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FRACTIONAL MAXIMAL OPERATOR AND ITS COMMUTATORS IN GENERALIZED MORREY SPACES ON HEISENBERG GROUP

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Abstract. In this paper we study the boundedness of the fractional maximal operator M_α on Heisenberg group $\mathbb{H}^n$ in the generalized Morrey spaces $M_{p,\varphi}(\mathbb{H}^n)$. We shall give a characterization for the strong and weak type Spanne and Adams type boundedness of M_α on the generalized Morrey spaces, respectively. Also we give a characterization for the Spanne and Adams type boundedness of fractional maximal commutator operator $M_{b,\alpha}$ on the generalized Morrey spaces.

1. Introduction

Heisenberg groups, in discrete and continuous versions, appear in many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology. We state some basic results about Heisenberg group. More detailed information can be found in [4, 7, 8] and the references therein. Let $\mathbb{H}^n$ be the $2n+1$-dimensional Heisenberg group. That is, $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, with multiplication

$$(z,t)\cdot(w,s) = (z+w, t+s+2Im(z\cdot\bar{w})),$$

where $z\cdot\bar{w} = \sum\limits_{j=1}^{n} z_j\bar{w}_j$. The inverse element of $u=(z,t)$ is $u^{-1}=(-z,-t)$ and we write the identity of $\mathbb{H}^n$ as $0=(0,0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}^n$, for $r>0$, by $\delta_r(z,t)=(rz,r^2t)$. These dilations are group automorphisms and the Jacobian determinant is r^Q, where $Q=2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. A homogeneous norm on $\mathbb{H}^n$ is given by

$$|(z,t)| = (|z|^2+|t|)^{1/2}.$$

With this norm, we define the Heisenberg ball centered at $u=(z,t)$ with radius r by $B(u,r)=\{v\in\mathbb{H}^n : |u^{-1}v|<r\}$, and we denote by $B_r = B(0,r) = \{v \in \mathbb{H}^n : |v|<r\}$ the open ball centered at 0, the identity element of $\mathbb{H}^n$, with radius r. The volume of the ball $B(u,r)$ is $C_Q r^Q$, where C_Q is the volume of the unit ball B_1.

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Using coordinates $u = (z,t) = (x+iy,t)$ for points in $\mathbb{H}^n$, the left-invariant vector fields X_j, Y_j and T on $\mathbb{H}^n$ equal to $\dfrac{\partial}{\partial x_j}$, $\dfrac{\partial}{\partial y_j}$ and $\dfrac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t},$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra of $\mathbb{H}^n$ with commutation relations

$$[Y_j, X_j] = 4T$$

for $j = 1, ..., n$, and all other commutators equal to 0.

Let $f \in L_1^{\rm loc}(\mathbb{H}^n)$. The fractional maximal operator M_α and the fractional integral operator I_α are defined by

$$M_\alpha f(u) = \sup_{r>0} |B(u,r)|^{-1+\alpha/Q} \int_{B(u,r)} |f(v)|\, dV(v),$$

$$I_\alpha f(u) = \int_{\mathbb{H}^n} \frac{f(v)dV(v)}{\left|u^{-1}v\right|^{Q-\alpha}}, \qquad 0 < \alpha < Q,$$

where Q is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$ and $|B(u,r)|$ is the Haar measure of the $\mathbb{H}^n$- ball $B(u,r)$. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator on $\mathbb{H}^n$. Recall that, for $0 < \alpha < Q$,

$$M_\alpha f(u) \le C_Q^{\frac{\alpha}{Q}-1}\ I_\alpha |f|(u).$$

The operators M_α and I_α play an important role in real and harmonic analysis and applications (see, for example [4] and [7]).

In the present work, we shall give a characterization for the Spanne and Adams type boundedness of the operator M_α on the generalized Morrey spaces, including weak versions. Also we give a characterization for the Spanne and Adams type boundedness of fractional maximal commutator operator $M_{b,\alpha}$ on the generalized Morrey spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized Morrey spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(\mathbb{H}^n)$ play an important role, see [9]. They were introduced by C. Morrey in 1938 [17]. The Morrey space in a Heisenberg group is defined as follows: for $1 \le p \le \infty$, $0 \le \lambda \le Q$, a function $f \in L_{p,\lambda}(\mathbb{H}^n)$ if $f \in L_p^{\rm loc}(\mathbb{H}^n)$ and

$$\|f\|_{L_{p,\lambda}} := \sup_{u\in\mathbb{H}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(u,r))} < \infty,$$

(If $\lambda = 0$, then $L_{p,0}(\mathbb{H}^n) = L_p(\mathbb{H}^n)$; if $\lambda = Q$, then $L_{p,Q}(\mathbb{H}^n) = L_\infty(\mathbb{H}^n)$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda}(\mathbb{H}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{H}^n$.)

