Riesz potential in the local Morrey–Lorentz spaces and some applications

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Research Article

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Abstract: In this paper, the necessary and sufficient conditions are found for the boundedness of the Riesz potential I_α in the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$. This result is applied to the boundedness of particular operators such as the fractional maximal operator, fractional Marcinkiewicz operator and fractional powers of some analytic semigroups on the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$.

Keywords: Local Morrey–Lorentz space, Riesz potential, Hardy operator

MSC 2010: Primary 42B20, 42B35; secondary 47G10

1 Introduction and main result

In a series of papers [3, 4, 14], the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}$ were introduced, the basic properties of these spaces were established, and the boundedness of the Hilbert transform H, the Hardy–Littlewood maximal operator M and the Calderón–Zygmund operators T on investigated spaces $M^{\mathrm{loc}}_{p,q;\lambda}$ was extensively studied, respectively. Some types of Morrey–Lorentz spaces were studied by some authors [16, 18, 20]. The present paper deals with the boundedness of the Riesz potential I_α defined by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0 < \alpha < n, \quad f \in L^{\mathrm{loc}}_1(\mathbb{R}^n),$$

in the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$.

Further we apply this result to particular operators such as a fractional maximal operator, fractional Marcinkiewicz operator and fractional powers of some analytic semigroups.

For each measurable function f on $(0,\infty)$ and each $t > 0$, the following operator

$$(S_\alpha f)(t) = t^{\frac{\alpha}{n}-1} \int\limits_0^t f(s)\,ds + \int\limits_t^\infty s^{\frac{\alpha}{n}-1} f(s)\,ds$$

was defined by A. P. Calderón [9]. The importance of S_α is based on the fact that it dominates the Riesz potential I_α.

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Theorem A ([19, 23]). *If the condition*

$$(S_\alpha f^*)(1) = \int_0^1 f^*(s)\,ds + \int_1^\infty s^{\frac{\alpha}{n}-1} f^*(s)\,ds < \infty \tag{1.1}$$

holds for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, *then the Riesz potential* $(I_\alpha f)(x)$, $x \in \mathbb{R}^n$, *exists almost everywhere. Furthermore, the inequality*

$$(I_\alpha f)^*(t) \le C S_\alpha(f^*)(t), \quad 0 < t < \infty, \tag{1.2}$$

is valid, where f^* *denotes the nonincreasing rearrangement of* f *defined by*

$$f^*(t) = \inf\{\lambda > 0 : \big|\{y \in \mathbb{R}^n : |f(y)| > \lambda\}\big| \le t\} \quad \text{for all } t \in (0,\infty),$$

and C *is a constant independent of* f *and* t.

The following is a well-known theorem on the Riesz potential.

Theorem B ([15]). *Let* $0 < \alpha < n$, $1 \le p \le \frac{n}{\alpha}$, $p < q < \infty$, $1 \le r \le \infty$ *and* $f \in L_{p,r}(\mathbb{R}^n)$ *satisfy condition* (1.1). *Then the Riesz potential* $I_\alpha f$ *exists almost everywhere. Furthermore,*

(i) *if* $1 < p < \frac{n}{\alpha}$, $1 \le r \le s \le \infty$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the Lorentz spaces* $L_{p,r}$ *to* $L_{q,s}$;

(ii) *if* $p = 1$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the Lorentz spaces* $L_{1,r}$ *to* WL_q.

The following theorem is the main result of our paper, in which we get an analogue of Theorem B for the boundedness of the Riesz potential in the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}$.

Theorem 1.1. *Let* $0 < \alpha < n$, $0 \le \lambda < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$, $\frac{r}{r+\lambda} \le p \le (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$ *and* $f \in M^{\mathrm{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ *satisfies the condition* (1.1). *Then the Riesz potential* $I_\alpha f$ *exists almost everywhere. Furthermore,*

(i) *if* $\frac{r}{r+\lambda} < p < (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the spaces* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $M^{\mathrm{loc}}_{q,s;\lambda}$;

(ii) *if* $p = \frac{r}{r+\lambda}$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the spaces* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $WM^{\mathrm{loc}}_{q,s;\lambda}$.

2 Preliminaries

We will use the following notation. For a Lebesgue measurable set $E \subset \mathbb{R}^n$ and $0 < p \le \infty$, $L_p(E)$ is the standard Lebesgue space of all functions f Lebesgue measurable on E for which

$$\|f\|_{L_p(E)} := \Big(\int_E |f(y)|^p\,dy\Big)^{\frac{1}{p}} < \infty \qquad \text{if } 0 < p < \infty,$$
$$\|f\|_{L_\infty(E)} := \sup\{\alpha : \big|\{y \in E : |f(y)| \ge \alpha\}\big| > 0\} < \infty \qquad \text{if } p = \infty.$$

Also, for an open set $E \subset \mathbb{R}^n$, $L_p^{\mathrm{loc}}(E)$ is the set of all functions f such that $f \in L_p(K)$ for any compact $K \subset E$. If $E = \mathbb{R}^n$, then, for brevity, we write L_p for $L_p(\mathbb{R}^n)$ and L_p^{loc} for $L_p^{\mathrm{loc}}(\mathbb{R}^n)$. The same convention refers to the case of weak Lebesgue spaces $WL_p(E)$, the space of all functions f Lebesgue measurable on E for which

$$\|f\|_{WL_p(E)} := \sup_{0<t\le|E|} t^{\frac{1}{p}} f^*(t) < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{WL_\infty} \equiv \|f\|_{L_\infty}, \qquad p = \infty.$$

In the following, we give the local Morrey spaces $LM_{p,\lambda}(0,\infty)$ which we use in proving our main results (see, e.g., [1, 3, 4, 21, 22]).

