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Generalized limits and sequence of matrices

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Abstract

Banach has proved that there exist positive linear regular functionals on m such that they are invariant under shift operator where m is the space of all bounded real sequences. It has also been shown that there exists positive linear regular functionals L on m such that $L(\chi_K) = 0$ for every characteristic sequence χ_K of sets, K, of natural density zero. Recently the comparison of such functionals and some applications have been examined. In this paper we define $S_{\mathfrak{B}}$ -limits and $\mathfrak{B}$-Banach limits where $\mathfrak{B}$ is a sequence of infinite matrices. It is clear that if $\mathfrak{B} = (A)$ then these definitions reduce to S_A-limits and A-Banach limits. We also show that the sets of all $S_{\mathfrak{B}}$ -limits and Banach limits are distinct but their intersection is not empty. Furthermore, we obtain that the generalized limits generated by $\mathfrak{B}$ where $\mathfrak{B}$ is strongly regular is equal to the set of Banach limits.

Keywords The Hahn–Banach extension theorem · Banach limit · $\mathfrak{B}$-statistical limit superior and inferior · Sequence of infinite matrices

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1 Introduction

Banach has proved the existence of positive linear regular functionals L on m such that they are invariant under shift operator, i.e., $L(\sigma(x)) = L(x)$; where

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$\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ and m is the space of all bounded real sequences [1,3,8,10]. These functionals are called Banach limits and it has been shown that the space of all almost convergent sequences can be characterized with the use of Banach limits [12].

In [17] the authors have noticed that the space of all bounded statistically convergent sequences can be represented as the set of all bounded sequences which have the same value under some generalized limits with the motivation of a result of Freedman [5]. It has also been proved that the set of such limits, called S_A-limits, and the set of Banach limits are distinct. The key role in the definition of S_A-limits is the A-density of $K \subseteq \mathbb{N}$ where A is a nonnegative, regular matrix. Since $\mathfrak{B}$-statistical convergence is a well studied concept of convergence and includes A-statistical convergence, lacunary statistical convergence, uniform statistical convergence as special cases, one can ask the $\mathfrak{B}$-statistical analogues of these results. In this study we examine the results analogues to those of [17].

An outline of the paper is as follows: The next section contains basic notations and definitions. In Sect. 3 we show the existence of $S_{\mathfrak{B}}$-limits and $\mathfrak{B}$-Banach limits which coincide with S_A-limits and A-Banach limits in the case $\mathfrak{B} = (A)$, respectively. In Sect. 4 we recall the definition of functionals that dominate or generate generalized limits. We also provide such functionals that dominate or generate $S_{\mathfrak{B}}$-limits. In the final section we present comparison results concerning the set of all $S_{\mathfrak{B}}$-limits and $\mathfrak{B}$-Banach limits.

2 Notations and definitions

Let c be the space of all convergent real sequences $x = (x_k)$. Note that m and c are normed by $\|x\| = \sup_n |x_n|$. Let Γ be the class of linear functionals γ on m which are nonnegative and regular, i.e, if $x_k \geq 0$ for all $k \in \mathbb{N}$ then $\gamma(x) \geq 0$, and also $\gamma(x) = \lim_k x_k$, for each $x \in c$ which are also called extended limits. If γ satisfies the equality $\gamma(\sigma(x)) = \gamma(x)$ for all $x \in m$ then γ is called a Banach limit.

Let $A = [a_{nk}]$ be an infinite matrix. The A-transform of a given sequence x is given by

$$(Ax)_n = \sum_k a_{nk}\, x_k, \tag{1}$$

where the series converges for each n, and denoted as $Ax = ((Ax)_n)$. If $\lim_A x := \lim_n (Ax)_n$ exists, then we say that x is A-summable. The space of all A-summable sequences is denoted by c_A, i.e., $c_A = \{x : \ \lim_A x \ exists\}$. We say that A is regular [4,14] if $\lim_n (Ax)_n = \lim_k x_k$ for each $x \in c$.