We also denote by $WL_{p,\lambda}(\mathbb{H}^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{H}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{H}^n)} = \sup_{u\in\mathbb{H}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(u,r))} < \infty,$$

where $WL_p(B(u,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(B(u,r))} = \sup_{t>0} t\, |\{v \in B(u,r) :\ |f(v)| > t\}|^{1/p}\,. \tag{2.1}$$

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1. Let $1 \le p < \infty$ and $\varphi(u,r)$ be a positive measurable function on $\mathbb{H}^n \times (0,\infty)$. The generalized Morrey space $M_{p,\varphi}(\mathbb{H}^n)$ is defined of all functions $f \in L_p^{loc}(\mathbb{H}^n)$ by the finite norm

$$\|f\|_{M_{p,\varphi}} = \sup_{u\in\mathbb{H}^n, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} \|f\|_{L_p(B(u,r))}.$$

Also the weak generalized Morrey space $WM_{p,\varphi}(\mathbb{H}^n)$ is defined of all functions $f \in L_p^{loc}(\mathbb{H}^n)$ by the finite norm

$$\|f\|_{WM_{p,\varphi}} = \sup_{u\in\mathbb{H}^n, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} \|f\|_{WL_p(B(u,r))}.$$

The following lemma in the Euclidean setting was proved in [2, 3]

Lemma 2.1. [6] *Let $\varphi(u,r)$ be a positive measurable function on $\mathbb{H}^n \times (0,\infty)$.*

(i) *If*

$$\sup_{t<r<\infty} \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} = \infty \quad \text{for some } t>0 \ \text{ and for all } u \in \mathbb{H}^n, \tag{2.2}$$

then $M_{p,\varphi}(\mathbb{H}^n) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau} \varphi(u,r)^{-1} = \infty \quad \text{for some } \tau>0 \ \text{ and for all } u \in \mathbb{H}^n, \tag{2.3}$$

then $M_{p,\varphi}(\mathbb{H}^n) = \Theta$.

Remark 2.1. We denote by Ω_p the sets of all positive measurable functions φ on $\mathbb{H}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{u\in\mathbb{H}^n} \Big\| \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} \Big\|_{L_\infty(t,\infty)} < \infty, \text{ and } \sup_{u\in\mathbb{H}^n} \Big\| \varphi(u,r)^{-1} \Big\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.1, we always assume that $\varphi \in \Omega_p$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C>0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

Let $1 \le p < \infty$. Denote by $\mathcal{G}_p$ the the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto t^{\frac{Q}{p}} \varphi(t) \in (0,\infty)$ is almost increasing.

Seemingly the requirement $\phi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai [18, p. 446] established that there exists a function ρ such that ρ itself is decreasing, that $\rho(t)t^{Q/p} \le \rho(r)r^{Q/p}$ for all $0 < t \le r < \infty$ and that $M_{p,\phi}(\mathbb{H}^n) = M_{p,\rho}(\mathbb{H}^n)$.

By elementary calculations we have the following, which shows particularly that the spaces $M_{p,\varphi}(\mathbb{H}^n)$ and $WM_{p,\varphi}(\mathbb{H}^n)$ are not trivial, see for example, [5].

Lemma 2.2. [6] *Let $\varphi \in \mathcal{G}_p$, $1 \le p < \infty$, $B_0 = B(u_0, r_0)$ and $\chi_{_{B_0}}$ is the characteristic function of the ball B_0, then $\chi_{_{B_0}} \in M_{p,\varphi}(\mathbb{H}^n)$. Moreover, there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{_{B_0}}\|_{WM_{p,\varphi}} \le \|\chi_{_{B_0}}\|_{M_{p,\varphi}} \le \frac{C}{\varphi(r_0)}.$$

The following theorem was proved in [14].

Theorem 2.1. [14] *Let $1 \le p < \infty$ and $\varphi_1, \varphi_2 \in \Omega_p$ satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(u,s)\, s^{\frac{Q}{p}} \le C\, \varphi_2(u,r), \tag{2.4}$$

where C does not depend on u and r. Then for $p > 1$, the operator M is bounded from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{p,\varphi_2}(\mathbb{H}^n)$ and for $p = 1$, the operator M is bounded from $M_{1,\varphi_1}(\mathbb{H}^n)$ to $WM_{1,\varphi_2}(\mathbb{H}^n)$.

Corollary 2.1. *Let $1 \le p < \infty$ and $\varphi \in \Omega_p$ satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \operatorname*{ess\,sup}_{t<s<\infty} \varphi(u,s)\, s^{\frac{Q}{p}} \le C\, \varphi(u,r), \tag{2.5}$$

where C does not depend on u and r. Then for $p > 1$, the operator M is bounded on $M_{p,\varphi}(\mathbb{H}^n)$ and for $p = 1$, the operator M is bounded from $M_{1,\varphi}(\mathbb{H}^n)$ to $WM_{1,\varphi}(\mathbb{H}^n)$.