Definition 2.1. Let $1 \le p < \infty$ and $0 \le \lambda \le 1$. We denote by $LM_{p,\lambda} \equiv LM_{p,\lambda}(0,\infty)$ the local Morrey space, the space of all functions $\varphi \in L_p^{\mathrm{loc}}(0,\infty)$ with finite quasinorm

$$\|\varphi\|_{LM_{p,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|\varphi\|_{L_p(0,r)}.$$

Also, by $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0,\infty)$ we denote the weak local Morrey space of all functions $\varphi \in WL_p^{\mathrm{loc}}(0,\infty)$ for which

$$\|\varphi\|_{WLM_{p,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|\varphi\|_{WL_p(0,r)} < \infty.$$

The local Morrey-type spaces $LM_{p\theta,w}$ were introduced by Guliyev in the doctoral thesis [12] in 1994 (see also [13]) defined by

$$\|\varphi\|_{LM_{p\theta,w}} = \big\|w(r)\|\varphi\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. If $\theta = \infty$ and $w = r^{-\frac{\lambda}{p}}$, then we get

$$LM_{p\theta,w} \equiv LM_{p,\lambda}.$$

The boundedness of the classical operators in $LM_{p\theta,w}$ was intensively studied in [6–8] and other works.

Lorentz spaces were introduced by Lorentz in 1950. These spaces are quasi-Banach spaces and generalizations of more familiar L_p spaces, also they appear to be useful in the general interpolation theory.

Definition 2.2 ([5]). The Lorentz space $L_{p,q} \equiv L_{p,q}(\mathbb{R}^n)$, $0 < p, q \le \infty$, is defined as the set of all measurable functions f on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{L_{p,q}} := \|\tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau)\|_{L_q(0,\infty)}.$$

The functional $\|\cdot\|_{L_{p,q}}$ is a norm if and only if either $1 \le q \le p$ or $p = q = \infty$. If $p = q = \infty$, then the space $L_{\infty,\infty}(\mathbb{R}^n)$ is denoted by $L_\infty(\mathbb{R}^n)$. Clearly, $L_{p,p} \equiv L_p$ and $L_{p,\infty} \equiv WL_p$.

Definition 2.3 ([4]). Let $0 < p, q \le \infty$ and $0 \le \lambda \le 1$. We denote by $M^{\mathrm{loc}}_{p,q;\lambda} \equiv M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$ the local Morrey–Lorentz spaces, the spaces of all measurable functions with finite quasinorm

$$\|f\|_{M^{\mathrm{loc}}_{p,q;\lambda}} := \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,r)}.$$

If $\lambda < 0$ or $\lambda > 1$, then $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Also,

$$M^{\mathrm{loc}}_{p,q;0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n) \quad \text{and} \quad M^{\mathrm{loc}}_{p,p;\lambda}(\mathbb{R}^n) \equiv M^{\mathrm{loc}}_{p;\lambda}(\mathbb{R}^n).$$

In the limiting case $\lambda = 1$ the space $M^{\mathrm{loc}}_{p,q;1}(\mathbb{R}^n)$ is the classical Lorentz space $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$. Note that

$$M^{\mathrm{loc}}_{p,\infty;\lambda} = L_{p,\infty} = WL_p.$$

We denote by $WM^{\mathrm{loc}}_{p,q;\lambda} \equiv WM^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$ the weak local Morrey–Lorentz spaces of all measurable functions with finite quasinorm

$$\|f\|_{WM^{\mathrm{loc}}_{p,q;\lambda}} := \sup_{t>0} t^{-\frac{\lambda}{q}} \|\tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau)\|_{WL_q(0,r)}.$$

We will use the boundedness of the following two Hardy operators to obtain the boundedness of the Riesz potential I_α in the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}$.

Definition 2.4 ([22]). Let φ be a measurable function on $(0,\infty)$ and γ and β be real numbers. The weighted Hardy operators with power weights acting on φ are defined by

$$H^\beta_{(\gamma)}\varphi(t) = t^{\gamma+\beta-1} \int_0^t \frac{\varphi(y)}{y^\gamma}\,dy, \quad \mathcal{H}^\beta_{(\gamma)}\varphi(t) = t^{\gamma+\beta} \int_t^\infty \frac{\varphi(y)}{y^{\gamma+1}}\,dy, \quad t > 0.$$

Definition 2.5 ([2]). Let φ be a measurable function on $(0,\infty)$, and let η be a real number. The Hardy operators P_η and $\mathcal{P}_\eta$ are defined by

$$P_\eta\varphi(t) = t^{-\eta}\int_0^t \varphi(s)\,ds, \quad \mathcal{P}_\eta\varphi(t) = t^{-\eta}\int_t^\infty \varphi(s)\,ds.$$

Throughout the paper we use the letter C for a positive constant independent of appropriate parameters and not necessarily the same at each occurrence. If $p \in [1,\infty]$, then the conjugate number p' is defined by $pp' = p + p'$.

The following theorem was proved in [2] by K. F. Andersen and B. Muckenhoupt.

Theorem C ([2]). *Suppose $1 \le p \le q < \infty$, u and v are nonnegative weight functions. Then the following (p,q) weak-type inequalities are valid.*

(i) *For $\eta > 0$, if*

$$B(\eta;a) = \sup_{\xi>0}\Big(\int_\xi^\infty \Big(\frac{\xi}{x}\Big)^a\Big(\frac{u(x)}{x^{\eta q}}\Big)\,dx\Big)^{\frac{1}{q}}\Big(\int_0^\xi v(x)^{-\frac{1}{p-1}}\,dx\Big)^{\frac{1}{p'}} \tag{2.1}$$

is finite for some $a > 0$, then (u,v) is a (p,q) weak-type weight pair for P_η:

$$\Big(\int_{\{t\in(0,\infty):|P_\eta\varphi(t)|>\mu\}} u(t)\,dt\Big)^{\frac{1}{q}} \le C\mu^{-1}\Big(\int_0^\infty |\varphi(t)|^p v(t)\,dt\Big)^{\frac{1}{p}}. \tag{2.2}$$

(ii) *For $\eta > 0$, if*

$$B(\eta) = \sup_{\xi>0}\xi^{-\eta}\Big(\int_0^\xi u(x)\,dx\Big)^{\frac{1}{q}}\Big(\int_\xi^\infty v(x)^{-\frac{1}{p-1}}\,dx\Big)^{\frac{1}{p'}} \tag{2.3}$$

is finite, then (u,v) is a (p,q) weak-type weight pair for $\mathcal{P}_\eta$:

$$\Big(\int_{\{t\in(0,\infty):|\mathcal{P}_\eta\varphi(t)|>\mu\}} u(t)\,dt\Big)^{\frac{1}{q}} \le C\mu^{-1}\Big(\int_0^\infty |\varphi(t)|^p v(t)\,dt\Big)^{\frac{1}{p}}, \tag{2.4}$$

and consequently, $\|\mathcal{P}_\eta\|_W = B(\eta)$.

