Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 110, pp. 1–24.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GLOBAL REGULARITY IN ORLICZ-MORREY SPACES OF SOLUTIONS TO NONDIVERGENCE ELLIPTIC EQUATIONS WITH VMO COEFFICIENTS

VAGIF S. GULIYEV, AYSEL A. AHMADLI,
MEHRIBAN N. OMAROVA, LUBOMIRA SOFTOVA

Communicated by Vicentiu D. Radulescu

Abstract. We show continuity in generalized Orlicz-Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ of sublinear integral operators generated by Calderón-Zygmund operator and their commutators with BMO functions. The obtained estimates are used to study global regularity of the solution of the Dirichlet problem for linear uniformly elliptic operator $\mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x)D_{ij}$ with discontinuous coefficients. We show that $\mathcal{L}u \in M_{\Phi,\varphi}$ implies the second-order derivatives belong to $M_{\Phi,\varphi}$.

1. Introduction

The classical Morrey spaces $L_{p,\lambda}$ are originally introduced in [37] to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and the Hölder spaces permits to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems. Recall that for a bounded domain $\Omega \subset \mathbb{R}^n$ satisfying the cone property, the space $L_{p,\lambda}$ with $1 \le p < \infty$ consists of all functions $f \in L_p(\Omega)$ such that

$$\|f\|_{L_{p,\lambda}(\Omega)} = \Big(\sup_{\mathcal{B}_r} \frac{1}{r^\lambda} \int_{\mathcal{B}_r \cap \Omega} |f(y)|^p \, dy\Big)^{1/p} < \infty,$$

where $\mathcal{B}_r$ ranges over all balls in $\mathbb{R}^n$ centered in some point $x \in \Omega$ and of radius $r > 0$. For the properties and applications of the classical Morrey spaces, we refer the readers to [7, 37, 41, 43] and the references there. Chiarenza and Frasca [8] showed the boundedness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity of fractional and classical Calderón-Zygmund operators in these spaces. Recall that integral operators of that kind appear in the representation formulae of the solutions of elliptic/parabolic equations and systems. Thus the continuity of the Calderón-Zygmund integrals implies regularity of the solutions in the corresponding spaces. Mizuhara[36] gave a generalization of these spaces considering a weight function $\omega(x, r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. He studied also a continuity in $L_{p,\omega}$ of some classical integral operators. Later Nakai

2010 *Mathematics Subject Classification.* 35J25, 35B40, 42B20, 42B35, 46E30.
Key words and phrases. Generalized Orlicz-Morrey spaces; Calderón-Zygmund integrals; commutators; VMO; elliptic equations; Dirichlet problem.

Submitted September 11, 2017. Published May 10, 2018.

extended the results of Chiarenza and Frasca in $L_{p,\omega}$ imposing certain integral and doubling conditions on ω (see [38]). Taking a weight $\omega = \varphi^p r^n$ the conditions of Mizuhara-Nakai become

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\,\varphi(x,r)^p, \quad C^{-1} \le \frac{\varphi(x,t)}{\varphi(x,r)} \le C, \quad \forall\, r \le t \le 2r,$$

where the constants do not depend on t, r and $x \in \mathbb{R}^n$.

In series of works, the first author studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators as Calderón-Zygmund, Riesz potental and others (see [18, 19, 21]). The following theorem obtained in [18] extends the results of Nakai in Morrey-type spaces with weight $\omega = \varphi r^n$ (for the definition of the spaces see § 3)

Theorem 1.1 ([18, 19]). *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty \varphi_1(x,r)\frac{dr}{r} \le C\varphi_2(x,t), \tag{1.1}$$

where C does not depend on x and t. Then the maximal operator M and the Calderón-Zygmund integral operators K are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to the weak space WM_{1,φ_2}.

Later this result was extended on spaces with weaker condition on the weight pair (φ_1, φ_2) (see [21], see also [11, 12, 13]). For more recent results on boundedness and continuity of singular integral operators in generalized Morrey and new functional spaces and their application in the differential equations theory see [2, 4, 5, 15, 16, 20, 25, 26, 35, 40, 42, 44, 48, 49, 51] and the references there.

