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On the boundedness of the B_n-maximal operator on B_n-Orlicz spaces

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Abstract. *In this paper we prove the weak and strong type boundedness of the B_n-maximal operator M_γ in B_n-Orlicz spaces $L_{\Phi,\gamma}(\mathbb{R}^n_+)$.*

Keywords. B_n-maximal operator · B_n-Orlicz space.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35.

1 Introduction

The fractional integral operators play an important role in the theory of harmonic analysis, differentiation theory and PDE's. The fractional integrals and related topics associated with the Laplace-Bessel differential operator have been research areas for many mathematicians such as I.A. Kipriyanov [14], L.N. Lyakhov [13], A.D. Gadjiev and I.A. Aliev [1], A. Serbetci and I. Ekincioglu [11], V.S. Guliyev [6,8] and others.

The first author [8] have introduced the maximal function, generated by the Laplace-Bessel differential operator (B_n-maximal function) and investigated the boundedness of B_n-maximal operator in $L_{p,\gamma}$-spaces. I. Kipriyanov and M. Klyuchantsev [16], [17] have introduced the singular integrals, generated by the Laplace-Bessel differential operator (B_n-singular integrals) and investigated the boundedness of B_n-singular operators in $L_{p,\gamma}$-spaces. I. Aliev and A. Gadjiev [3], A. Gadjiev and E.V. Guliyev [5] and I. Ekincioglu [2] have studied the boundedness of B_n-singular integrals in weighted $L_{p,\gamma}$-spaces.

Let $\mathbb{R}^n_+$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x=(x_1,\ldots,x_n)$ defined by the inequalities $x_n>0$, and γ is a fixed positive number. Denote by $L_{p,\gamma}\equiv L_{p,\gamma}(\mathbb{R}^n_+)$ the set of all classes of measurable functions f with finite norm

$$\|f\|_{L_{p,\gamma}}=\left(\int_{\mathbb{R}^n_+}|f(x)|^p x_n^\gamma dx\right)^{\frac{1}{p}},\ \ 1\le p<\infty.$$

If $p=\infty$, we assume

$$L_{\infty,\gamma}(\mathbb{R}^n_+)=L_\infty(\mathbb{R}^n_+)=\{f:\|f\|_{L_{\infty,\gamma}}=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n_+}|f(x)|<\infty\}.$$

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For measurable set $E \subset \mathbb{R}_+^n$ let $|E|_\gamma = \int_E x_n^\gamma dx$, then $|E(0,r)|_\gamma = \omega(n,\gamma) r^{n+\gamma}$, where $\omega(n,\gamma) = |E(0,1)|_\gamma$.

The operator of generalized shift (B_n shift operator) is defined by the following way (see [16], [20]):

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2}\right) \sin^{\gamma-1}\alpha d\alpha,$$

where $C_\gamma = \pi^{-\frac{1}{2}} \Gamma\left(\gamma + \frac{1}{2}\right) \Gamma^{-1}(\gamma)$.

Note that the generalized shift operator T^y is closely related to the Δ_{B_n} Laplace-Bessel differential operator ([14])

$$\Delta_{B_n} = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + B_n,$$

where $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \ \gamma > 0$ (see [16], [20]).

Furthermore, T^y generates the corresponding B_n-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}_+^n} f(y)\, T^y g(x)\, y_n^\gamma dy.$$

The fractional integrals and related topics associated with the Laplace-Bessel differential operator have been research areas for many mathematicians such as I.A. Kipriyanov [14], L.N. Lyakhov [13], A.D. Gadjiev and I.A. Aliev [1], A.D. Gadjiev and V.S. Guliyev [4], A. Serbetci and I. Ekincioglu [11], V.S. Guliyev [6]-[10] and others.

Let $f \in L_{1,\gamma}^{\rm loc}(\mathbb{R}_+^n)$. The B_n-maximal function $M_\gamma f$ is defined by (see[6])

$$M_\gamma f(x) = \sup_{r>0} |B(0,r)|^{-1} \int_{B(0,r)} T^y |f(x)|\, y_n^\gamma\, dy.$$

The following theorems is valid (see [6]).