3. Fractional maximal operator in the spaces $M_{p,\varphi}(\mathbb{H}^n)$

3.1. Spanne type result. The following theorem is valid.

Theorem 3.1. [14] *Let $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ and the pair (φ_1, φ_2) satisfy the condition*

$$\sup_{t<r<\infty} r^{-\frac{Q}{q}} \operatorname*{ess\,sup}_{r<s<\infty} \varphi_1(u,s) s^{\frac{Q}{p}} \le C\, \varphi_2(u,t), \tag{3.1}$$

where C does not depend on u and t. Then for $p > 1$ the operator M_α is bounded from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$ and for $p = 1$ the operator M_α is bounded from $M_{1,\varphi_1}(\mathbb{H}^n)$ to $WM_{q,\varphi_2}(\mathbb{H}^n)$.

Remark 3.1. Note that, in the Euclidean setting Theorem 3.1 was proved in [15], see also [10, 11, 12, 13].

For proving our main results, we need the following estimate.

Lemma 3.1. *If $B_0 := B(u_0, r_0)$, then $r_0^\alpha \le C_Q^{-\frac{\alpha}{Q}} 2^{Q-\alpha} M_\alpha \chi_{B_0}(u)$ for every $u \in B_0$.*

Proof. It is well known that

$$\mathrm{M}_\alpha f(u) \le 2^{Q-\alpha} M_\alpha f(u), \tag{3.2}$$

where $\mathrm{M}_\alpha(f)(u) = \sup\limits_{B \ni u} |B|^{-1+\frac{\alpha}{Q}} \int_B |f(v)| dV(v)$.

Now let $u \in B_0$. By using (3.2), we get

$$\begin{aligned} M_\alpha \chi_{B_0}(u) &\ge 2^{\alpha-Q} \mathrm{M}_\alpha \chi_{B_0}(u) \ge 2^{\alpha-Q} \sup_{B \ni u} |B|^{-1+\frac{\alpha}{Q}} |B \cap B_0| \\ &\ge 2^{\alpha-Q} |B_0|^{-1+\frac{\alpha}{Q}} |B_0 \cap B_0| = C_Q^{\frac{\alpha}{Q}} 2^{\alpha-Q} r_0^\alpha. \end{aligned}$$

□

The following theorem is one of our main results.

Theorem 3.2. *Let $0 \le \alpha < Q$, $p, q \in [1, \infty)$, $\varphi_1 \in \Omega_p$ and $\varphi_2 \in \Omega_q$.*

1. If $1 \le p < \frac{Q}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, then the condition (3.1) is sufficient for the boundedness of M_α from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $WM_{q,\varphi_2}(\mathbb{H}^n)$. Moreover, if $1 < p < \frac{Q}{\alpha}$, the condition (3.1) is sufficient for the boundedness of M_α from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$.

2. If the function $\varphi_1 \in \mathcal{G}_p$, then the condition

$$t^\alpha \varphi_1(t) \le C \varphi_2(t), \tag{3.3}$$

for all $t > 0$, where $C > 0$ does not depend t, is necessary for the boundedness of M_α from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $WM_{q,\varphi_2}(\mathbb{H}^n)$ and $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$.

3. Let $1 \le p < \frac{Q}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. If $\varphi_1 \in \mathcal{G}_p$, then the condition (3.3) is necessary and sufficient for the boundedness of M_α from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $WM_{q,\varphi_2}(\mathbb{H}^n)$. Moreover, if $1 < p < \frac{Q}{\alpha}$, then the condition (3.3) is necessary and sufficient for the boundedness of M_α from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$.

Proof. The first part of the theorem proved in Theorem 3.1.

We shall now prove the second part. Let $B_0 = B(u_0, t_0)$ and $x \in B_0$. By Lemma 3.1 we have $t_0^\alpha \le C M_\alpha \chi_{B_0}(x)$. Therefore, by Lemma 2.2 and Lemma 3.1

$$t_0^\alpha \lesssim |B_0|^{-\frac{1}{p}} \|M_\alpha \chi_{B_0}\|_{L_q(B_0)} \lesssim \varphi_2(t_0) \|M_\alpha \chi_{B_0}\|_{M_{q,\varphi_2}} \lesssim \varphi_2(t_0) \|\chi_{B_0}\|_{M_{p,\varphi_1}} \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)}$$

or

$$t_0^\alpha \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)} \text{ for all } t_0 > 0 \Longleftrightarrow t_0^\alpha \varphi_1(t_0) \lesssim \varphi_2(t_0) \text{ for all } t_0 > 0.$$

Since this is true for every $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

Remark 3.2. If we take $\varphi_1(t) = t^{\frac{\lambda-Q}{p}}$ and $\varphi_2(t) = t^{\frac{\mu-Q}{q}}$ at Theorem 3.2, then condition (3.3) is equivalent to $0 < \lambda < Q - \alpha p$ and $\frac{\lambda}{p} = \frac{\mu}{q}$, respectively. Therefore, we get the following Spanne result for Morrey spaces on Heisenberg groups.

Corollary 3.1. *Let* $0 \le \alpha < Q$, $1 \le p < \frac{Q}{\alpha}$, $0 < \lambda < Q - \alpha p$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. *Then the operator* M_α *is bounded from* $L_{p,\lambda}(\mathbb{H}^n)$ *to* $WL_{q,\mu}(\mathbb{H}^n)$ *if and only if* $\frac{\lambda}{p} = \frac{\mu}{q}$. *Moreover, if* $1 < p < \frac{Q}{\alpha}$, *then the operator* M_α *is bounded from* $L_{p,\lambda}(\mathbb{H}^n)$ *to* $L_{q,\mu}(\mathbb{H}^n)$ *if and only if* $\frac{\lambda}{p} = \frac{\mu}{q}$.