Throughout this paper the following notation will be used:
$D_i u = \partial u/\partial x_i$, $Du = (D_1 u, \dots, D_n u)$ means the gradient of u,
$D_{ij}u = \partial^2 u/\partial x_i \partial x_j$, $D^2 u = \{D_{ij}u\}_{ij=1}^n$ is the Hessian matrix of u,
$\mathcal{B}_r = \mathcal{B}(x_0, r) = \{x \in \mathbb{R}^n : \ |x - x_0| < r\}$, $\mathcal{B}_r^c = \mathbb{R}^n \setminus \mathcal{B}_r$, $2\mathcal{B}_r = \mathcal{B}(x_0, 2r)$,
$\mathbb{S}^{n-1}$ is a unit sphere in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a domain and $\Omega_r = \Omega \cap \mathcal{B}_r(x)$, $x \in \Omega$,
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x = (x', x_n), x' \in \mathbb{R}^{n-1}, x_n > 0\}$,
$\mathcal{B}_r^+ \equiv \mathcal{B}^+(x^0, r) = \mathcal{B}(x^0, r) \cap \mathbb{R}^n_+$, $2\mathcal{B}_r^+ = \mathcal{B}^+(x^0, 2r)$ where $x^0 = (x', 0)$.

The standard summation convention on repeated upper and lower indices is adopted. The letter C is used for various positive constants and may change from one occurrence to another. In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries on Orlicz and Orlicz-Morrey spaces

Definition 2.1. A function $\Phi : [0, +\infty] \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = \Phi(\infty) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, +\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \ge s$.

We say that $\Phi \in \Delta_2$, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \le C_a\Phi(t)$ for all $t > 0$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \quad r \ge 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

The following two indices

$$q_\Phi = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \quad p_\Phi = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}$$

of Φ, where $\varphi(t)$ is the right-continuous derivative of Φ, are well known in the theory of Orlicz spaces. As is well known,

$$p_\Phi < \infty \iff \Phi \in \Delta_2,$$

and the function Φ is strictly convex if and only if $q_\Phi > 1$. If $0 < q_\Phi \le p_\Phi < \infty$, then $\frac{\Phi(t)}{t^{q_\Phi}}$ is increasing and $\frac{\Phi(t)}{t^{p_\Phi}}$ is decreasing on $(0, \infty)$.

Lemma 2.2 ([29, Lemma 1.3.2]). *Let $\Phi \in \Delta_2$. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \le b \frac{\Phi(t_1)}{t_1^p}$$

for $0 < t_1 < t_2$.

Lemma 2.3 ([47, Proposition 62.20]). *Let Φ be a Young function with canonical representation*

$$\Phi(t) = \int_0^t \varphi(s) ds, \quad t \ge 0.$$

(1) *Assume that $\Phi \in \Delta_2$. More precisely $\Phi(2t) \le A\Phi(t)$ for some $A \ge 2$. If $p > 1 + \log_2 A$, then*

$$\int_t^\infty \frac{\varphi(s)}{s^p} ds \lesssim \frac{\Phi(t)}{t^p}, \quad t > 0.$$

(2) *Assume that $\Phi \in \nabla_2$. Then*

$$\int_0^t \frac{\varphi(s)}{s} ds \lesssim \frac{\Phi(t)}{t}, \quad t > 0.$$

Recall that a function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \le \Phi(t) \le c\omega(ct), \ t \in [0, \infty).$$

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \quad \text{for } 0 < r < +\infty. \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, +\infty)$ and bijective from $[0, +\infty)$ to itself.

Definition 2.4. For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \Big\{ f \in L_1^{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|) dx < +\infty \text{ for some } k > 0 \Big\}$$

is called Orlicz space. The space $L_\Phi^{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Note that $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\Big\},$$

see, for example [45, Section 3, Theorem 10], so that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big)dx \le 1.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let

$$m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 2.5. The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) = \{f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\Big\{\lambda > 0\ : \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda}, t\Big) \le 1\Big\}.$$

For Young functions Φ and Ψ, we write $\Phi \approx \Psi$ if there exists a constant $C \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \quad \text{for all } r \ge 0.$$

If $\Phi \approx \Psi$, then $L_\Phi(\mathbb{R}^n) = L_\Psi(\mathbb{R}^n)$ with equivalent norms. We note that, for Young functions Φ and Ψ, if there exist $C, R \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \quad \text{for } r \in (0, R^{-1}) \cup (R, \infty),$$

then $\Phi \approx \Psi$.