The smallest choice of constants C in (2.2) and (2.4), called the weak norms of P_η and $\mathcal{P}_\eta$, is denoted by $\|P_\eta\|_W$ and $\|\mathcal{P}_\eta\|_W$, respectively. Furthermore,

$$\Big[\frac{a}{\eta q + a}\Big]^{\frac{1}{q}} B(\eta;a) \le \|P_\eta\|_W \le \Big[\frac{\eta q + a}{\eta}\Big]^{\frac{1}{q}}(q')^{\frac{1}{p'}} B(\eta;a),$$
$$\Big[\frac{a}{a - \eta q}\Big]^{\frac{1}{q}} B(\eta;a) \le \|\mathcal{P}_\eta\|_W \le \Big[\frac{\eta q - a}{\eta}\Big]^{\frac{1}{q}}(q')^{\frac{1}{p'}} B(\eta;a).$$

Note that, taking $u(\tau) = v(\tau) = \chi_{(0,t)}(\tau)$ in inequalities (2.2) and (2.4), we get the inequalities

$$\Big(\int_{\{\tau\in(0,t):|P_\eta\varphi(\tau)|>\mu\}} dt\Big)^{\frac{1}{q}} \le C\mu^{-1}\Big(\int_0^t |\varphi(\tau)|^p\,dt\Big)^{\frac{1}{p}},$$

$$\Big(\int_{\{\tau\in(0,t):|\mathcal{P}_\eta\varphi(t)|>\mu\}} dt\Big)^{\frac{1}{q}} \le C\mu^{-1}\Big(\int_0^t |\varphi(\tau)|^p\,dt\Big)^{\frac{1}{p}}.$$

In the following two lemmas we give the boundedness of the Hardy operators $H^\alpha_{(\gamma)}$ and $\mathcal{H}^\alpha_{(\gamma)}$ on Morrey and weak Morrey spaces.

Lemma 2.6 ([22]). *Let*

$$0<\lambda<1,\quad 0<\beta<1-\lambda,\quad 1\le r<\frac{1-\lambda}{\beta}\quad \textit{and}\quad \frac{1}{r}-\frac{1}{s}=\frac{\beta}{1-\lambda}.$$

If $\gamma<\frac{1}{r'}+\frac{\lambda}{r}$, then $H^{\beta}_{(\gamma)}$ is bounded from $LM_{r,\lambda}(0,\infty)$ to $LM_{s,\lambda}(0,\infty)$, and if $\gamma>\frac{\lambda-1}{r}$, then the operator $\mathcal{H}^{\beta}_{(\gamma)}$ is bounded from $LM_{r,\lambda}(0,\infty)$ to $LM_{s,\lambda}(0,\infty)$.

Lemma 2.7. *Let*

$$0<\lambda<1,\quad 0<\beta<1-\lambda,\quad 1\le r<\frac{1-\lambda}{\beta}\quad \textit{and}\quad \frac{1}{r}-\frac{1}{s}=\frac{\beta}{1-\lambda}.$$

If $\gamma=\frac{1}{r'}+\frac{\lambda}{r}$, then $H^{\beta}_{(\gamma)}$ is bounded from $LM_{r,\lambda}(0,\infty)$ to $WLM_{s,\lambda}(0,\infty)$, and if $\gamma=\frac{\lambda-1}{r}$, then the operator $\mathcal{H}^{\beta}_{(\gamma)}$ is bounded from $LM_{r,\lambda}(0,\infty)$ to $WLM_{s,\lambda}(0,\infty)$.

Proof. It is sufficient to show that the following statement is valid:

$$\|H^{\beta}_{(\gamma)}\varphi\|_{WL_{s,\lambda}(0,\infty)}\le C\|\varphi\|_{L_{r,\lambda}(0,\infty)}\iff \sup_{t>0} t^{-\frac{\lambda}{s}+\frac{\lambda}{r}}\frac{\|H^{\beta}_{(\gamma)}\varphi\|_{WL_s(0,t)}}{\|\varphi\|_{L_r(0,t)}}\le C<\infty. \tag{2.5}$$

Let $\gamma=\frac{1}{r'}+\frac{\lambda}{r}$. We have

$$\begin{aligned}\|H^{\beta}_{(\gamma)}\varphi\|_{WL_s(0,\infty)}&=\|\chi_{(0,t)}(\tau)H^{\beta}_{(\gamma)}\varphi(\tau)\|_{WL_s(0,\infty)}=\sup_{\mu>0}\mu\Big(\int\limits_{\{\tau\in(0,t):|H^{\beta}_{(\gamma)}\varphi(\tau)|>\mu\}}d\tau\Big)^{\frac{1}{s}}\\&=\sup_{\mu>0}\mu\Big|\Big\{\tau\in(0,t):\tau^{\frac{\lambda-1}{r}+\beta}\int_0^{\tau}\varphi(y)y^{-1+\frac{1-\lambda}{r}}\,dy>\mu\Big\}\Big|^{\frac{1}{s}}.\end{aligned}$$