Theorem 1.1 *1) Let $f \in L_{1,\gamma}(\mathbb{R}_+^n)$. Then for any $\tau > 0$*

$$\left|\left\{x \in \mathbb{R}_+^n : M_\gamma f(x) > \tau\right\}\right|_\gamma \le \frac{C_3}{\tau} \int_{\mathbb{R}_+^n} |f(x)|\, x_n^\gamma dx,$$

where constant C_3 does not depend from f.

2) Let $f \in L_{p,\gamma}(\mathbb{R}_+^n)$, $1 < p \le \infty$. Then $M_\gamma f(x) \in L_{p,\gamma}(\mathbb{R}_+^n)$ and

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_4 \|f\|_{L_{p,\gamma}},$$

where constant C_4 does not depend from f.

Corollary 1.1 *If $f \in L_{p,\gamma}(\mathbb{R}_+^n)$, $1 \le p \le \infty$, then*

$$\lim_{\varepsilon \to 0} |B(0,\varepsilon)|_\gamma^{-1} \int_{B(0,\varepsilon)} T^y f(x)\, y_n^\gamma dy = f(x)$$

for almost all $x \in \mathbb{R}_+^n$.

Remark 1.1 Let's note, that Theorem 1.1 in a one-dimensional case, that is at $n = 1$ is prove in [25], and in a multivariate case $n \ge 2$ in [6] (see [7]).

It is well known that maximal operator play an important role in harmonic analysis (see [24]). Harmonic analysis associated to the Fourier-Bessel transform and the Laplace-Bessel differential operator gives rise to convolutions with a relevant generalized translation. In the framework of this analysis we study Hardy-Littlewood maximal functions (B_n-maximal functions) in the relevant Orlicz-Bessel space (B_n-Orlicz space). We prove the weak and strong type boundedness of the B_n-maximal operator M_B in B_n-Orlicz spaces $L_{\Phi,\gamma}(\mathbb{R}^n_+)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 On Young Functions and Orlicz Spaces

Orlicz space was first introduced by Orlicz in [21,22] as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

First, we recall the definition of Young functions.

Definition 2.1 *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \qquad r \geq 0$$

for some $C > 1$.

Lemma 2.1 *[19] Let Φ be a Young function with canonical representation*

$$\Phi(t) = \int_0^t \varphi(s)ds, t > 0.$$

(1) *Assume that $\Phi \in \Delta_2$. More precisely $\Phi(2t) \leq A\Phi(t)$ for some $A \geq 2$. Set $\beta = \log_2 A$. If $p > \beta + 1$, then the following inequality is valid:*

$$\int_t^\infty \frac{\varphi(s)}{s^p} \lesssim \frac{\Phi(t)}{t^p},\ t > 0.$$

(2) *Assume that $\Phi \in \nabla_2$. Then the following inequality is valid:*

$$\int_0^t \frac{\varphi(s)}{s} \lesssim \frac{\Phi(t)}{t},\ t > 0.$$

Definition 2.2 *(Orlicz Space). For a Young function Φ, the set*

$$L_{\Phi,\gamma}(\mathbb{R}^n_+) = \left\{ f \in L^{\rm loc}_{1,\gamma}(\mathbb{R}^n_+) : \int_{\mathbb{R}^n_+} \Phi(k|f(x)|)\, x_n^\gamma\, dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L_{\Phi,\gamma}(\mathbb{R}^n_+) = L_{p,\gamma}(\mathbb{R}^n_+)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_{\Phi,\gamma}(\mathbb{R}^n_+) = L_\infty(\mathbb{R}^n_+)$. The space $L^{\rm loc}_{\Phi,\gamma}(\mathbb{R}^n_+)$ is defined as the set of all functions f such that $f\chi_B \in L_{\Phi,\gamma}(\mathbb{R}^n_+)$ for all balls $B \subset \mathbb{R}^n_+$.

$L_{\Phi,\gamma}(\mathbb{R}^n_+)$ is a Banach space with respect to the norm

$$\|f\|_{L_{\Phi,\gamma}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n_+} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, x_n^\gamma\, dx \leq 1 \right\}.$$

For a measurable function f on $\mathbb{R}^n_+$ and $t > 0$, let $m(f,t)_\gamma = \big|\{x \in \mathbb{R}^n_+ : |f(x)| > t\}\big|_\gamma$.