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \quad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty.$$

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ +\infty, & r = +\infty. \end{cases} \tag{2.2}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1 + r)\log(1 + r) - r$.

Remark 2.6. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. Also, if Φ is a Young function, then $\Phi \in \nabla_2$ if and only if Φ^γ be quasiconvex for some $\gamma \in (0, 1)$ (see, for example [29, p. 15]).

It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \quad \text{for } r \ge 0. \tag{2.3}$$

The following analogue of the Hölder inequality is known.

Theorem 2.7 ([50]). *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\|fg\|_{L_1(\mathbb{R}^n)} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}.$$

Note that Young functions satisfy the property

$$\Phi(\alpha t) \le \alpha\Phi(t) \tag{2.4}$$

for all $0 < \alpha < 1$ and $0 \le t < \infty$, which is a consequence of the convexity: $\Phi(\alpha t) = \Phi(\alpha t + (1-\alpha)0) \le \alpha\Phi(t) + (1-\alpha)\Phi(0) = \alpha\Phi(t)$.

Lemma 2.8 ([3, 34]). *Let Φ be a Young function and B a ball in $\mathbb{R}^n$. Then*

$$\|\chi_B\|_{WL_\Phi(\mathbb{R}^n)} = \|\chi_B\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.7 and Lemma 2.8.

Lemma 2.9. *For a Young function Φ and $B = B(x,r)$, we have*

$$\|f\|_{L_1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}.$$

Definition 2.10. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $M_{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space, the space of all functions $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1})\,\|f\|_{L_\Phi(B(x,r))}.$$

Also by $WM_{\Phi,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Orlicz-Morrey space of all functions $f \in WL^{\rm loc}_\Phi(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1})\,\|f\|_{WL_\Phi(B(x,r))} < \infty,$$

where $WL_\Phi(B(x,r))$ denotes the weak L_Φ-space of measurable functions f for which

$$\|f\|_{WL_\Phi(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_\Phi(\mathbb{R}^n)}.$$

According to this definition, we recover the spaces $M_{p,\varphi}$ and $WM_{p,\varphi}$ under the choice $\Phi(r) = r^p$:

$$M_{p,\varphi} = M_{\Phi,\varphi}\big|_{\Phi(r)=r^p}, \quad WM_{\Phi,\lambda} = WM_{\Phi,\varphi}\big|_{\Phi(r)=r^p}.$$

3. Definitions and statement of the problem

In the present section we give the definitions of the functional spaces to which the coefficients and the data of the problem belong. The domain $\Omega \subset \mathbb{R}^n$ supposed to be bounded with $\partial\Omega \in C^{1,1}$.

Definition 3.1. Let $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $1 \le p < \infty$. The generalized Orlicz-Morrey space $M_{\Phi,\varphi}(\Omega)$ consists of all $f \in L^{\rm loc}_\Phi(\Omega)$

$$\|f\|_{M_{\Phi,\varphi}(\Omega)} = \sup_{x\in\Omega, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|\mathcal{B}(x,r)|^{-1})\|f\|_{L_\Phi(\Omega\cap\mathcal{B}(x,r))}$$

For any bounded domain Ω we define $M_{\Phi,\varphi}(\Omega)$ taking $f \in L_\Phi(\Omega)$ and Ω_r instead of $\mathcal{B}(x,r)$ in the norm above.

The generalized Sobolev-Orlicz-Morrey space $W_{2,\Phi,\varphi}(\Omega)$ consists of all Sobolev functions $u \in W_{2,\Phi}(\Omega)$ with distributional derivatives $D^s u \in M_{\Phi,\varphi}(\Omega)$, endowed with the norm

$$\|u\|_{W_{2,\Phi,\varphi}(\Omega)} = \sum_{0\le |s|\le 2} \|D^s f\|_{M_{\Phi,\varphi}(\Omega)}.$$

The space $W_{2,\Phi,\varphi}(\Omega) \cap W^0_{1,\Phi}(\Omega)$ consists of all functions $u \in W_{2,\Phi}(\Omega) \cap W^0_{1,\Phi}(\Omega)$ with $D^s u \in M_{\Phi,\varphi}(\Omega)$, and is endowed by the same norm. Recall that $W^0_{1,\Phi}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W_{1,\Phi}$.