3.2. Adams type results. We got the Adams-Guliyev (see, [15]) type result for the operator M_α in the space $M_{p,\varphi}(\mathbb{H}^n)$ in [14, Theorem 3.3]. The following is a result of Adams-Gunawan (see, [16]) type for the operator M_α in the space $M_{p,\varphi}(\mathbb{H}^n)$.

Theorem 3.3. (Adams type result). *Let* $0 < \alpha < Q$, $1 \le p < q < \infty$ *and* $\varphi \in \Omega_p$ *satisfy condition* (2.5) *and*

$$r^\alpha \varphi(u,r) + \sup_{r<t<\infty} t^\alpha \, \varphi(u,t) \le C\varphi(u,r)^{\frac{p}{q}}, \tag{3.4}$$

where C *does not depend on* $u \in \mathbb{H}^n$ *and* $r > 0$. *Then for* $p > 1$, *the operator* M_α *is bounded from* $M_{p,\varphi}(\mathbb{H}^n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$ *and for* $p = 1$, *the operator* M_α *is bounded from* $M_{1,\varphi}(\mathbb{H}^n)$ *to* $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{H}^n)$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi}(\mathbb{H}^n)$.

Write $f = f_1 + f_2$, where $f_1(v) = f\chi_{2B}(v)$, $f_2(v) = f\chi_{\complement_{(2B)}}(v)$ and $B = B(u,r)$. Then

$$M_\alpha f(v) \le M_\alpha f_1(v) + M_\alpha f_2(v).$$

For $M_\alpha f_1(v)$, following Hedberg's trick (see for instance [20], p. 354), for all $v \in \mathbb{H}^n$ we obtain $M_\alpha f_1(v) \le C_1 r^\alpha M f(v)$.

Let v be an arbitrary point in B. If $B(v,t) \cap {}^\complement(B(u,2r)) \ne \emptyset$, then $t > r$. Indeed, if $z \in B(v,t) \cap {}^\complement(B(u,2r))$, then

$$t > |v^{-1}z| \ge |u^{-1}z| - |u^{-1}v| > 2r - r = r.$$

On the other hand, $B(v,t) \cap {}^\complement B(u,2r) \subset B(u,2t)$. Indeed, if $z \in B(v,t) \cap {}^\complement(B(u,2r))$, then we get $|u^{-1}z| \le |v^{-1}z| + |u^{-1}y| < t + r < 2t$.

Hence

$$\begin{aligned} M_\alpha f_2(v) &= \sup_{t>0} \frac{1}{|B(v,t)|^{1-\frac{\alpha}{Q}}} \int_{B(v,t)\cap {}^\complement(B(u,2r))} |f(z)| dV(z) \\ &\le 2^{Q-\alpha} \sup_{t>r} \frac{1}{|B(u,2t)|^{1-\frac{\alpha}{Q}}} \int_{B(u,2t)} |f(z)| dV(z) \\ &= 2^{Q-\alpha} \sup_{t>2r} \frac{1}{|B(u,t)|^{1-\frac{\alpha}{Q}}} \int_{B(u,t)} |f(z)| dV(z) \\ &\le 2^{Q-\alpha} C_Q^{\frac{\alpha}{Q}} \sup_{r<t<\infty} \frac{t^\alpha}{|B(u,t)|^{\frac{1}{p}}} \|f\|_{L_p(B(u,t))}. \end{aligned} \tag{3.5}$$

Then from conditions (3.4) and (3.5) and the technique in [19, p. 6492] we have

$$\begin{aligned} M_\alpha f_2(v) &\lesssim r^\alpha Mf(v) + \sup_{r<t<\infty} \frac{t^\alpha}{|B(u,t)|^{\frac{1}{p}}} \|f\|_{L_p(B(u,t))} \\ &\le r^\alpha Mf(v) + \sup_{r<t<\infty} t^{\alpha-\frac{Q}{p}} \|f\|_{L_p(B(u,t))} \\ &\le r^\alpha Mf(v) + \|f\|_{M_{p,\varphi}} \sup_{r<t<\infty} t^\alpha \varphi(u,t) \\ &\lesssim \min\{\varphi(u,r)^{\frac{p}{q}-1} Mf(v), \varphi(u,r)^{\frac{p}{q}} \|f\|_{M_{p,\varphi}}\} \\ &\lesssim \sup_{s>0} \min\{s^{\frac{p}{q}-1} Mf(v), s^{\frac{p}{q}} \|f\|_{M_{p,\varphi}}\} \\ &= (Mf(v))^{\frac{p}{q}} \|f\|_{M_{p,\varphi}}^{1-\frac{p}{q}}, \end{aligned} \tag{3.6}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Corollary 2.1 and (3.6), we get