Let $\mathfrak{B} = (B_i)$ be a sequence of infinite matrices with $B_i = (b_{nk}^{(i)})$. Then $x \in m$ is said to be $F_{\mathfrak{B}}-$convergent (or $\mathfrak{B}-$summable) to the value L if

$$\lim_n (B_i x)_n = \lim_n \sum_k b_{nk}^{(i)} x_k = L, \text{ uniformly in } i \geq 0.$$

In this case we write $\mathfrak{B} - lim x = L$ (see, [2] and [16]). The method $\mathfrak{B}$ is regular if and only if

- $\| \mathfrak{B} \| < \infty$
- $\lim_n b_{nk}^{(i)} = 0$, for all $k \geq 1$, uniformly in i
- $\lim_n \sum_k b_{nk}^{(i)} = 1$, uniformly in i

where $\|\mathfrak{B}\| := \sup_{n,i} \sum_k |b_{nk}^{(i)}| < \infty$ means that there exists a constant M such that $\sum_k |b_{nk}^{(i)}| \leq M$ for all n, i and the series $\sum_k |b_{nk}^{(i)}|$ converges uniformly in i for each n.

Kolk [11] introduced the following.

An index set K is said to have $\mathfrak{B}$-density denoted by $\delta_{\mathfrak{B}}(K) = d$, if the characteristic sequence of K is $\mathfrak{B}-$summable to d, i.e.

$$\lim_n \sum_{k \in K} b_{nk}^{(i)} = d, \text{uniformly in } i;$$

where by an index set we mean a set $K = \{k_r\} \subseteq \mathbb{N}$, $k_r < k_{r+1}$ for all r. Throughout the paper the statement $\delta_{\mathfrak{B}}(K) \neq 0$ will mean either $\delta_{\mathfrak{B}}(K) > 0$ or that the $\mathfrak{B}$-density of K does not exist.

Let $\mathfrak{R}^+$ denote the set of all regular methods $\mathfrak{B}$ with $b_{nk}^{(i)} \geq 0$ for all n, k and i. Let $\mathfrak{B} \in \mathfrak{R}^+$. A sequence $x = (x_k)$ is called $\mathfrak{B}$-statistically convergent to the number l, if for every $\varepsilon > 0$

$$\delta_{\mathfrak{B}}(\{k : |x_k - l| \geq \varepsilon\}) = 0$$

and we write $st_{\mathfrak{B}} - \lim x = l$. We denote the space of all $\mathfrak{B}-$statistically convergent sequences by $st(\mathfrak{B})$. In particular, if $\mathfrak{B} = (C_1)$, then $\mathfrak{B}$-statistical convergence is reduced to the usual statistical convergence. For $\mathfrak{B} = (I)$, the identity matrix, then $\mathfrak{B}$-statistical convergence is also reduced to the usual convergence. For $\mathfrak{B} = (A)$, it is reduced to A-statistical convergence. For $\mathfrak{B} = \mathfrak{B}_1$, it is reduced to uniform statistical convergence where $\mathfrak{B}_1 = (b_{nk}^{(i)})$

$$b_{nk}^{(i)} = \begin{cases} \frac{1}{n}, & \text{if } 1 + i \leq k \leq n + i \\ 0, & \text{otherwise.} \end{cases}$$

Following [5] and [17] we give

Definition 1 Let L be a linear functional on m that satisfies the following properties:

(1) $L(x) \geq 0$, if $x_k \geq 0$ for all k,
(2) $L(x) = \lim_k x_k$ for $x \in c$,
(3) For every $K \subseteq \mathbb{N}$ such that $\delta_{\mathfrak{B}}(K) = 0$ implies that $L(\chi_K) = 0$.

We will call such a functional an $S_{\mathfrak{B}}$-limit, and we will denote their collection by $SL_{\mathfrak{B}}$.

If $\mathfrak{B} = (C_1)$, then any such L will be called an S-limit and their collection will be denoted by SL. Freedman [5] proved that the space of all bounded statistically convergent sequences can be characterized with the use of S-limits.

It is well known that Banach has proved the existence of Banach limits. In [17] the authors have introduced the notion of an A-Banach limit as a generalization of Banach limits. In a similar way we introduce the following

Definition 2 ($\mathfrak{B}$-*Banach limits*) Let L be a bounded linear functional on m that satisfies the following conditions:

(1) $L(x) \geq 0$ if $x_k \geq 0$ for all k.
(2) $L(x) = \lim_k x_k$ if $x \in c$,
(3) $L(x) \leq \lim_n \sup_i \sup_j \sup \sum b_{nk}^{(i)} x_{k+j}$ for every $x \in m$.