In (2.1), if we take $\eta=\frac{1-\lambda}{r}-\beta=\frac{1-\lambda}{s}>0$, $u(\tau)=\chi_{(0,t)}(\tau)$, $v(\tau)=\chi_{(0,t)}(\tau)\tau^{r+\lambda-1}$, then we get

$$\begin{aligned}B(\eta,a)&=\sup_{\xi>0}\xi^{\frac{a}{s}}\Big(\int_{\xi}^{\infty}\chi_{(0,t)}(\tau)\tau^{-a}\tau^{-s(\frac{1-\lambda}{s})}\,d\tau\Big)^{\frac{1}{s}}\Big(\int_0^{\xi}\chi_{(0,t)}(\tau)\tau^{(r+\lambda-1)^{-\frac{1}{r-1}}}\,d\tau\Big)^{\frac{1}{r'}}\\&=\sup_{0<\xi<t}\xi^{\frac{a}{s}}\Big(\int_{\xi}^{t}\tau^{-a}\tau^{\lambda-1}\,d\tau\Big)^{\frac{1}{s}}\Big(\int_0^{\xi}\chi_{(0,t)}(\tau)\tau^{-\frac{r}{r-1}-\frac{\lambda}{r-1}+\frac{1}{r-1}}\,d\tau\Big)^{\frac{1}{r'}}\\&\le C\sup_{0<\xi<t}\xi^{\frac{a}{s}}\xi^{-\frac{a}{s}+\frac{\lambda}{s}-1-\frac{\lambda}{r}+\frac{1}{r}+1-\frac{1}{r}}=Ct^{-\frac{\lambda}{r}+\frac{\lambda}{s}}.\end{aligned}$$

Due to Theorem C (i), we can replace $\frac{\|H^{\beta}_{(\gamma)}\varphi\|_{WL_s(0,t)}}{\|\varphi\|_{L_r(0,t)}}$ in (2.5) by the above expression. Then we get

$$C\sup_{t>0}t^{-\frac{\lambda}{r}+\frac{\lambda}{s}-\frac{\lambda}{s}+\frac{\lambda}{r}}=C<\infty.$$

Now let $\gamma=\frac{\lambda-1}{r}$. We will carry out the proof by using similar methods to those used in the proof of the boundedness of $H^{\beta}_{(\gamma)}$. We have

$$\begin{aligned}\|\mathcal{H}^{\beta}_{(\gamma)}\varphi\|_{WL_s(0,\infty)}&=\|\chi_{(0,t)}(\tau)\mathcal{H}^{\beta}_{(\gamma)}\varphi(\tau)\|_{WL_s(0,\infty)}=\sup_{\mu>0}\mu\Big(\int\limits_{\{\tau\in(0,t):|\mathcal{H}^{\beta}_{(\gamma)}\varphi(\tau)|>\mu\}}d\tau\Big)^{\frac{1}{s}}\\&=\sup_{\mu>0}\mu\Big|\Big\{\tau\in(0,t):\tau^{\frac{\lambda-1}{r}+\beta}\int_{\tau}^{\infty}\varphi(y)y^{-1+\frac{1-\lambda}{r}}\,dy>\mu\Big\}\Big|^{\frac{1}{s}}.\end{aligned}$$

Taking into account (2.3), if we take $\eta=\frac{1-\lambda}{r}-\beta>0$, $u(\tau)=\chi_{(0,t)}(\tau)$, $v(\tau)=\chi_{(0,t)}(\tau)\,\tau^{r+\lambda-1}$, we get

$$\begin{aligned}B(\eta)&=\sup_{\xi>0}\xi^{\frac{1-\lambda}{r}+\beta}\Big(\int_0^{\xi}\chi_{(0,t)}(\tau)d\tau\Big)^{\frac{1}{s}}\Big(\int_{\xi}^{\infty}(\chi_{(0,t)}(\tau)(\tau^{r+\lambda-1})^{-\frac{1}{r-1}}\,d\tau\Big)^{\frac{1}{r'}}\\&\le C\sup_{0<\xi<t}\xi^{\frac{\lambda-1}{r}+\beta+\frac{1}{s}-\frac{\lambda}{(r-1)r'}}=Ct^{-\frac{\lambda}{r}+\frac{\lambda}{s}}.\end{aligned}$$

Due to Theorem C (ii), we can replace $\frac{\|H^{\beta}_{(\gamma)}\varphi\|_{WL_s(0,t)}}{\|\varphi\|_{L_r(0,t)}}$ in (2.5) by the above expression. Then we get

$$C\sup_{t>0} t^{-\frac{\lambda}{r}+\frac{\lambda}{s}-\frac{\lambda}{s}+\frac{\lambda}{r}} = C < \infty,$$

which completes the proof. □

3 Proof of Theorem 1.1

In this section, the necessary and sufficient conditions are found for the boundedness of the Riesz potential I_α in the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$ by using related rearrangement inequality, Lemmas 2.6 and 2.7.

Proof. If f satisfies (1.1), then by Theorem A the Riesz potential $I_\alpha f(x)$, $x \in \mathbb{R}^n$, exists almost everywhere.

(i) *Sufficiency.* Let $\frac{r}{r+\lambda} < p < (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$. By using inequality (1.2), we get

$$\begin{aligned}\|I_\alpha f\|_{M^{\mathrm{loc}}_{q,s,\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} (I_\alpha f)^*(\tau)\|_{L_s(0,t)} \\ &\le C\sup_{t>0} t^{-\frac{\lambda}{s}} \Big\|\tau^{\frac{1}{q}-\frac{1}{s}}\Big(\tau^{\frac{\alpha}{n}-1}\int_0^\tau f^*(y)\,dy + \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)\,dy\Big)\Big\|_{L_s(0,t)} \\ &\le C\sup_{t>0} t^{-\frac{\lambda}{s}} \Big\|\tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1}\int_0^\tau f^*(y)\,dy\Big\|_{L_s(0,t)} + C\sup_{t>0} t^{-\frac{\lambda}{s}} \Big\|\tau^{\frac{1}{q}-\frac{1}{s}}\int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)\,dy\Big\|_{L_s(0,t)} \\ &= I_1 + I_2.\end{aligned}$$

Let us estimate I_1:

$$I_1 = C\sup_{t>0} t^{-\frac{\lambda}{s}} \Big\|\tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1}\int_0^\tau f^*(y)\,dy\Big\|_{L_s(0,t)} = C\|H^{\beta}_{(\gamma)} g\|_{L_{s,\lambda}(0,\infty)}.$$

We take $\gamma = \frac{1}{p} - \frac{1}{r}$ and consider the Hardy operator $H^{\beta}_{(\gamma)}$ and $g(t) = t^{\frac{1}{p}-\frac{1}{r}} f^*(t)$. Therefore we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} - \frac{1}{p} + \frac{\alpha}{n}.$$

By Lemma 2.6, we have $\beta = (1-\lambda)(\frac{1}{r} - \frac{1}{s})$, and then we obtain $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$.