$$\begin{aligned} \|M_\alpha f\|_{M_{q,\varphi^{\frac{p}{q}}}} &\lesssim \|f\|_{M_{p,\varphi}}^{1-\frac{p}{q}} \|(Mf(\cdot))^{\frac{p}{q}}\|_{M_{q,\varphi^{\frac{p}{q}}}} \\ &= \|f\|_{M_{p,\varphi}}^{1-\frac{p}{q}} \|Mf\|_{M_{p,\varphi}}^{\frac{p}{q}} \lesssim \|f\|_{M_{p,\varphi}}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\|M_\alpha f\|_{WM_{q,\varphi^{\frac{p}{q}}}} \lesssim \|f\|_{M_{p,\varphi}}^{1-\frac{p}{q}} \|Mf\|_{M_{p,\varphi}}^{\frac{p}{q}} \lesssim \|f\|_{\mathcal{M}_{p,\varphi}},$$

if $1 \le p < q < \infty$. □

The following theorem is one of our main results.

Theorem 3.4. *Let $0 < \alpha < Q$, $1 \le p < q < \infty$ and $\varphi \in \Omega_p$.*

1. *If $\varphi(u,t)$ satisfy condition (2.5), then the condition (3.4) is sufficient for the boundedness of M_α from $M_{p,\varphi}(\mathbb{H}^n)$ to $WM_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$. Moreover, if $p > 1$, then the condition (3.4) is sufficient for the boundedness of M_α from $M_{p,\varphi}(\mathbb{H}^n)$ to $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$.*
2. *If $\varphi \in \mathcal{G}_p$, then the condition*

$$r^\alpha \varphi(r) \le C\varphi(r)^{\frac{p}{q}}, \tag{3.7}$$

for all $r > 0$, where $C > 0$ does not depend r, is necessary for the boundedness of M_α from $M_{p,\varphi}(\mathbb{H}^n)$ to $WM_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$ and from $M_{p,\varphi}(\mathbb{H}^n)$ to $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$, if $p > 1$.

3. *If $\varphi \in \mathcal{G}_p$, then the condition (3.7) is necessary and sufficient for the boundedness of M_α from $M_{p,\varphi}(\mathbb{H}^n)$ to $WM_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$. Moreover, if $p > 1$, then the condition (3.7) is necessary and sufficient for the boundedness of M_α from $M_{p,\varphi}(\mathbb{H}^n)$ to $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$.*

Proof. The first part of the theorem is a corollary of Theorem 3.3.

We shall now prove the second part. Let $B_0 = B(u_0, t_0)$ and $x \in B_0$. By Lemma 3.1 we have $t_0^\alpha \le CM_\alpha \chi_{B_0}(x)$. Therefore, by Lemma 2.2 and Lemma 3.1

we have

$t_0^\alpha \lesssim |B_0|^{-\frac{1}{q}} \|M_\alpha \chi_{B_0}\|_{L_q(B_0)} \lesssim \varphi(t_0)^{\frac{p}{q}} \|M_\alpha \chi_{B_0}\|_{M_{q,\varphi^{\frac{p}{q}}}} \lesssim \varphi(t_0)^{\frac{p}{q}} \|\chi_{B_0}\|_{M_{p,\varphi}} \lesssim \varphi(t_0)^{\frac{p}{q}-1}$

or

$$t_0^\alpha \varphi(t_0)^{1-\frac{p}{q}} \lesssim 1 \text{ for all } t_0 > 0 \Longleftrightarrow t_0^\alpha \varphi(t_0) \lesssim \varphi(t_0)^{\frac{p}{q}}.$$

Since this is true for every $u \in \mathbb{H}^n$ and $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

Remark 3.3. If we take $\varphi(t) = t^{\frac{\lambda - Q}{q}}$ at Theorem 3.4, then condition (3.4) is equivalent to $0 < \lambda < Q - \alpha p$ and condition (3.7) is equivalent to $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$. Therefore, we get the following Adams result for Morrey spaces in Heisenberg groups.

Corollary 3.2. *Let $0 < \alpha < Q$, $1 \le p < q < \infty$ and $0 < \lambda < Q - \alpha p$. Then the operator M_α is bounded from $L_{p,\lambda}(\mathbb{H}^n)$ to $WL_{q,\lambda}(\mathbb{H}^n)$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$. Moreover, if $1 < p < q < \infty$, then the operator M_α is bounded from $L_{p,\lambda}(\mathbb{H}^n)$ to $L_{q,\lambda}(\mathbb{H}^n)$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$.*

Remark 3.4. Note that, in the case $\mathbb{H}^n = \mathbb{R}^n$ the sufficient part of the Corollary 3.2 was proved in [1].