Any such L will be called a $\mathfrak{B}$-Banach limit, and the collection of all such functionals will be denoted by $BL_{\mathfrak{B}}$.

If $\mathfrak{B} = (C_1)$, then we get $BL_{C_1} = BL$ where BL is the set of all Banach limits. Lorentz [12] has proved that every Banach limit agree on the set of all almost convergent sequences which is denoted by ac.

3 The existence of $\mathfrak{B}$-limits and $\mathfrak{B}$-Banach limits

In this section we will present some results concerning the properties of these functionals.

If for any bounded real sequence x with $\lim_A x = l$ also implies $\lim_A \sigma x = l$ then the matrix A is translative [14]. A regular matrix A is (boundedly) translative [14] if and only if

$$\lim_{n\to\infty} \sum_k |a_{n,k+1} - a_{nk}| \ = \ 0.$$

It is known that the bounded convergence field of any regular summability method cannot be equal to ac. A regular matrix A is (boundedly) translative if and only if $A \in (ac, c, p)$, i.e, A sums all almost convergent sequences and preserves their Banach limits [14]. These methods are called strongly regular. This concept has been extended for a sequence of matrices $\mathfrak{B}$ by Bell [2].

A sequence of matrices $\mathfrak{B}$ is called strongly regular, if whenever a sequence x is almost convergent to l, then $\mathfrak{B}x$ converges to l. It has been proved by Bell [2] that a regular sequence of matrices $\mathfrak{B} = (B_i)$ is strongly regular if and only if

$$\lim_n \sum_k |b_{n,k+1}^{(i)} - b_{nk}^{(i)}| \ = 0, \text{ uniformly in } i.$$

Now we are ready to prove the following theorem which is an analog of Theorem 2.1 in [17]. It shows not only the existence of $S_{\mathfrak{B}}$-limits and $\mathfrak{B}$-Banach limits but also a little more than it.

Theorem 1 *Let $\mathfrak{B} \in \mathfrak{R}^+$. Then both $\mathfrak{B}$-Banach limits and $S_{\mathfrak{B}}$-limits exist. Furthermore, the following results hold:*

(1) $BL_{\mathfrak{B}} \cap SL_{\mathfrak{B}} \neq \emptyset$.
(2) $BL = BL_{\mathfrak{B}}$ *if and only if* $\mathfrak{B}$ *is strongly regular.*
(3) $BL \cap SL_{\mathfrak{B}} \neq \emptyset$ *when* $\mathfrak{B}$ *is strongly regular. In particular, if an almost convergent sequence is also* $\mathfrak{B}$*-statistically convergent then the two limits are equal.*

Proof Define the sublinear functional as follows

$$Q_{\mathfrak{B}}(x) = \lim_n \sup_i \sup \sum_k b_{nk}^{(i)} x_k, \qquad x \in m.$$

By the regularity of $\mathfrak{B}$, we see that $Q_{\mathfrak{B}}(x) = \lim_k x_k$ for each $x \in c$. With the use of Hahn–Banach theorem one can find a bounded linear functional R defined on m so that

$$-Q_{\mathfrak{B}}(-x) \leq R(x) \leq Q_{\mathfrak{B}}(x), \quad x \in m. \tag{2}$$

Denote the set of all such R by $\mathcal{L}_{\mathfrak{B}}$. We will also show $\mathcal{L}_{\mathfrak{B}} \subseteq SL_{\mathfrak{B}}$ which says a little more than the theorem's statement.

It is easy to see that $R(x) \geq 0$ for every $x \geq 0$ and $R(x) = \ell(x) = \lim_k x_k$ for every $x \in c$. Next if E is a set with $\mathfrak{B}$-density zero, then by (2) one can see that $0 \leq R(\chi_E) \leq Q_{\mathfrak{B}}(\chi_E) = 0$. Hence, R is an $S_{\mathfrak{B}}$-limit. It also belongs to $BL_{\mathfrak{B}}$ since $Q_{\mathfrak{B}}(x) \leq \limsup_n \sup_i \sup_j \sum_k b_{nk}^{(i)} x_{k+j}$, for all $x \in m$, which means that $\mathcal{L}_{\mathfrak{B}} \subseteq BL_{\mathfrak{B}} \cap SL_{\mathfrak{B}}$.