Definition 3.2. Let $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function, the generalized weak Morrey space $WM_{\Phi,\varphi}(\Omega)$ consists of all measurable functions such that

$$\|f\|_{WM_{\Phi,\varphi}(\Omega)} = \sup_{x\in\Omega, r>0} \varphi(x,r)^{-1} \Phi^{-1}(|\mathcal{B}(x,r)|^{-1}) \|f\|_{WL_\Phi(\Omega\cap\mathcal{B}(x,r))},$$

where $WL_\Phi(\Omega \cap \mathcal{B}(x,r))$ denotes the weak L_Φ-space of measurable functions f for which

$$\|f\|_{WL_\Phi(\mathcal{B}(x,r))} \equiv \|f\chi_{\Omega\cap\mathcal{B}(x,r)}\|_{WL_\Phi(\mathbb{R}^n)}.$$

For a bounded domain Ω we define the space $WM_{\Phi,\varphi}(\Omega)$ taking $f \in WL_\Phi(\Omega)$.

Definition 3.3. Let $a \in L_1^{\rm loc}(\mathbb{R}^n)$ and $a_{\mathcal{B}_r} = \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} a(y)dy$ is the mean integral of a. We say that

- $a \in BMO$ (bounded mean oscillation, [31]) if
$$\|a\|_* = \sup_{R>0} \sup_{\mathcal{B}_r, r\le R} \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} |a(y) - a_{\mathcal{B}_r}|dy < +\infty.$$
The quantity $\|a\|_*$ is a norm in BMO modulo constant function under which BMO is a Banach space;
- $a \in VMO$ (*vanishing mean oscillation*, [46]) if $a \in BMO$ and
$$\lim_{R\to 0} \gamma_a(R) = \lim_{R\to 0} \sup_{\mathcal{B}_r, r\le R} \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} |a(y) - a_{\mathcal{B}_r}|dy = 0.$$
The quantity $\gamma_a(R)$ is called VMO-modulus of a.

For any bounded domain $\Omega \subset \mathbb{R}^n$ we define $BMO(\Omega)$ and $VMO(\Omega)$ taking $a \in L_1(\Omega)$ and Ω_r instead of $\mathcal{B}_r$ in the definition above.

According to [1, 32], having a function $a \in BMO(\Omega)$ or $VMO(\Omega)$ it is possible to extend it in the whole $\mathbb{R}^n$ preserving its BMO-norm or VMO-modulus, respectively. In the following we use this property without explicit references. Any bounded uniformly continuous function $f \in BUC$ with modulus of continuity $\omega_f(r)$ is also VMO and $\gamma_f(r) \equiv \omega_f(r)$. Besides that, BMO and VMO contain also discontinuous functions and the following example shows the inclusion $W_{1,n}(\mathbb{R}^n) \subset VMO \subset BMO$.

Example 3.4. $f_\alpha(x) = |\log|x||^\alpha \in VMO$ for any $\alpha \in (0,1)$; $f_\alpha \in W_{1,n}(\mathbb{R}^n)$ for $\alpha \in (0, 1-1/n)$, $f_\alpha \notin W_{1,n}(\mathbb{R}^n)$ for $\alpha \in [1-1/n, 1)$; $f(x) = |\log|x|| \in BMO\backslash VMO$; $\sin f_\alpha(x) \in VMO \cap L_\infty(\mathbb{R}^n)$.