4. Fractional maximal commutator operator in the spaces $M_{p,\varphi}(\mathbb{H}^n)$

4.1. Spanne type result. We recall the definition of the space of $BMO(\mathbb{H}^n)$.

Definition 4.1. Suppose that $b \in L_1^{\rm loc}(\mathbb{H}^n)$, and let

$$\|f\|_* = \sup_{u \in \mathbb{H}^n, r>0} \frac{1}{|B(u,r)|} \int_{B(u,r)} |b(v) - b_{B(u,r)}| dV(v) < \infty,$$

where

$$b_{B(u,r)} = \frac{1}{|B(u,r)|} \int_{B(u,r)} b(v) dV(v).$$

Define

$$BMO(\mathbb{H}^n) = \{b \in L_1^{\rm loc}(\mathbb{H}^n) \ : \ \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{H}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

The following lemma is valid.

Remark 4.1. [7, 20] (1) Let $b \in BMO(\mathbb{H}^n)$. Then

$$\|b\|_* \approx \sup_{u \in \mathbb{H}^n, r>0} \left(\frac{1}{|B(u,r)|} \int_{B(u,r)} |b(v) - b_{B(u,r)}|^p dV(v) \right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(2) Let $b \in BMO(\mathbb{H}^n)$. Then there is a constant $C > 0$ such that

$$\left| b_{B(u,r)} - b_{B(u,\tau)} \right| \le C \|b\|_* \log \frac{\tau}{r} \text{ for } 0 < 2r < \tau, \tag{4.2}$$

where C is independent of f, u, r and τ.

For the fractional maximal commutator operator $M_{b,\alpha}$

$$M_{b,\alpha}(f)(u) = \sup_{\tau>0} |B(u,\tau)|^{-1+\frac{\alpha}{Q}} \int_{B(u,\tau)} |b(u)-b(v)||f(v)|dV(v)$$

the following statement is true [14].

Theorem 4.1. [14] *Let $1<p<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $b\in BMO(\mathbb{H}^n)$ and $\varphi_1\in\Omega_p$, $\varphi_2\in\Omega_q$ satisfies the condition*

$$\sup_{r<t<\infty} t^{\alpha-\frac{Q}{p}} \log\left(e+\frac{t}{r}\right) \operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(u,s)\, s^{\frac{Q}{p}} \le C\,\varphi_2(u,r), \tag{4.3}$$

where C does not depend on u and r.

Then the operator $M_{b,\alpha}$ is bounded from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$. Moreover

$$\|M_{b,\alpha}f\|_{M_{q,\varphi_2}} \lesssim \|b\|_*\, \|f\|_{M_{p,\varphi_1}}.$$

In the case $\alpha=0$ and $\varphi_1=\varphi_2$ from Theorem 4.1 we get the following corollary.

Corollary 4.1. *Let $1<p<\infty$, $b\in BMO(\mathbb{H}^n)$ and $\varphi\in\Omega_p$ satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \log\left(e+\frac{t}{r}\right) \operatorname*{ess\,sup}_{t<s<\infty} \varphi(u,s)\, s^{\frac{Q}{p}} \le C\,\varphi(u,r), \tag{4.4}$$

where C does not depend on u and r.

Then the operator $M_b\equiv M_{b,0}$ is bounded on $M_{p,\varphi}(\mathbb{H}^n)$.

For proving our main results, we need the following estimate.

Lemma 4.1. *If $b\in L_1^{\rm loc}(\mathbb{H}^n)$ and $B_0:=B(u_0,r_0)$, then*

$$r_0^\alpha\,|b(u)-b_{B_0}| \le 2^{\alpha-Q}C_Q^{\frac{\alpha}{Q}} M_{b,\alpha}\chi_{B_0}(u) \text{ for every } u\in B_0.$$

Proof. It is well-known that

$$\mathrm{M}_{b,\alpha}f(u) \le 2^{Q-\alpha}M_{b,\alpha}f(u), \tag{4.5}$$

where $\mathrm{M}_{b,\alpha}(f)(u) := \sup_{B\ni u}|B|^{-1+\frac{\alpha}{Q}}\int_B|b(u)-b(v)|\,|f(v)|\,dV(v)$.

Now let $x\in B_0$. By using (4.5), we get

$$\begin{aligned} M_{b,\alpha}\chi_{B_0}(u) &\ge 2^{\alpha-Q}\,\mathrm{M}_{b,\alpha}\chi_{B_0}(u) \\ &= 2^{\alpha-Q}\sup_{B\ni x}|B|^{-1+\frac{\alpha}{Q}}\int_{B\cap B_0}|b(u)-b(v)|dV(v) \\ &\ge 2^{\alpha-Q}|B_0|^{-1+\frac{\alpha}{Q}}\int_{B_0\cap B_0}|b(u)-b(v)|dV(v) \\ &\ge 2^{\alpha-Q}|B_0|^{\frac{\alpha}{Q}}\,\left||B_0|^{-1}\int_{B_0}(b(u)-b(v))dV(v)\right| \\ &= 2^{\alpha-Q}C_Q^{\frac{\alpha}{Q}}\,r_0^\alpha\,|b(u)-b_{B_0}|. \end{aligned}$$

□

The following theorem is one of our main results.

Theorem 4.2. *Let $p, q \in [1, \infty)$, $0 \le \alpha < Q$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ and $b \in BMO(\mathbb{H}^n)$.*

1. *Let $1 < p < \frac{Q}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, then the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{\frac{Q}{q}} \operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(s) s^{\frac{Q}{p}} \le C \varphi_2(r)$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$.