(2) If $S \in BL_{\mathfrak{B}}$ and $\mathfrak{B}$ is strongly regular, then we get

$$|S(\sigma x - x)| \leq \|x\| \lim_n \sup_i \sum_k |b_{n,k+1}^{(i)} - b_{n,k}^{(i)}| = 0.$$

This gives us $BL_{\mathfrak{B}} \subseteq BL$. By taking into account the regularity of $\mathfrak{B}$, one can easily obtain that $BL \subseteq BL_{\mathfrak{B}}$ with the use of a similar argument in Theorem 19 (c) in [15].

Conversely, if $S \in BL_{\mathfrak{B}} = BL$, then for any sequence which is almost convergent to l, we must have $S(x) = \ell$, for every $S \in BL_{\mathfrak{B}}$ which implies that

$$0 = \lim_n \inf_i \inf_j \inf \sum_k b_{nk}^{(i)}(x_{k+j} - \ell) = \lim_n \sup_i \sup_j \sup \sum_k b_{nk}^{(i)}(x_{k+j} - \ell) = 0.$$

This implies that $\lim_n \sum_k b_{nk}^{(i)}(x_k - \ell) = 0$, uniformly in i. Therefore, $\mathfrak{B}$ is strongly regular.

(3) Let $\mathfrak{B}$ be strongly regular. From (2), it is obvious that $BL \cap SL_{\mathfrak{B}} \neq \varnothing$ and if a sequence is almost convergent and also $\mathfrak{B}$-statistically convergent then the two limits must be equal.

□

We will also provide an example which shows it is possible to have $BL \cap SL_{\mathfrak{B}} = \emptyset$ when $\mathfrak{B}$ is not strongly regular.

Let us recall the concepts of $\mathfrak{B}-$statistical limit superior and $\mathfrak{B}-$statistical limit inferior to examine further relationships between various generalized limits. Following [6], [7] Mursaleen and Edely [13] have defined

$$st_{\mathfrak{B}} - \limsup x = \begin{cases} \sup G_x, & \text{if } G_x \neq \emptyset \\ -\infty, & \text{if } G_x = \emptyset, \end{cases}$$

where $G_x = \{b \in \mathbb{R} : \delta_{\mathfrak{B}}(\{k \in \mathbb{N} : \ x_k > b\}) \neq 0\}$. Also the $\mathfrak{B}$-statistical limit inferior of x is given by

$$st_{\mathfrak{B}} - \liminf x = \begin{cases} \inf F_x, & \text{if } F_x \neq \emptyset \\ +\infty, & \text{if } F_x = \emptyset, \end{cases}$$

where $F_x = \{a \in \mathbb{R} : \delta_{\mathfrak{B}}(\{k \in \mathbb{N} : \ x_k < a\}) \neq 0\}$.

The next result is an analog of Proposition 2.2 in [17].

Proposition 1 *If* $\mathfrak{B} \in \mathfrak{R}^+$ *and* $P_{\mathfrak{B}}(x) := st_{\mathfrak{B}} - \limsup x$, *then the following results hold.*

(a) $-P_{\mathfrak{B}}(-x) = st_{\mathfrak{B}} - \liminf x$, *for all* $x \in m$.
(b) $P_{\mathfrak{B}}(x + y) \leq P_{\mathfrak{B}}(x) + P_{\mathfrak{B}}(y)$, *for any* $x, y \in m$.
(c) $P_{\mathfrak{B}}(\alpha x) = \alpha P_{\mathfrak{B}}(x)$ *for any* $\alpha \geq 0$ *and* $x \in m$.