Definition 2.3 *The weak Orlicz space*

$$WL_{\Phi,\gamma}(\mathbb{R}^n_+) = \{f \in L^{\rm loc}_{1,\gamma}(\mathbb{R}^n_+) : \|f\|_{WL_{\Phi,\gamma}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL_{\Phi,\gamma}} = \inf \Big\{ \lambda > 0\ :\ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big)_\gamma\ \leq 1 \Big\}.$$

We note that $\|f\|_{WL_{\Phi,\gamma}} \leq \|f\|_{L_{\Phi,\gamma}}$,

$$\sup_{t>0} \Phi(t) m(f,t)_\gamma = \sup_{t>0} t\, m(f, \Phi^{-1}(t))_\gamma = \sup_{t>0} t\, m(\Phi(|f|), t)_\gamma$$

and

$$\int_{\mathbb{R}^n_+} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_{\Phi,\gamma}}}\Big)\, x_n^\gamma\, dx \leq 1, \qquad \sup_{t>0} \Phi(t) m\Big(\frac{f}{\|f\|_{WL_{\Phi,\gamma}}},\ t\Big)_\gamma \leq 1. \tag{2.2}$$

The following analogue of the Hölder's inequality is well known (see, for example, [23]).

Theorem 2.1 *Let the functions f and g measurable on $\mathbb{R}^n_+$. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_{\mathbb{R}^n_+} |f(x)g(x)|\, x_n^\gamma\, dx \le 2\|f\|_{L_{\Phi,\gamma}} \|g\|_{L_{\widetilde{\Phi},\gamma}}.$$

By elementary calculations we have the following property.

Lemma 2.2 *Let Φ be a Young function and B be a balls in $\mathbb{R}^n_+$. Then*

$$\|\chi_B\|_{L_{\Phi,\gamma}} = \|\chi_B\|_{WL_{\Phi,\gamma}} = \frac{1}{\Phi^{-1}\left(|B|_\gamma^{-1}\right)}.$$

Proof. Note that, by elementary calculus the following equalities are valid

$$\begin{aligned}
\|\chi_B\|_{L_{\Phi,\gamma}} &= \inf\left\{\lambda > 0 : \int_B \Phi\left(\frac{1}{\lambda}\right) y_n^\gamma\, dy \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \Phi\left(\frac{1}{\lambda}\right) \int_B y_n^\gamma\, dy \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{\lambda} \le \Phi^{-1}\left(|B|_\gamma^{-1}\right)\right\} \\
&= \inf\left\{\lambda > 0 : \lambda \ge \frac{1}{\Phi^{-1}\left(|B|_\gamma^{-1}\right)}\right\} \\
&= \frac{1}{\Phi^{-1}\left(|B|_\gamma^{-1}\right)},
\end{aligned}$$

and

$$\begin{aligned}
\|\chi_B\|_{WL_{\Phi,\gamma}} &= \inf\left\{\lambda > 0 : \sup_{t>0} \Phi\left(\frac{t}{\lambda}\right) \left|\{x \in \mathbb{R}^n_+ : |\chi_B(x)| > t\}\right|_\gamma \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \sup_{0<t<1} \Phi\left(\frac{t}{\lambda}\right) \left|\{x \in \mathbb{R}^n_+ : |\chi_B(x)| > t\}\right|_\gamma \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \Phi\left(\frac{1}{\lambda}\right) \le |B|_\gamma^{-1}\right\} \\
&= \inf\left\{\lambda > 0 : \lambda \ge \frac{1}{\Phi^{-1}\left(|B|_\gamma^{-1}\right)}\right\} \\
&= \frac{1}{\Phi^{-1}\left(|B|_\gamma^{-1}\right)}.
\end{aligned}$$

By Theorem 2.1, Lemma 2.2 and (2.1) we get the following estimate.