In the Sections 4, 6 and 7 we study continuity in the spaces $M_{\Phi,\varphi}$ of certain sublinear integrals and their commutators with BMO functions. These results unified withe known estimates in $L_p(\mathbb{R}^n)$ permit to obtain continuity of the Calderón-Zygmund operators in $M_{p,\varphi}(\mathbb{R}^n)$ that is shown in § 8. The last section is dedicated to the Dirichlet problem for a linear uniformly elliptic operator with VMO coefficients. This problem is firstly studied by Chiarenza, Frasca and Longo. In their pioneer works [9], [10] they prove unique strong solvability of

$$\begin{gathered}\mathcal{L}u \equiv a^{ij}(x)D_{ij}u = f(x) \quad \text{a.a. } x \in \Omega, \\ u \in W_{2,p}(\Omega) \cap W^0_{1,p}(\Omega), \quad p \in (1,\infty)\end{gathered} \tag{3.1}$$

providing such way the classical theory on operators with continuous coefficients to those with discontinuous ones. Later their results are extended in the Sobolev-Morrey spaces $W_{2,p,\lambda}(\Omega) \cap W^0_{1,p}(\Omega)$, $\lambda \in (1,n)$ (see [15], [16]). In the present work we show that $\mathcal{L}u \in M_{\Phi,\varphi}(\Omega)$ implies the same regularity of the second order derivatives $D_{ij}u$. The weight $\varphi(x,r)$ satisfies an integral condition weaker than (1.1).

4. Sublinear operators and commutators generated by singular integrals in the space $M_{\Phi,\varphi}(\mathbb{R}^n)$

In this section we present results obtained in [27] concerning continuity of sublinear operators generated by singular integrals as Calderón-Zygmund. Let T be a sublinear operator such that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$ holds

$$|Tf(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\, dy, \tag{4.1}$$

where C is independent of f.

Theorem 4.1. *Let Φ any Young function, $\varphi_1, \varphi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions such that for any $x \in \mathbb{R}^n$ and for any $t > 0$,*

$$\int_r^\infty \Big(\operatorname{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Big) \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\varphi_2(x,r) \tag{4.2}$$

and T be sublinear operator satisfying (4.1).

(i) *If T bounded on $L_\Phi(\mathbb{R}^n)$, then T is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ and*

$$\|Tf\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n)} \le C\|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n)}.$$

(ii) *If T bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$, then it is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ and*

$$\|Tf\|_{WM_{\Phi,\varphi_2}(\mathbb{R}^n)} \le C\|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n)}$$

with constants independent of f.

Note that condition (4.2) is weaker than the one in Theorem 1.1. Indeed, if condition (1.1) holds then

$$\int_r^\infty \Big(\operatorname{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Big) \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le \int_r^\infty \varphi_1(x,t) \frac{dt}{t}$$

that implies (4.2). We give also two examples of admissible pairs of functions.

Example 4.2. For $\beta \in (0,n)$ consider the weight functions

$$\varphi_1(r) = \frac{r^\beta}{\Phi^{-1}\big(r^{-n}\big)} \big|\sin\big(\max\big\{1, \frac{\pi}{r}\big\}\big)\big|, \quad \varphi_2(r) = \frac{r^{2\beta}}{\Phi^{-1}\big(r^{-n}\big)}.$$

If $r\in(0,1)$ then $\operatorname{ess\,inf}_{r<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}=0$ and

$$\int_r^\infty\Big(\operatorname{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Big)\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}=\begin{cases}0 & r\in(0,1)\\ \frac{r^\beta}{\Phi^{-1}\big(r^{-n}\big)} & r\in(1,\infty)\end{cases}$$
$$\le C\varphi_2(r).$$

Hence the pair (φ_1,φ_2) satisfies (4.2) but not (1.1).

Example 4.3. For $\beta\in(0,n)$ consider the functions

$$\varphi_1(r)=\frac{r^{-\beta}}{\chi_{(1,\infty)}(r)\Phi^{-1}\big(r^{-n}\big)},\quad \varphi_2(r)=\frac{1+r^\beta}{\Phi^{-1}\big(r^{-n}\big)}.$$

They satisfy condition (4.2) but not (1.1).

Consider now the commutator $T_af=T[a,f]=aTf-T(af)$ such that for any $f\in L_\Phi(\mathbb{R}^n)$ with a compact support and $x\notin \operatorname{supp} f$ holds

$$|T_af(x)|\le C\int_{\mathbb{R}^n}|a(x)-a(y)|\frac{|f(y)|}{|x-y|^n}dy, \tag{4.3}$$

where C is independent of f and x. Suppose in addition that T_a is bounded in $L_\Phi(\mathbb{R}^n)$ satisfying the estimate $\|T_af\|_{L_\Phi(\mathbb{R}^n)}\le C\|a\|_*\|f\|_{L_\Phi(\mathbb{R}^n)}$. Then the following result holds (see [14, 27]).