Proof Since $\mathfrak{B}$ is nonnegative and

$$\begin{aligned} P_{\mathfrak{B}}(-x) &= \sup\left\{b : \limsup_n \sup_i \sum_{k:-x_k>b} b_{nk}^{(i)} > 0\right\} \\ &= -\inf\left\{-b : \limsup_n \sup_i \sum_{k:x_k<-b} b_{nk}^{(i)} > 0\right\} = -st_{\mathfrak{B}} - \liminf_n x, \end{aligned}$$

we can obtain the proof of (a). For (c), in case of $\alpha > 0$, we have

$$\begin{aligned} P_{\mathfrak{B}}(\alpha x) &= \sup\left\{b : \limsup_n \sup_i \sum_{k:\alpha x_k>b} b_{nk}^{(i)} > 0\right\} \\ &= \alpha \sup\left\{\frac{b}{\alpha} : \limsup_n \sup_i \sum_{k:x_k>b/\alpha} b_{nk}^{(i)} > 0\right\} = \alpha P_{\mathfrak{B}}(x). \end{aligned}$$

It is easy to see that for $\alpha = 0$, $P_{\mathfrak{B}}(\alpha x) = \alpha P_{\mathfrak{B}}(x)$.

For part (b), let $P_{\mathfrak{B}}(x)=\ell_x$ and $P_{\mathfrak{B}}(y)=\ell_y$. For any $\varepsilon>0$, we therefore have that

$$\lim_n \sup_i \sum_{k:x_k>\ell_x+\frac{\varepsilon}{2}} b_{nk}^{(i)}=0, \qquad \lim_n \sup_i \sum_{k:y_k>\ell_y+\frac{\varepsilon}{2}} b_{nk}^{(i)}=0.$$

Therefore, we have

$$\lim_n \sup_i \sum_{k:x_k+y_k>\ell_x+\ell_y+\varepsilon} b_{nk}^{(i)} \le \lim_n \sup_i \sum_{k:x_k>\ell_x+\frac{\varepsilon}{2}} b_{nk}^{(i)} + \lim_n \sup_i \sum_{k:y_k>\ell_y+\frac{\varepsilon}{2}} b_{nk}^{(i)}=0.$$

This gives that $P_{\mathfrak{B}}(x+y)\le\ell_x+\ell_y+\varepsilon$ for all $\varepsilon>0$ which completes the proof. □

4 Domination and generation

In this section we study the relationship between sublinear functional $P_{\mathfrak{B}}$ and $SL_{\mathfrak{B}}$.

Now let us recall the definitions of functionals that generate (dominate) generalized limits [15]. By m^* we denote the algebraic dual of m.

Definition 3 Let S and V be sublinear functionals on m and let $\mathcal{L}$ be a collection of bounded linear functionals on m. Then

(i) S is said to generate $\mathcal{L}$ if for any $T\in m^*$ and $T(x)\le S(x)$ for all $x\in m$ together imply that $T\in\mathcal{L}$.
(ii) V is said to dominate $\mathcal{L}$ if for every $T\in\mathcal{L}$ we have $T(x)\le V(x)$ for all $x\in m$.

A sublinear functional, S, on m generates $\mathcal{L}$ if and only if $S(x)\le W(x)$ for all x, where

$$W(x) := \sup\{T(x): T\in\mathcal{L}\}, \text{ for all } x\in m.$$

It is easy to see that a sublinear functional, S, dominates $\mathcal{L}$ if and only if $S(x)\ge W(x)$ for all $x\in m$. Considering these two statements together, a sublinear functional S on m both generates and dominates L-limits if and only if S equals W. The following theorem shows that $P_{\mathfrak{B}}$ both generates and dominates $SL_{\mathfrak{B}}$-limits. Motivated by [17] we have

Theorem 2 *Let $\mathfrak{B}=(b_{nk}^{(i)})\in\mathfrak{R}^+$. Then the following results hold.*

(i) *$P_{\mathfrak{B}}$ both generates and dominates $SL_{\mathfrak{B}}$. Therefore,*

$$P_{\mathfrak{B}}(x)=\sup\{T(x): T\in SL_{\mathfrak{B}}\}, \textit{ for all } x\in m.$$

(ii) *$Q_{\mathfrak{B}}$ generates $SL_{\mathfrak{B}}$. Furthermore, if $\mathfrak{B}$ sums a divergent $0,1$ sequence to a number $l\in(0,1)$ then $Q_{\mathfrak{B}}$ does not dominate $SL_{\mathfrak{B}}$.*
(iii) *Q_I dominates $SL_{\mathfrak{B}}$ where I is the identity matrix and $\mathfrak{B}=(I)$. Furthermore, if $\mathfrak{B}$ sums a divergent $0,1$ sequence to zero then Q_I does not generate $SL_{\mathfrak{B}}$.*