Lemma 2.3 *For a Young function Φ and for the balls $B = B(x,r)$ the following inequality is valid:*

$$\int_B |f(y)|\, y_n^\gamma\, dy \le 2|B|_\gamma \Phi^{-1}\left(|B|_\gamma^{-1}\right) \|f\|_{L_{\Phi,\gamma}(B)}.$$

3 Boundedness of the B_n-maximal operator in B_n-Orlicz spaces $L_{\Phi,\gamma}(\mathbb{R}^n_+)$

In this section the boundedness of the maximal operator M_γ in B_n-Orlicz spaces $L_{\Phi,\gamma}(\mathbb{R}^n_+)$ have been obtained.

Theorem 3.1 *Let Φ any Young function. Then the B_n-maximal operator M_γ is bounded from $L_{\Phi,\gamma}(\mathbb{R}^n_+)$ to $WL_{\Phi,\gamma}(\mathbb{R}^n_+)$ and for $\Phi \in \nabla_2$ bounded in $L_{\Phi,\gamma}(\mathbb{R}^n_+)$.*

Proof. At first proved that the B_n-maximal operator M_γ is bounded from $L_{\Phi,\gamma}(\mathbb{R}^n_+)$ to $WL_{\Phi,\gamma}(\mathbb{R}^n_+)$.

We take $f \in L_{\Phi,\gamma}(\mathbb{R}^n_+)$ satisfying $\|f\|_{L_{\Phi,\gamma}} = 1$ so that the modular

$$\rho_{\Phi,\gamma}(f) := \int_{\mathbb{R}^n_+} \Phi(|f(x)|)\, x_n^\gamma \, dx \le 1.$$

We know that by Jensen inequality

$$\Phi\left(\frac{1}{|B|_\gamma}\int_B |f(y)|\, y_n^\gamma \, dy\right) \le \frac{1}{|B|_\gamma}\int_B \Phi(|f(y)|)\, y_n^\gamma \, dy \tag{3.1}$$

for all balls B. Using (3.1) and definition of B_n-maximal operator we have

$$\Phi(M_\gamma f(x)) \le M_\gamma[(\Phi \circ f)(x)]. \tag{3.2}$$

Using (3.2) and weak $(1,1)_\gamma$ boundedness of the B_n-maximal operator we get

$$\begin{aligned}
\big|\{x \in \mathbb{R}^n_+ : M_\gamma f(x) > t\}\big|_\gamma &= \big|\{x \in \mathbb{R}^n_+ : \Phi(M_\gamma f(x)) > \Phi(t)\}\big|_\gamma \\
&\le \big|\{x \in \mathbb{R}^n_+ : M_\gamma(\Phi \circ f)(x) > \Phi(t)\}\big|_\gamma \\
&\le \frac{C}{\Phi(t)}\int_{\mathbb{R}^n_+} \Phi(|f(x)|)\, x_n^\gamma \, dx \\
&\le \frac{C}{\Phi(t)} \le \frac{1}{\Phi(\frac{t}{C\|f\|_{L_\Phi}})},
\end{aligned}$$

since $\|f\|_{L_\Phi} = 1$ and $\frac{1}{C}\Phi(t) \ge \Phi\left(\frac{t}{C}\right)$, if $C \ge 1$.

Since $\|\cdot\|_{L_{\Phi,\gamma}}$ norm is homogeneous the inequality

$$\big|\{x \in \mathbb{R}^n_+ : M_\gamma f(x) > t\}\big|_\gamma \le \frac{1}{\Phi(\frac{t}{C\|f\|_{L_{\Phi,\gamma}}})}$$

is true for every $f \in L_{\Phi,\gamma}(\mathbb{R}^n_+)$.

Now proved that for $\Phi \in \nabla_2$ the B_n-maximal operator M_γ is bounded in $L_{\Phi,\gamma}(\mathbb{R}^n_+)$.

