RE is hypergeometric type and the symmetry number is 3. So,

(i) Applying the transformations $n \to 3k$ and $a_{3k} \to c_k$ to RE we get

$$c_{k+1} = -\frac{c_k}{3^2\left(k+\frac{1}{3}\right)(k+1)}.$$

This is of the form (2.1) and it is the first solution of the DE. So with setting the coefficient $c_0 = 1$ we obtain the result as

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{3^n n![1 \cdot 4 \cdots (3n-2)]}.$$

(ii) Applying the transformations $n \to\ 3k + 1$ and $a_{3k+1} \to c_k$ to RE we get

$$c_{k+1} = -\frac{(k+1)c_k}{3^2\left(k+\frac{2}{3}\right)\left(k+\frac{4}{3}\right)(k+1)}.$$

This is of the form (2.1) and it is the second linearly independent solution of the DE. So with setting the coefficient $c_0 = 1$ we obtain the result as

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{[2 \cdot 5 \cdots (3n-1)][4 \cdot 7 \cdots (3n+1)]}.$$

(iii) Applying the transformations $n \to 3k+2$ and $a_{3k+2} \to c_k$ to RE we get

$$c_{k+1} = -\frac{c_k}{3^2\left(k+\frac{5}{3}\right)(k+1)}.$$

This is of the form (2.1) and it is the third linearly independent solution of the DE. So with setting the coefficient $c_0 = 1$ we obtain the result as

$$y_3(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{3^n n! [5 \cdot 8 \cdots (3n+2)]}.$$

Finally the general solution of the given DE can be found as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x),$$

where C_1, C_2 and C_3 are arbitrary constants.

Example 3.4. Consider the equation $y''' + x^2 y'' + xy' - y = 0$. $x = 0$ is an ordinary point. If we apply the transformation (2.4) we get the RE as

$$a_{n+3} = -\frac{(n-1)(n+1)a_n}{(n+1)(n+2)(n+3)}.$$

The RE is hypergeometric type and the symmetry number is 3. Applying the transformations $n \to 3k$ and $a_{3k} \to c_k$, $n \to 3k+1$ and $a_{3k+1} \to c_k$, $n \to 3k+2$ and $a_{3k+2} \to c_k$ to RE respectively we get

$$c_{k+1} = -\frac{\left(k-\frac{1}{3}\right)c_k}{3\left(k+\frac{2}{3}\right)(k+1)}, \tag{3.1}$$

$$c_{k+1} = -\frac{kc_k}{3\left(k+\frac{4}{3}\right)(k+1)}$$

and

$$c_{k+1} = -\frac{\left(k-\frac{1}{3}\right)c_k}{3\left(k+\frac{4}{3}\right)\left(k+\frac{5}{3}\right)}.$$

From Eq. (3.1) we get the first solution as

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{3n}}{3^n n!(3n-1)}.$$

With similar calculations and setting the constant $c_0 = 1$, we get the second and third solutions as

$$y_2(x) = x$$

and

$$y_3(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{[5 \cdot 8 \cdots (3n+2)](3n+1)}.$$

So the general solution of the given differential equation is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x).$$

The Maple code of Algorithm 3.1 is similar with the code in [1]. The reader can be constructed the code of Algorithm 3.1 with basic adaptations.

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