Theorem 4.4. *Let Φ any Young function, $a\in BMO$, $\varphi_1,\varphi_2:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ be measurable functions such that for any $x\in\mathbb{R}^n$ and for any $t>0$,*

$$\int_r^\infty\Big(1+\ln\frac{t}{r}\Big)\Big(\operatorname{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Big)\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\le C\varphi_2(x,r), \tag{4.4}$$

where C does not depend on x and r. Suppose T_a be a sublinear operator satisfying (4.3) and bounded on $L_\Phi(\mathbb{R}^n)$. Then the operator T_a is bounded from M_{Φ,φ_1} to M_{Φ,φ_2}

$$\|T_af\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n)}\le C\|a\|_*\|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n)}.$$

5. Nonsingular integral operators in the Orlicz space $L_\Phi(\mathbb{R}^n_+)$

The following theorem was proved in [10].

Theorem 5.1. *Let $x\in\mathbb{R}^n_+$ and*

$$\widetilde{K}f(x)=\int_{\mathbb{R}^n_+}\frac{|f(y)|}{|\tilde{x}-y|^n}\,dy,\quad \tilde{x}=(x',-x_n). \tag{5.1}$$

Then there exists a constant C independent of f, such that

$$\|\widetilde{K}f\|_{L_p(\mathbb{R}^n_+)}\le C_p\|f\|_{L_p(\mathbb{R}^n_+)},\quad 1<p<\infty,$$
$$\|\widetilde{K}f\|_{WL_1(\mathbb{R}^n_+)}\le C\|f\|_{L_1(\mathbb{R}^n_+)}.$$

Theorem 5.2. *Let Φ be a Young function and $\widetilde{K}$ be a nonsingular integral operator, defined by (5.1). If $\Phi\in\Delta_2\cap\nabla_2$, then the operator $\widetilde{K}$ is bounded on $L_\Phi(\mathbb{R}^n_+)$ and if $\Phi\in\Delta_2$, then the operator $\widetilde{K}$ is bounded from $L_\Phi(\mathbb{R}^n_+)$ to $WL_\Phi(\mathbb{R}^n_+)$.*

Proof. First we prove that for $\Phi \in \Delta_2$ the nonsingular integral operator $\widetilde{K}$ is bounded from $L_\Phi(\mathbb{R}^n_+)$ to $WL_\Phi(\mathbb{R}^n_+)$.

We take $f \in L_\Phi(\mathbb{R}^n_+)$ satisfying $\|f\|_{L_\Phi} = 1$. Fix $\lambda > 0$ and define $f_1 = \chi_{\{|f|>\lambda\}} \cdot f$ and $f_2 = \chi_{\{|f|\le\lambda\}} \cdot f$. Then $f = f_1 + f_2$. We have

$$
\begin{gathered}
|\{|\widetilde{K}f| > \lambda\}| \le |\{|\widetilde{K}f_1| > \lambda/2\}| + |\{|\widetilde{K}f_2| > \lambda/2\}|, \\
\Phi(\lambda)|\{|\widetilde{K}f| > \lambda\}| \le |\Phi(\lambda)\{|\widetilde{K}f_1| > \lambda/2\}| + \Phi(\lambda)|\{|\widetilde{K}f_2| > \lambda/2\}|.
\end{gathered}
$$

We know that from the weak (1,1) boundedness and L_p, $p > 1$ boundedness of $\widetilde{K}$,

$$
\begin{gathered}
\{|\widetilde{K}(\chi_{\{|f|>\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda} \int_{\{|f|>\lambda\}} |f|, \\
\{|\widetilde{K}(\chi_{\{|f|\le\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda^p} \int_{\{|f|\le\lambda\}} |f|^p.
\end{gathered}
$$