Hence the operator $H^{\beta}_{(\gamma)}$ is bounded from the Morrey space $L_{r,\lambda}(0,\infty)$ to $L_{s,\lambda}(0,\infty)$ under the condition $\gamma = \frac{1}{p} - \frac{1}{r} < \frac{1}{r'} + \frac{\lambda}{r}$. Then we get

$$I_1 \le C\|H^{\beta}_{(\gamma)} g\|_{L_{s,\lambda}(0,\infty)} \le C\|g\|_{L_{r,\lambda}(0,\infty)} = C\sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{\frac{1}{p}-\frac{1}{r}} f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}. \tag{3.1}$$

Now we consider I_2:

$$I_2 = C\sup_{t>0} t^{-\frac{\lambda}{s}} \Big\|\tau^{\frac{1}{q}-\frac{1}{s}}\int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)\,dy\Big\|_{L_s(0,t)} = C\|\mathcal{H}^{\beta}_{(\gamma)} g\|_{L_{s,\lambda}(0,\infty)}.$$

We take $\gamma = \frac{1}{p} - \frac{1}{r} - \frac{\alpha}{n}$ in the Hardy operator $\mathcal{H}^{\beta}_{(\gamma)}$ and $g(t) = t^{\frac{1}{p}-\frac{1}{r}} f^*(t)$. Therefore we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} - \frac{1}{p} + \frac{\alpha}{n}.$$

By Lemma 2.6, we have $\beta = (1-\lambda)(\frac{1}{r} - \frac{1}{s})$, then we obtain $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$.

Hence the operator $\mathcal{H}^{\beta}_{(\gamma)}$ is bounded from the Morrey space $L_{r,\lambda}(0,\infty)$ to $L_{s,\lambda}(0,\infty)$ under the condition $\frac{\lambda-1}{r} < \gamma = \frac{1}{p} - \frac{1}{r} - \frac{\alpha}{n}$. Then we get

$$\sup_{t>0} t^{-\frac{\lambda}{s}} \|\mathcal{H}^{\beta}_{(\gamma)} g\|_{L_s(0,t)} \le C\|g\|_{L_{r,\lambda}(0,\infty)} = C\sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{\frac{1}{p}-\frac{1}{r}} f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}. \tag{3.2}$$

From inequalities (3.1) and (3.2), we obtain the boundedness of the operator I_α from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $M^{\mathrm{loc}}_{q,s;\lambda}$.

Necessity. Suppose that the operator I_α is bounded from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $M^{\mathrm{loc}}_{q,s;\lambda}$, and $\frac{r}{r+\lambda} \le p \le (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$. Define $f_\tau(x) =: f(\tau x)$ for $\tau > 0$. Then $f^*_\tau(t) = f^*(t\tau^n)$ and

$$\begin{aligned}
\|f_\tau\|_{M^{\mathrm{loc}}_{p,r;\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{r}} \|y^{\frac{1}{p}-\frac{1}{r}} f^*_\tau(y)\|_{L_r(0,t)} = \sup_{t>0} t^{-\frac{\lambda}{r}} \|y^{\frac{1}{p}-\frac{1}{r}} f^*(y\tau^n)\|_{L_r(0,t)} \\
&= \sup_{t>0} t^{-\frac{\lambda}{r}} \tau^{-\frac{n}{p}} \|y^{\frac{1}{p}-\frac{1}{r}} f^*(y)\|_{L_r(0,t\tau^n)} = \tau^{-\frac{n}{p}+\frac{n\lambda}{r}} \sup_{t>0} (t\tau^n)^{-\frac{\lambda}{r}} \|y^{\frac{1}{p}-\frac{1}{r}} f^*(y)\|_{L_r(0,t\tau^n)} \\
&= \tau^{-n\left(\frac{1}{p}-\frac{\lambda}{r}\right)} \|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}.
\end{aligned}$$

Also, $(I_\alpha f_\tau)(x) = \tau^{-\alpha}(I_\alpha f)(\tau^n x)$, $(I_\alpha f_\tau)^*(t) = \tau^{-\alpha}(I_\alpha f)^*(t\tau^n)$ and

$$\begin{aligned}
\|I_\alpha f_\tau\|_{M^{\mathrm{loc}}_{q,s;\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{s}} \|y^{\frac{1}{q}-\frac{1}{s}} (I_\alpha f_\tau)^*(y)\|_{L_s(0,t)} \\
&= \tau^{-\alpha} \sup_{t>0} t^{-\frac{\lambda}{s}} \|y^{\frac{1}{q}-\frac{1}{s}} (I_\alpha f)^*(y\tau^n)\|_{L_s(0,t)} \\
&= \tau^{-\alpha} \sup_{t>0} t^{-\frac{\lambda}{s}} \Big(\int_0^t (y\tau^n)^{\frac{s}{q}-1} ((I_\alpha f)^*(y\tau^n))^s \, d((y\tau^n)) \Big)^{\frac{1}{s}} \tau^{-\frac{n}{q}} \\
&= \tau^{-\alpha-\frac{n}{q}+\frac{n\lambda}{s}} (t\tau^n)^{-\frac{\lambda}{s}} \|y^{\frac{1}{q}-\frac{1}{s}} I_\alpha f^*(y)\|_{L_s(0,t)} \\
&= \tau^{-\alpha-n(\frac{1}{q}-\frac{\lambda}{s})} \|I_\alpha f\|_{M^{\mathrm{loc}}_{q,s;\lambda}}.
\end{aligned}$$