Proof Let $T \in SL_{\mathfrak{B}}$. If there exists a sequence $x \in m$ so that $T(x) > P_{\mathfrak{B}}(x)$, then without loss of generality we may assume that $x_k \geq 0$ for all k. Then take $p \in (P_{\mathfrak{B}}(x), T(x))$ and take $E = \{k : x_k > p\}$ which implies

$$\limsup_n \sup_i \sum_{k:k\in E} b_{nk}^{(i)} = 0.$$

Hence, $\delta_{\mathfrak{B}}(E) = 0$. Therefore, one can obtain

$$\begin{aligned} T(x) &= T(x\chi_E) + T(x\chi_{E^c}) \\ &\leq \|x\| T(\chi_E) + pT(\chi_{E^c}) \\ &\leq pT(e) \;=\; p \;<\; T(x) \end{aligned}$$

where $e = (1, 1, 1, \ldots)$. Then this is a contradiction and we have that $T(x) \leq P_{\mathfrak{B}}(x)$ for all $x \in m$ which means that $P_{\mathfrak{B}}$ dominates $SL_{\mathfrak{B}}$. The fact that $P_{\mathfrak{B}}$ generates $SL_{\mathfrak{B}}$ follows by an identical proof as that of part (1) of Theorem 1. So we omit the details here.

(ii) It is already known that $Q_{\mathfrak{B}}$ generates $SL_{\mathfrak{B}}$, by part (1) of Theorem 1. In order to show that $Q_{\mathfrak{B}}$ does not dominate $SL_{\mathfrak{B}}$, one can find a sequence, x, of zero's and one's which $\mathfrak{B}$ sums to a number $\ell \in (0, 1)$. Let $E \subseteq \mathbb{N}$ so that $\chi_E = x$. Note that $P_{\mathfrak{B}}(\chi_E) = 1$. Furthermore,

$$Q_{\mathfrak{B}}(\chi_E) = \lim_n \sup_i \sum_{k:k\in E} b_{nk}^{(i)} = \ell < 1 = P_{\mathfrak{B}}(\chi_E) = \sup\{T(\chi_E) : \; T \in SL_{\mathfrak{B}}\}.$$

By part (i), we can write the last equality. Therefore, $Q_{\mathfrak{B}}$ does not dominate $SL_{\mathfrak{B}}$.

(iii) Since $P_{\mathfrak{B}}(x) \leq Q_I(x) = \limsup_n \sup_i \sum_k I_{nk}^{(i)} x_k = \limsup_n x_n$ for all $x \in m$, and $P_{\mathfrak{B}}$ dominates $SL_{\mathfrak{B}}$, it must be that Q_I dominates $SL_{\mathfrak{B}}$. In order to show that Q_I cannot generate $SL_{\mathfrak{B}}$, we construct a positive regular functional L so that $L(x) \leq Q_I(x)$ for all $x \in m$ but $L \notin SL_{\mathfrak{B}}$. To produce it, let $E \subseteq \mathbb{N}$ be an infinite set so that $\delta_{\mathfrak{B}}(E) = 0$. Denote the members of E as $j_1 < j_2 < \cdots$. Define a new nonnegative regular matrix $\mathfrak{B} = (b_{nk}^{(i)})$ where $b_{nk} = 1$ when $k = j_n$ and $b_{nk} = 0$ for other values of k, for all i. Using the resulting $Q_{\mathfrak{B}}$, and the linear functional $\lim_{\mathfrak{B}}$ on $m \cap c_{\mathfrak{B}}$, by the Hahn–Banach theorem, we obtain a bounded linear functional, L, on m so that $L(x) = \lim_{\mathfrak{B}} x$ on $m \cap c_{\mathfrak{B}}$. Certainly, $Q_{\mathfrak{B}}(x) \leq Q_I(x)$, and hence $L(x) \leq Q_I(x)$ for all $x \in m$. However, $L(\chi_E) = \lim_{\mathfrak{B}} \chi_E = 1$. On the other hand, $\delta_{\mathfrak{B}}(E) = 0$ which implies that for every $T \in SL_{\mathfrak{B}}$ we must have $T(\chi_E) = 0$. Hence, $L \notin SL_{\mathfrak{B}}$. □

In order to sum a divergent sequence of 0's and 1's to a number $l \in (0, 1)$ for a matrix, there are some well known sufficient conditions [9].