Since $f_1 \in WL_1(\mathbb{R}^n_+)$ and $\frac{\Phi(\lambda)}{\lambda}$ increasing we have

$$
\begin{aligned}
\Phi(\lambda)\big|\big\{x \in \mathbb{R}^n_+ : |\widetilde{K}f_1(x)| > \frac{\lambda}{2}\big\}\big| &\lesssim \frac{\Phi(\lambda)}{\lambda} \int_{\mathbb{R}^n_+} |f_1(x)|dx \\
&= \frac{\Phi(\lambda)}{\lambda} \int_{\{x\in\mathbb{R}^n_+ : |f(x)|>\lambda\}} |f(x)|dx \\
&\lesssim \int_{\mathbb{R}^n_+} |f(x)| \frac{\Phi(|f(x)|)}{|f(x)|} dx \\
&= \int_{\mathbb{R}^n_+} \Phi(|f(x)|)dx.
\end{aligned}
$$

By Lemma 2.2 and $f_2 \in L_p(\mathbb{R}^n_+)$ we have

$$
\begin{aligned}
\Phi(\lambda)\,\big|\big\{x \in \mathbb{R}^n_+ : |\widetilde{K}f_2(x)| > \frac{\lambda}{2}\big\}\big| &\lesssim \frac{\Phi(\lambda)}{\lambda^p} \int_{\mathbb{R}^n_+} |f_2(x)|^p dx \\
&= \frac{\Phi(\lambda)}{\lambda^p} \int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p dx \\
&\lesssim \int_{\mathbb{R}^n_+} |f(x)|^p \frac{\Phi(|f(x)|)}{|f(x)|^p} dx \\
&= \int_{\mathbb{R}^n_+} \Phi(|f(x)|)dx.
\end{aligned}
$$

Thus we obtain

$$
|\{x \in \mathbb{R}^n_+ : |\widetilde{K}f(x)| > \lambda\}| \le \frac{C}{\Phi(\lambda)} \int_{\mathbb{R}^n_+} \Phi(|f(x)|)dx \le \frac{1}{\Phi\Big(\frac{\lambda}{C\|f\|_{L_\Phi}}\Big)}.
$$

Since $\|\cdot\|_{L_\Phi}$ norm is homogeneous this inequality is true for every $f \in L_\Phi(\mathbb{R}^n_+)$.

Now proved that for $\Phi \in \Delta_2 \cap \nabla_2$ the nonsingular integral operator $\widetilde{K}$ is bounded in $L_\Phi(\mathbb{R}^n_+)$. As before we use distribution functions.

$$
\int_{\mathbb{R}^n_+} \Phi\Big(\frac{\widetilde{K}f(x)}{\Lambda}\Big)dx = \frac{1}{\Lambda} \int_0^\infty \varphi\big(\frac{\lambda}{\Lambda}\big)|\{x \in \mathbb{R}^n_+ : |\widetilde{K}f(x)| > \lambda\}|d\lambda
$$

$$= \frac{2}{\Lambda}\int_0^\infty \varphi(\frac{2\lambda}{\Lambda})|\{x\in\mathbb{R}^n_+ : |\widetilde{K}f(x)|>2\lambda\}|d\lambda.$$

What is different from the estimate for the maximal operator is the point that $\widetilde{K}$ is not $L_\infty(\mathbb{R}^n_+)$ bounded. Let $p>1$ be sufficiently large. Then

$$\begin{aligned}|\{x\in\mathbb{R}^n_+ : \widetilde{K}f(x)>2\lambda\}| &\le |\{x\in\mathbb{R}^n_+ : |\widetilde{K}(\chi_{\{|f|>\lambda\}}\cdot f)(x)|>\lambda\}| \\ &\quad + |\{x\in\mathbb{R}^n_+ : |\widetilde{K}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)|>\lambda\}|.\end{aligned}$$

By the weak $(1,1)$ boundedness and L_p-boundedness of $\widetilde{K}$ (see Theorem 5.1) we have

$$|\{x\in\mathbb{R}^n_+ : |\widetilde{K}(\chi_{\{|f|>\lambda\}}\cdot f)(x)|>\lambda\}| \lesssim \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|>\lambda\}}|f(x)|dx,$$

$$|\{x\in\mathbb{R}^n_+ : |\widetilde{K}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)|>\lambda\}| \lesssim \frac{1}{\lambda^p}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}}|f(x)|^p dx.$$