2. *If $\varphi_1 \in \mathcal{G}_p$, then the condition (3.3) is necessary for the boundedness of $M_{b,\alpha}$ from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$.*

3. *Let $1 < p < \frac{Q}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. If $\varphi_1 \in \mathcal{G}_p$ satisfies the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{\alpha} \varphi_1(t) \le C r^{\alpha} \varphi_1(r) \tag{4.6}$$

for all $r > 0$, where $C > 0$ does not depend on r, then the condition (3.3) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $M_{p,\varphi_1}(\mathbb{H}^n)$ to $M_{q,\varphi_2}(\mathbb{H}^n)$.

Proof. The first part of the theorem is a Theorem 4.1.

We shall now prove the second part. Let $B_0 = B(u_0, r_0)$ and $x \in B_0$. By Lemma 4.1 we have $r_0^{\alpha} |b(u) - b_{B_0}| \lesssim M_{b,\alpha} \chi_{B_0}(u)$. Therefore, by Remark 4.1

$$\begin{aligned} r_0^{\alpha} &\lesssim \frac{\|M_{b,\alpha} \chi_{B_0}\|_{L_q(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L_q(B_0)}} \lesssim \|M_{b,\alpha} \chi_{B_0}\|_{L_q(B_0)} |B_0|^{-\frac{1}{q}} \\ &\lesssim \varphi_2(r_0) \|M_{b,\alpha} \chi_{B_0}\|_{M_{q,\varphi_2}} \lesssim \varphi_2(r_0) \|\chi_{B_0}\|_{M_{p,\varphi_1}} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}. \end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and second parts of the theorem. □

4.2. Adams type result. In this section we shall give a characterization for the Adams type boundedness of the operator $M_{b,\alpha}$ in generalized Morrey spaces defined on Heisenberg groups.

The following lemma is the analogue of the Hedberg's trick for $M_{b,\alpha}$.

Lemma 4.2. *If $0 < \alpha < Q$ and $f, b \in L_1^{\rm loc}(\mathbb{H}^n)$, then for all $u \in \mathbb{H}^n$ and $r > 0$ we get*

$$\int_{B(u,r)} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} |b(u) - b(v)| dV(v) \lesssim r^{\alpha} M_b f(u). \tag{4.7}$$

Proof.

$$\begin{aligned} &\int_{B(u,r)} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} |b(u) - b(v)| dV(v) \\ &= \sum_{j=0}^{\infty} \int_{B(u,2^{-j}r) \setminus B(u,2^{-j-1}r)} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} |b(u) - b(v)| dV(v) \\ &\lesssim \sum_{j=0}^{\infty} (2^{-j}r)^{\alpha} (2^{-j}r)^{-n} \int_{B(u,2^{-j}r)} |f(v)||b(u) - b(v)| dV(v) \lesssim r^{\alpha} M_b f(u). \end{aligned}$$

□

The following is a result of Adams type.

Theorem 4.3. *Let* $1 < p < q < \infty$, $0 < \alpha < Q$, $b \in BMO(\mathbb{H}^n)$ *and let* $\varphi(u,\cdot) \in \Omega_p$ *satisfy the conditions* (4.4) *and*

$$r^{\alpha}\,\varphi(u,r) + \sup_{r<t<\infty} \log\Big(e + \frac{t}{r}\Big)\, t^{\alpha}\,\varphi(u,t) \le C\varphi(u,r)^{\frac{p}{q}}, \tag{4.8}$$

where C *does not depend on* $u \in \mathbb{H}^n$ *and* $r > 0$.

Then $M_{b,\alpha}$ *is bounded from* $M_{p,\varphi}(\mathbb{H}^n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and $f \in M_{p,\varphi}(\mathbb{H}^n)$. For arbitrary $x \in \mathbb{H}^n$, set $B = B(u,r)$ for the ball centered at u and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

For $z \in B$ we have

$$\begin{aligned} M_{b,\alpha} f_2(z) &\lesssim \sup_{t>0} t^{\alpha-Q} \int_{B(z,t)} |b(v) - b(z)|\,|f_2(v)| dV(v) \\ &\approx \sup_{t>2r} t^{\alpha-Q} \int_{B(z,t)} |b(v) - b(z)|\,|f_2(v)| dV(v). \end{aligned}$$

Analogously section 4.1, for all $p \in (1,\infty)$ and $z \in B$ we get

$$M_{b,\alpha} f_2(z) \lesssim \sup_{t>2r} t^{\alpha-\frac{Q}{p}} \Big(1 + \log\frac{t}{r}\Big) \|f\|_{L_p(B(u,t))}. \tag{4.9}$$