Let $\Lambda > 0$ and $f \in L_{\Phi,\gamma}(\mathbb{R}^n_+)\backslash\{0\}$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^n_+} \Phi\left(\frac{M_\gamma f(x)}{\Lambda}\right) x_n^\gamma\, dx &= \int_{\mathbb{R}^n_+} \int_0^{\frac{M_\gamma f(x)}{\Lambda}} \varphi(s) ds\, x_n^\gamma\, dx \\ &= \int_{\mathbb{R}^n_+} \int_0^{\infty} \chi_{\{s\in[0,\infty):\frac{M_\gamma f(x)}{\Lambda}>s\}} \varphi(s) ds\, x_n^\gamma\, dx \\ &= \int_0^{\infty} \varphi(s) \int_{\mathbb{R}^n_+} \chi_{\{x\in\mathbb{R}^n_+:M_\gamma f(x)>\Lambda s\}}\, x_n^\gamma\, dx ds \\ &= \frac{1}{\Lambda}\int_0^{\infty} \varphi\left(\frac{\lambda}{\Lambda}\right) \left|\{x\in\mathbb{R}^n_+ : M_\gamma f(x)>\lambda\}\right|_\gamma d\lambda \\ &= \frac{2}{\Lambda}\int_0^{\infty} \varphi\left(\frac{2\lambda}{\Lambda}\right) \left|\{x\in\mathbb{R}^n_+ : M_\gamma f(x)>\lambda\}\right|_\gamma d\lambda.\end{aligned}$$

From the maximal inequality (see [7, Lemma X])

$$\left|\{x\in\mathbb{R}^n_+ : M_\gamma f(x)>2\lambda\}\right|_\gamma \lesssim \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n_+:|f(x)|>\lambda\}} |f(x)|\, x_n^\gamma\, dx$$

and change the order of integration

$$\begin{aligned}\int_{\mathbb{R}^n_+} \Phi\left(\frac{M_\gamma f(x)}{\Lambda}\right) x_n^\gamma\, dx &\lesssim \frac{1}{\Lambda}\int_0^{\infty} \varphi\left(\frac{2\lambda}{\Lambda}\right) \left(\int_{\{x\in\mathbb{R}^n_+:|f(x)|>\lambda\}} |f(x)|\, x_n^\gamma\, dx\right)\frac{d\lambda}{\lambda} \\ &\lesssim \frac{1}{\Lambda}\int_{\mathbb{R}^n_+} |f(x)| \left(\int_0^{|f(x)|} \varphi\left(\frac{2\lambda}{\Lambda}\right)\frac{d\lambda}{\lambda}\right) x_n^\gamma\, dx \\ &\lesssim \frac{1}{\Lambda}\int_{\mathbb{R}^n_+} |f(x)| \left(\int_0^{2\Lambda^{-1}|f(x)|} \varphi(\lambda)\frac{d\lambda}{\lambda}\right) x_n^\gamma\, dx.\end{aligned}$$

Now we use Lemma 2.1 which yields

$$\left(\int_0^{2\Lambda^{-1}|f(x)|} \varphi(\lambda)\frac{d\lambda}{\lambda}\right) \lesssim |f(x)|^{-1}\Lambda\Phi\left(\frac{2|f(x)|}{\Lambda}\right),$$

if $f(x)\neq 0$. Recall that $k\Phi(t)\leq \Phi(kt)$ for $k\geq 1$ and $t>0$, assuming Φ convex. Therefore, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n_+} \Phi\left(\frac{M_\gamma f(x)}{\Lambda}\right) x_n^\gamma\, dx &\leq c_0\int_{\mathbb{R}^n_+} \Phi\left(\frac{2|f(x)|}{\Lambda}\right) x_n^\gamma\, dx \\ &\leq \int_{\mathbb{R}^n_+} \Phi\left(\frac{c_0|f(x)|}{\Lambda}\right) x_n^\gamma\, dx.\end{aligned}$$

Here c_0 is a constant we would like to shed light on. Choosing $\Lambda = c_0\|f\|_{L_{\Phi,\gamma}}$, we obtain

$$\int_{\mathbb{R}^n_+} \Phi\left(\frac{M_\gamma f(x)}{\Lambda}\right) x_n^\gamma\, dx \leq 1.$$

This means

$$\|M_\gamma f\|_{L_{\Phi,\gamma}} \leq \Lambda = c_0\|f\|_{L_{\Phi,\gamma}}$$

from the definition of the norm.

Remark 3.1 Note that Theorem 3.1 in the case $\Phi(t)=t^p$, $1\leq p<\infty$ were proved in [6, 8].

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