Let $\mathfrak{B} \in \mathfrak{R}^+$ and consider a class, $\tau_{\mathfrak{B}}^*$ consists of those $\mathfrak{C} = [c_{nk}^{(i)}]$ such that

(i) $\mathfrak{C}$ is nonnegative.
(ii) $\lim_n \sum_k c_{nk}^{(i)} = 1$, uniformly in i.
(iii) For every $K \subseteq \mathbb{N}$ with $\delta_{\mathfrak{B}}(K) = 0$ implies that $\delta_{\mathfrak{C}}(K) = 0$.

Theorem 3 *Let $\mathfrak{B} \in \mathfrak{R}^+$ and let $\mathfrak{C} = [c_{nk}^{(i)}]$ be a nonnegative method with*

$$\sup_{n,i} \sum_k c_{nk}^{(i)} < \infty.$$

Then the following results hold.

(i) *$Q_{\mathfrak{C}}$ generates $SL_{\mathfrak{B}}$ if and only if $\mathfrak{C} \in \tau_{\mathfrak{B}}^*$.*
(ii) *If $Q_{\mathfrak{C}}$ dominates $SL_{\mathfrak{B}}$ then* $\liminf_n \inf_i \sum_k c_{nk}^{(i)} \le 1 \le \limsup_n \sup_i \sum_k c_{nk}^{(i)}$.

Proof (i) Let $\mathfrak{C} \in \tau_{\mathfrak{B}}^*$ and $L \in m^*$ so that $L(x) \le Q_{\mathfrak{C}}(x)$ for all $x \in m$. Since $\mathfrak{C}$ is nonnegative, L is positive. Since $Q_{\mathfrak{C}}(e) = 1$, we have $L(e) = 1$ as well. Also, for every $K \subseteq \mathbb{N}$ for which $\delta_{\mathfrak{B}}(K) = 0$, we have $\delta_{\mathfrak{C}}(K) = 0$. This gives that $L(\chi_K) = 0$. For $x \in c$ with $\ell = \lim_k x_k$ we have $L(x) = \lim_k x_k = Q_{\mathfrak{C}}(x)$. That is, $Q_{\mathfrak{C}}$ generates $SL_{\mathfrak{B}}$. Conversely, assume that $Q_{\mathfrak{C}}$ generates $SL_{\mathfrak{B}}$. Hence, it must be that $Q_{\mathfrak{C}}(x) \le P_{\mathfrak{B}}(x)$ for all $x \in m$ which gives that $\lim_n \sup_i \sum_k c_{nk}^{(i)} = 1$. Also, if $K \subseteq \mathbb{N}$ such that $\delta_{\mathfrak{B}}(K) = 0$, then

$$0 = -P_{\mathfrak{B}}(-\chi_K) \le -Q_{\mathfrak{C}}(-\chi_K) \le Q_{\mathfrak{C}}(\chi_K) \le P_{\mathfrak{B}}(\chi_K) = 0.$$

That is, $\delta_{\mathfrak{C}}(K) = 0$. Hence, $\mathfrak{C} \in \tau_{\mathfrak{B}}^*$.

(ii) This follows easily from $P_{\mathfrak{B}}(e) = 1$ and $P_{\mathfrak{B}}(x) \le Q_{\mathfrak{C}}(x)$ for all $x \in m$. □

5 Comparison of $SL_{\mathfrak{B}}$ and $BL_{\mathfrak{B}}$

In this section we examine the characteristic features of the two types of generalized limits. We have already proved that, when $\mathfrak{B} \in \mathfrak{R}^+$,

$$\mathcal{L}_{\mathfrak{B}} \subseteq SL_{\mathfrak{B}} \cap BL_{\mathfrak{B}}.$$

We also show that there exist such functionals that generate (dominate) $SL_{\mathfrak{B}}$ but does not generate (dominate) $BL_{\mathfrak{B}}$ when $\mathfrak{B}$ is strongly regular, and conversely.