Since the operator I_α is bounded from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $M^{\mathrm{loc}}_{q,s;\lambda}$, we have $\|I_\alpha f\|_{M^{\mathrm{loc}}_{q,s;\lambda}} \le C\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}$, where C is independent of f. Then we get

$$\begin{aligned}
\|I_\alpha f\|_{M^{\mathrm{loc}}_{q,s;\lambda}} &= \tau^{\alpha+n(\frac{1}{q}-\frac{\lambda}{s})} \|I_\alpha f_\tau\|_{M^{\mathrm{loc}}_{q,s;\lambda}} \le C\tau^{\alpha+n(\frac{1}{q}-\frac{\lambda}{s})} \|f_\tau\|_{M^{\mathrm{loc}}_{p,r;\lambda}} \\
&= \tau^{\alpha+n(\frac{1}{q}-\frac{\lambda}{s})-n(\frac{1}{p}-\frac{\lambda}{r})} \|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}} = \tau^{\alpha+n(\frac{1}{q}-\frac{1}{p})+n\lambda(\frac{1}{r}-\frac{1}{s})} \|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}.
\end{aligned}$$

- If $\frac{1}{p} < \frac{1}{q} + \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$, then, for all $f \in M^{\mathrm{loc}}_{p,r;\lambda}$, we have $\|I_\alpha f\|_{M^{\mathrm{loc}}_{q,s;\lambda}} = 0$ as $\tau \to 0$.
- If $\frac{1}{p} > \frac{1}{q} + \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$, then, for all $f \in M^{\mathrm{loc}}_{p,r;\lambda}$, we have $\|I_\alpha f\|_{M^{\mathrm{loc}}_{q,s;\lambda}} = 0$ as $\tau \to \infty$.

If $\frac{1}{p} - \frac{1}{q} \ne \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$ for all $f \in M^{\mathrm{loc}}_{p,r;\lambda}$, we have $I_\alpha f(x) = 0$ for almost every $x \in \mathbb{R}^n$, which is impossible. Therefore we get $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$.

(ii) *Sufficiency.* For the limiting case $p = \frac{r}{r+\lambda}$, $1 \le r \le s < \infty$, suppose $f \in M^{\mathrm{loc}}_{p,r;\lambda}$. By using inequality (1.2) and Minkowski's inequality, we get

$$\begin{aligned}
\|I_\alpha f\|_{WM^{\mathrm{loc}}_{q,s;\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} (I_\alpha f)^*(\tau)\|_{WL_s(0,t)} \\
&\le C \sup_{t>0} t^{-\frac{\lambda}{s}} \Big\| \tau^{\frac{1}{q}-\frac{1}{s}} \Big(\tau^{\frac{\alpha}{n}-1} \int_0^\tau f^*(y)\,dy + \int_\tau^\infty f^*(y) y^{\frac{\alpha}{n}-1}\,dy \Big) \Big\|_{WL_s(0,t)} \\
&\le C \sup_{t>0} t^{-\frac{\lambda}{s}} \Big\| \tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1} \int_0^\tau f^*(y)\,dy \Big\|_{WL_s(0,t)} + C \sup_{t>0} t^{-\frac{\lambda}{s}} \Big\| \tau^{\frac{1}{q}-\frac{1}{s}} \int_\tau^\infty f^*(y) y^{\frac{\alpha}{n}-1}\,dy \Big\|_{WL_s(0,t)} \\
&= N_1 + N_2.
\end{aligned}$$

Let us estimate N_1:

$$N_1 = C \sup_{t>0} t^{-\frac{\lambda}{s}} \Big\| \tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1} \int_0^\tau f^*(y)\,dy \Big\|_{WL_s(0,t)} = C\|H^\beta_{(\gamma)} h\|_{WL_{s,\lambda}(0,\infty)}.$$

We take $\gamma = 1 + \frac{\lambda-1}{r}$ in the Hardy operator $H^\beta_{(\gamma)}$ and $h(t) = t^{1+\frac{\lambda-1}{r}} f^*(t)$. Therefore we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} + \frac{\alpha}{n} - 1 - \frac{\lambda}{r}.$$

By Lemma 2.7, we have $\beta = (1-\lambda)(\frac{1}{r}-\frac{1}{s})$, and then we obtain $1-\frac{1}{q}=\frac{\alpha}{n}-\frac{\lambda}{s}$. Thus we have

$$N_1 \le C\|H^{\beta}_{(\gamma)}h\|_{WL_{s,\lambda}(0,\infty)} \le C\|h\|_{L_{r,\lambda}(0,\infty)} = C\sup_{t>0} t^{-\frac{\lambda}{r}}\|\tau^{1+\frac{\lambda-1}{r}}f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}. \tag{3.3}$$

Now we consider N_2:

$$N_2 = C\sup_{t>0} t^{-\frac{\lambda}{s}}\Big\|\tau^{\frac{1}{q}-\frac{1}{s}}\int_\tau^\infty f^*(y)y^{\frac{\alpha}{n}-1}\,dy\Big\|_{WL_s(0,t)} = C\|\mathcal{H}^{\beta}_{(\gamma)}h\|_{WL_{q,\lambda}(0,\infty)}.$$

We take $\gamma = 1+\frac{\lambda-1}{r}-\frac{\alpha}{n}$ in the Hardy operator $\mathcal{H}^{\beta}_{(\gamma)}$ and $h(t)=t^{1+\frac{\lambda-1}{r}}f^*(t)$. Therefore we get

$$\beta = \frac{1}{q}-\frac{1}{s}+\frac{1}{r}+\frac{\alpha}{n}-1-\frac{\lambda}{r}.$$

By Lemma 2.7, we have $\beta=(1-\lambda)(\frac{1}{r}-\frac{1}{s})$, and then we obtain $1-\frac{1}{q}=\frac{\alpha}{n}-\frac{\lambda}{s}$. Hence the operator $\mathcal{H}^{\beta}_{(\gamma)}$ is bounded from the Morrey spaces $L_{r,\lambda}(0,\infty)$ to $WL_{s,\lambda}(0,\infty)$. Then we get

$$N_2 \le C\|\mathcal{H}^{\beta}_{(\gamma)}h\|_{WL_{s,\lambda}(0,\infty)} \le C\|h\|_{L_{r,\lambda}(0,\infty)} = C\sup_{t>0} t^{-\frac{\lambda}{r}}\|\tau^{1+\frac{\lambda-1}{r}}f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}. \tag{3.4}$$

From inequalities (3.3) and (3.4), we obtain the boundedness of the Riesz potential operator I_α from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $WM^{\mathrm{loc}}_{q,s;\lambda}$.