Then from conditions (4.8) and (4.9) we get

$$\begin{aligned} M_{b,\alpha} f(z) &\lesssim r^{\alpha}\, M_b f(z) + \|b\|_* \sup_{t>2r} t^{\alpha-\frac{Q}{p}} \Big(1 + \log\frac{t}{r}\Big) \|f\|_{L_p(B(u,t))} \\ &\le r^{\alpha}\, M_b f(z) + \|b\|_* \,\|f\|_{M_{p,\varphi}} \sup_{t>r} \Big(1 + \log\frac{t}{r}\Big)\, t^{\alpha}\,\varphi(u,t) \\ &\lesssim r^{\alpha}\, M_b f(z) + \|b\|_*\, \varphi(u,r)^{\frac{p}{q}}\, \|f\|_{M_{p,\varphi}} \\ &\lesssim \sup_{s>0} \min\big\{ s^{\frac{p}{q}-1} M_b f(z),\, s^{\frac{p}{q}} \|f\|_{M_{p,\varphi}} \big\} \\ &= (M_b f(z))^{\frac{p}{q}}\, \|f\|_{M_{p,\varphi}}^{1-\frac{p}{q}}, \end{aligned} \tag{4.10}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Corollary 4.1 and (4.10), we get

$$\begin{aligned} \|M_{b,\alpha} f\|_{M_{q,\varphi^{\frac{p}{q}}}} &\lesssim \|f\|_{M_{p,\varphi}}^{1-\frac{p}{q}}\, \|(M_b f(\cdot))^{\frac{p}{q}}\|_{M_{q,\varphi^{\frac{p}{q}}}} \\ &= \|f\|_{M_{p,\varphi}}^{1-\frac{p}{q}}\, \|M_b f\|_{M_{p,\varphi}}^{\frac{p}{q}} \lesssim \|f\|_{M_{p,\varphi}}. \end{aligned}$$

□

The following theorem is one of our main results.

Theorem 4.4. *Let* $0 < \alpha < Q$, $1 \le p < q < \infty$, $\varphi \in \Omega_p$ *and* $b \in BMO(\mathbb{H}^n)$.

1. *If* $1 < p < \infty$ *and* $\varphi(t)$ *satisfies*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{-\frac{n}{p}} \operatorname*{ess\,sup}_{t<s<\infty} \varphi(s)\, s^{\frac{n}{p}} \le C\varphi(r)$$

for all $r > 0$ *and* $C > 0$ *does not depend on* r, *then the condition*

$$r^{\alpha}\, \varphi(r) + \sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{\alpha}\, \varphi(t) \le C\varphi(r)^{\frac{p}{q}}$$

for all $r > 0$ *and* $C > 0$ *does not depend on* r, *is sufficient for the boundedness of* $M_{b,\alpha}$ *from* $M_{p,\varphi}(\mathbb{H}^n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$.

2. *If* $\varphi \in \mathcal{G}_p$, *then the condition*

$$r^{\alpha}\, \varphi(r) \le C\varphi(r)^{\frac{p}{q}} \tag{4.11}$$

for all $r > 0$ *and* $C > 0$ *does not depend on* r, *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $M_{p,\varphi}(\mathbb{H}^n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$.

3. *Let* $1 < p < q < \infty$. *If* $\varphi \in \mathcal{G}_p$ *satisfies the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{\alpha}\, \varphi(t) \le C r^{\alpha}\, \varphi(r)$$

for all $r > 0$ *and* $C > 0$ *does not depend on* r, *then the condition* (4.11) *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $M_{p,\varphi}(\mathbb{H}^n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(\mathbb{H}^n)$.

Proof. The first part of the theorem is a corollary of Theorem 4.3.

We shall now prove the second part. Let $B_0 = B(u_0, r_0)$ and $x \in B_0$. By Lemma 4.1 we have $r_0^{\alpha}|b(u) - b_{B_0}| \lesssim M_{b,\alpha}\chi_{B_0}(u)$. Therefore, by Remark 4.1 and Lemma 2.2

$$\begin{aligned} r_0^{\alpha} &\lesssim \frac{\|M_{b,\alpha}\chi_{B_0}\|_{L_q(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L_q(B_0)}} \lesssim \|M_{b,\alpha}\chi_{B_0}\|_{L_q(B_0)} |B_0|^{-\frac{1}{q}} \\ &\lesssim \varphi(r_0)^{\frac{p}{q}}\, \|M_{b,\alpha}\chi_{B_0}\|_{M_{q,\varphi^{\frac{p}{q}}}} \lesssim \varphi(r_0)^{\frac{p}{q}}\, \|\chi_{B_0}\|_{M_{p,\varphi}} \lesssim \varphi(r_0)^{\frac{p}{q}-1}. \end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and second parts of the theorem. □

In the case $\varphi(u, r) = r^{\frac{\lambda-Q}{p}}$, $0 < \lambda < Q$ from Theorem 4.4 we get the following Adams type result for the commutator of fractional maximal operator.

Corollary 4.2. *Let* $0 < \alpha < Q$, $1 < p < q < \infty$, $0 < \lambda < Q - \alpha p$ *and* $b \in BMO(\mathbb{H}^n)$. *Then the operator* $M_{b,\alpha}$ *is bounded from* $L_{p,\lambda}(\mathbb{H}^n)$ *to* $L_{q,\lambda}(\mathbb{H}^n)$ *if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$.

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