Theorem 4 *Let $\mathfrak{B} \in \mathfrak{R}^+$ and strongly regular then neither $SL_{\mathfrak{B}}$ nor $BL_{\mathfrak{B}}$ contains the other.*

Proof In order to see that there exists a functional Ψ in $SL_{\mathfrak{B}}$ but not in $BL_{\mathfrak{B}}$, we can use the same sequence in the proof of Theorem 4.1 in [17]. Again, to see that $SL_{\mathfrak{B}}$ does not contain $BL_{\mathfrak{B}}$, it is sufficient to take $\mathfrak{B} = (A)$ where A is the same as in the proof of Theorem 4.1 in [17]. □

As a result of the above theorem we can give the following corollary.

Corollary 1 *Let $\mathfrak{B}$ be strongly regular. Then one can find a sublinear functional which generates(dominates) $S_{\mathfrak{B}}$-limits but does not generate(dominate) Banach limits, also a sublinear functional which generates(dominates) Banach limits but does not generate(dominate) $S_{\mathfrak{B}}$-limits.*

Now the following example shows that BL and $SL_{\mathfrak{B}}$ can be mutually exclusive.

Example 1 Let us consider the matrix A as in [17]. Then define $\mathfrak{B} = \left[b_{nk}^{(i)}\right]$ as follows:

$$b_{nk}^{(1)} = \begin{bmatrix} 0 & 0 & a_{11} & a_{12} & \cdots \\ 0 & 0 & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & a_{n1} & a_{n2} & \cdots \\ 0 & 0 & \vdots & \vdots & \ddots \end{bmatrix}, \quad b_{nk}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{11} & a_{12} & \cdots \\ 0 & 0 & 0 & 0 & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & a_{n1} & a_{n2} & \cdots \\ 0 & 0 & 0 & 0 & \vdots & \vdots & \ddots \end{bmatrix}$$

and $b_{nk}^{(i)}$ by adding $2i$ columns of $0's$ from the beginning for every i. One can easily see that $\mathfrak{B} \in \mathfrak{R}^+$. Let $x = (1, 0, 1, 0, 1, 0, \ldots)$ then $\lim_{\mathfrak{B}} x = 1$. Notice that, $\lim_{\mathfrak{B}} \sigma(x) = 0$, which implies that $\mathfrak{B}$ is not strongly regular. Note that x is $\mathfrak{B}$-statistically convergent to 1 since, for small ε, the set $\{k : |x_k - 1| > \varepsilon\}$ consists of even numbers whose $\mathfrak{B}$-density is zero. Similarly, $\sigma(x)$ is $\mathfrak{B}$-statistically convergent to 0, since, for small ε, the set $\{k : |\sigma(x)_k - 0| > \varepsilon\}$ consists of even numbers whose $\mathfrak{B}$-density is zero.

Note that for any Banach limit L we have $L(1, -1, 1, -1, \ldots) = 0$. Now to see that no $L \in SL_{\mathfrak{B}}$ can be a Banach limit in BL, if there exists an $L \in SL_{\mathfrak{B}}$ as well as L is a Banach limit then

$$1 - 0 = L(x - \sigma x) = L(1, -1, 1, -1, \ldots) = 0.$$

Hence, in this case $BL \cap SL_{\mathfrak{B}} = \emptyset$.

By Theorem 1 we immediately obtain that $BL_{\mathfrak{B}} \neq BL$, since $\mathfrak{B}$ is not strongly regular. However, $\mathfrak{B} \in \mathfrak{R}^+$ then we have $BL \subset BL_{\mathfrak{B}}$ where the inclusion must, therefore, be strict. A simple example can also be constructed from [17] to see the strict inclusion. Consider the matrix A as in the example of [17] and then let $\mathfrak{B} = (A)$. Take $x_k = 0$ for k even, $x_k = 1$ for k odd and $\frac{k+1}{2}$ odd, and $x_k = 2$ for k odd and $\frac{k+1}{2}$ even. It is known from [17] that this sequence is almost convergent to $\frac{3}{4}$ but not A-almost convergent. Therefore it is not $\mathfrak{B}$-summable. Hence, there must exist $L_1, L_2 \in BL_{\mathfrak{B}}$ for which $L_1(x) \neq L_2(x)$, making at least one of them not in BL. Therefore this completes the construction of the example for the strict inclusion.

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