Necessity. Suppose that the operator I_α is bounded from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $WM^{\mathrm{loc}}_{q,s;\lambda}$ for $p=\frac{r}{r+\lambda}$.

Define $f_\tau(x) =: f(\tau x)$ for $\tau>0$. Then $\|f_\tau\|_{M^{\mathrm{loc}}_{r/(r+\lambda),r;\lambda}} = \tau^{-n}\|f\|_{M^{\mathrm{loc}}_{r/(r+\lambda),r;\lambda}}$ and

$$\begin{aligned}\|I_\alpha f_\tau\|_{WM^{\mathrm{loc}}_{q,s;\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{s}}\|y^{\frac{1}{q}-\frac{1}{s}}(I_\alpha f_\tau)^*(y)\|_{WL_s(0,t)}\\ &= \tau^{-\alpha}\sup_{t>0} t^{-\frac{\lambda}{s}}\|y^{\frac{1}{q}-\frac{1}{s}}(I_\alpha f)^*(y\tau^n)\|_{WL_s(0,t)}\\ &= \tau^{-\alpha-\frac{n}{q}+\frac{n\lambda}{s}}(t\tau^n)^{-\frac{\lambda}{s}}\|y^{\frac{1}{q}-\frac{1}{s}}I_\alpha f^*(y)\|_{WL_s(0,t)}\\ &= \tau^{-\alpha-n(\frac{1}{q}-\frac{\lambda}{s})}\|I_\alpha f\|_{WM^{\mathrm{loc}}_{q,s;\lambda}}.\end{aligned}$$

Since the operator I_α is bounded from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $WM^{\mathrm{loc}}_{q,s;\lambda}$, we have $\|I_\alpha f\|_{WM^{\mathrm{loc}}_{q,s;\lambda}} \le C\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}$, where C is independent of f. Then we get

$$\begin{aligned}\|I_\alpha f\|_{WM^{\mathrm{loc}}_{q,s;\lambda}} &= \tau^{\alpha+n(\frac{1}{q}-\frac{\lambda}{s})}\|I_\alpha f_\tau\|_{WM^{\mathrm{loc}}_{q,s;\lambda}} \le C\tau^{\alpha+n(\frac{1}{q}-\frac{\lambda}{s})}\|f_\tau\|_{M^{\mathrm{loc}}_{p,r;\lambda}}\\ &= \tau^{\alpha+n(\frac{1}{q}-\frac{\lambda}{s})-n}\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}} = \tau^{\alpha+n(\frac{1}{q}-1-\frac{\lambda}{r})+n\lambda(\frac{1}{r}-\frac{1}{s})}\|f\|_{M^{\mathrm{loc}}_{p,r;\lambda}}.\end{aligned}$$

- If $1<\frac{1}{q}+\frac{\alpha}{n}-\frac{\lambda}{s}$, then, for all $f\in M^{\mathrm{loc}}_{r/(r+\lambda),r;\lambda}$, we have $\|I_\alpha f\|_{WM^{\mathrm{loc}}_{q,s;\lambda}}=0$ as $\tau\to 0$.
- If $1>\frac{1}{q}+\frac{\alpha}{n}-\frac{\lambda}{s}$, then, for all $f\in M^{\mathrm{loc}}_{r/(r+\lambda),r;\lambda}$, we have $\|I_\alpha f\|_{WM^{\mathrm{loc}}_{q,s;\lambda}}=0$ as $\tau\to\infty$.

If $1\ne\frac{1}{q}+\frac{\alpha}{n}-\frac{\lambda}{s}$, then, for all $f\in M^{\mathrm{loc}}_{p,r;\lambda}$, we have $I_\alpha f(x)=0$ for almost every $x\in\mathbb{R}^n$, which is impossible. Therefore we get the equality $1-\frac{1}{q}=\frac{\alpha}{n}-\frac{\lambda}{s}$ and the proof of the theorem is completed. □

Remark 3.1. Note that for the limiting case $\lambda=1$, the space $M^{\mathrm{loc}}_{p,q;\lambda}$ is the classical Lorentz space $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}$ (see [4]). The boundedness of I_α in $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}$ is investigated in [23].

4 Some applications

Theorem 1.1 can be applied to various operators that are estimated from above by the Riesz potentials. In this section, we apply the theorem to the fractional maximal operator, fractional Marcinkiewicz operator and the fractional powers of some analytic semigroups.

4.1 Fractional maximal operator

For $0 \le \alpha < n$, we define the fractional maximal operator

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}-1} \int_{B(x,t)} |f(y)|\, dy,$$

where $B(x,t)$ is the open ball centered at x of radius t for $x \in \mathbb{R}^n$, and $|B(x,t)|$ is a Lebesgue measure of $B(x,t)$ such that $|B(x,t)| = \omega_n t^n$, in which ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. The fractional maximal operator M_α is closely related to a Riesz potential operator such that

$$M_\alpha f(x) \le \omega_n^{\frac{\alpha}{n}-1} (I_\alpha |f|)(x). \tag{4.1}$$

From inequality (4.1) we get the following corollary.

Corollary 4.1. *Let* $0 < \alpha < n$, $0 \le \lambda < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$,

$$\frac{r}{r+\lambda} \le p \le \Big(\frac{\lambda}{r} + \frac{\alpha}{n}\Big)^{-1} \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = \lambda\Big(\frac{1}{r} - \frac{1}{s}\Big) + \frac{\alpha}{n}.$$

(i) *If* $\frac{r}{r+\lambda} < p < (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the fractional maximal operator* M_α *from the space* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $M^{\mathrm{loc}}_{q,s;\lambda}$.

(ii) *If* $p = \frac{r}{r+\lambda}$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$ *is necessary and sufficient for the boundedness of the operator* M_α *from the space* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $WM^{\mathrm{loc}}_{q,s;\lambda}$.

Proof. Let $0 < \alpha < n$, $0 \le \lambda < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$ and $\frac{r}{r+\lambda} \le p \le (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$.

Sufficiency. The sufficiency parts of (i) and (ii) follow from Theorem 1.1 and inequality (4.1).

Necessity. (i) Suppose that the operator M_α is bounded from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $M^{\mathrm{loc}}_{q,s;\lambda}$ for $\frac{r}{r+\lambda} < p < (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$. Then we have

$$M_\alpha f_\tau(x) = \tau^{-\alpha} M_\alpha f(\tau x) \quad \text{and} \quad \|M_\alpha f_\tau\|_{M^{\mathrm{loc}}_{q,s;\lambda}} = \tau^{-\alpha-n(\frac{1}{q}-\frac{\lambda}{s})} \|M_\alpha f\|_{M^{\mathrm{loc}}_{q,s;\lambda}}.$$

By the same argument as in Theorem 1.1, we obtain

$$\frac{1}{p} - \frac{1}{q} = \lambda\Big(\frac{1}{r} - \frac{1}{s}\Big) + \frac{\alpha}{n}.$$

(ii) Suppose that the operator M_α is bounded from $M^{\mathrm{loc}}_{p,r;\lambda}$ to $WM^{\mathrm{loc}}_{q,s;\lambda}$ for $p = \frac{r}{r+\lambda}$. Then we have

$$M_\alpha f_\tau(x) = \tau^{-\alpha} M_\alpha f(\tau x) \quad \text{and} \quad \|M_\alpha f_\tau\|_{WM^{\mathrm{loc}}_{q,s;\lambda}} = \tau^{-\alpha-n(\frac{1}{q}-\frac{\lambda}{s})} \|M_\alpha f\|_{WM^{\mathrm{loc}}_{q,s;\lambda}}.$$

Hence we obtain the equality $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$. □

4.2 Fractional Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, i.e.,

$$\Omega(tx) = \Omega(x) \quad \text{for any } t > 0,\ x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, i.e.,

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \le 1$, that is, there exists a constant $C > 0$ such that

$$|\Omega(x') - \Omega(y')| \le C|x' - y'|^\gamma \quad \text{for any } x', y' \in S^{n-1}.$$

In 1958, Stein [24] defined the fractional Marcinkiewicz integral of higher dimension $\mu_{\Omega,\alpha}$ as

$$\mu_{\Omega,\alpha}(f)(x) = \Big(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3}\Big)^{\frac{1}{2}},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)\,dy.$$

The continuity of the Marcinkiewicz operator μ_Ω was extensively studied in [10, 11, 17, 25]. Let H be the space

$$H = \Big\{h : \|h\| = \Big(\int_0^\infty |h(t)|^2 \frac{dt}{t^3}\Big)^{\frac{1}{2}} < \infty\Big\}.$$

Then it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|$.

By Minkowski's inequality and the conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \Big(\int_{|x-y|}^\infty \frac{dt}{t^3}\Big)^{\frac{1}{2}} dy \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy = I_\alpha(|f|)(x).$$

Then we have the following corollary.

Corollary 4.2. *Let* $0 < \alpha < n$, $0 \le \lambda < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$,

$$\frac{r}{r+\lambda} \le p \le \Big(\frac{\lambda}{r} + \frac{\alpha}{n}\Big)^{-1} \quad \textit{and} \quad \frac{1}{p} - \frac{1}{q} = \lambda\Big(\frac{1}{r} - \frac{1}{s}\Big) + \frac{\alpha}{n}.$$

(i) *If* $\frac{r}{r+\lambda} < p < (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$ *and* $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$, *then the fractional Marcinkiewicz operator* $\mu_{\Omega,\alpha}$ *is bounded from the space* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $M^{\mathrm{loc}}_{q,s;\lambda}$.

(ii) *If* $p = \frac{r}{r+\lambda}$ *and* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$, *then the operator* $\mu_{\Omega,\alpha}$ *is bounded from the space* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $WM^{\mathrm{loc}}_{q,s;\lambda}$.

4.3 Fractional powers of some analytic semigroups

Suppose that L is a linear operator on L_2 that generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, i.e.,

$$|p_t(x,y)| \le \frac{c_1}{t^{\frac{n}{2}}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{4.2}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x, y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\frac{\alpha}{2}}$ of the operator L are defined by

$$L^{-\frac{\alpha}{2}} f(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\frac{\alpha}{2}+1}}.$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\frac{\alpha}{2}}$ is the Riesz potential I_α. Property (4.2) is satisfied for large classes of differential operators. In [7], other examples of operators which are estimates from above by the Riesz potentials are given. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (4.2), it follows that $|L^{-\frac{\alpha}{2}} f(x)| \le C I_\alpha(|f|)(x)$. Hence we get the following corollary.

Corollary 4.3. *Let* $0 < \alpha < n$, $0 \le \lambda < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$,

$$\frac{r}{r+\lambda} \le p \le \Big(\frac{\lambda}{r} + \frac{\alpha}{n}\Big)^{-1} \quad \textit{and} \quad \frac{1}{p} - \frac{1}{q} = \lambda\Big(\frac{1}{r} - \frac{1}{s}\Big) + \frac{\alpha}{n}.$$

(i) *If* $\frac{r}{r+\lambda} < p < (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$ *and* $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$, *then the operator* $L^{-\frac{\alpha}{2}}$ *is bounded from the space* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $M^{\mathrm{loc}}_{q,s;\lambda}$.

(ii) *If* $p = \frac{r}{r+\lambda}$ *and* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$, *then the operator* $L^{-\frac{\alpha}{2}}$ *is bounded from the space* $M^{\mathrm{loc}}_{p,r;\lambda}$ *to* $WM^{\mathrm{loc}}_{q,s;\lambda}$.

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