Using the same calculation used for the maximal operator works for the first term,

$$\frac{1}{\Lambda}\int_0^\infty \varphi\Big(\frac{2\lambda}{\Lambda}\Big)\{|\widetilde{K}(\chi_{\{|f|>\lambda\}}\cdot f)|>\lambda\}|d\lambda \le \int_{\mathbb{R}^n_+}\Phi\Big(\frac{c|f|}{\Lambda}\Big). \tag{5.2}$$

For the second term a similar computation still works, but we use that $\Phi\in\Delta_2$,

$$\begin{aligned}&\frac{1}{\Lambda}\int_0^\infty \varphi\Big(\frac{2\lambda}{\Lambda}\Big)\{|\widetilde{K}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)|>\lambda\}|d\lambda \\ &\lesssim \frac{1}{\Lambda}\int_0^\infty \varphi\Big(\frac{2\lambda}{\Lambda}\Big)\Big(\int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}}|f(x)|^p dx\Big)\frac{d\lambda}{\lambda^p} \\ &\lesssim \frac{1}{\Lambda}\int_{\mathbb{R}^n_+}|f(x)|^p\Big(\int_{|f(x)|}^\infty \varphi\Big(\frac{2\lambda}{\Lambda}\Big)\frac{d\lambda}{\lambda^p}\Big)dx.\end{aligned}$$

Using Lemma 2.3 (1), we have

$$\begin{aligned}&\frac{1}{\Lambda}\int_0^\infty \varphi\Big(\frac{2\lambda}{\Lambda}\Big)\{|\widetilde{K}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)|>\lambda\}|d\lambda \\ &\lesssim \int_{\mathbb{R}^n_+}\Phi\Big(\frac{2|f(x)|}{\Lambda}\Big)dx \le \int_{\mathbb{R}^n_+}\Phi\Big(\frac{c|f(x)|}{\Lambda}\Big)dx.\end{aligned} \tag{5.3}$$

Thus, putting together (5.2) and (5.3), we obtain

$$\int_{\mathbb{R}^n_+}\Phi\Big(\frac{\widetilde{K}f(x)}{\Lambda}\Big)dx \le \int_{\mathbb{R}^n_+}\Phi\Big(\frac{c_0|f(x)|}{\Lambda}\Big)dx.$$

Again we shall label the constant we want to distinguish from other less important constants. As before, if we set $\Lambda = c_2\|f\|_{L_\Phi}(\mathbb{R}^n_+)$, then we obtain

$$\int_{\mathbb{R}^n_+}\Phi\Big(\frac{\widetilde{K}f(x)}{\Lambda}\Big)dx \le 1.$$

Hence the operator norm of $\widetilde{T}$ is less than c_2. $\square$

6. Sublinear operators generated by nonsingular integral operators in the space $M_{\Phi,\varphi}(\mathbb{R}^n_+)$

We use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight. The following theorem was proved in [22, 23] and in the case $w = 1$ in [6].

Theorem 6.1. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)H^*_w g(t) \le C \sup_{t>0} v_1(t)g(t) \tag{6.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{6.2}$$

Moreover, the value $C = B$ is the best constant for (6.1).

Remark 6.2. In (6.1) and (6.2) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

For any $x \in \mathbb{R}^n_+$ define $\tilde{x} = (x', -x_n)$ and recall that $x^0 = (x', 0)$. Let $\widetilde{T}$ be a sublinear operator such that for any $f \in L_1(\mathbb{R}^n_+)$ with a compact support holds

$$|\widetilde{T} f(x)| \le C \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy. \tag{6.3}$$

Lemma 6.3. *Let Φ any Young function, $f \in L^{\rm loc}_\Phi(\mathbb{R}^n_+)$, be such that*

$$\int_1^\infty \|f\|_{L_\Phi(\mathcal{B}^+(x^0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} < \infty \tag{6.4}$$

and $\widetilde{T}$ be a sublinear operator satisfying (6.3).

(i) *If $\widetilde{T}$ bounded on $L_\Phi(\mathbb{R}^n_+)$, then*

$$\|\widetilde{T} f\|_{L_\Phi(\mathcal{B}^+(x^0,r))} \le \frac{C}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}^+(x^0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{6.5}$$

(ii) *If $\widetilde{T}$ bounded from $L_\Phi(\mathbb{R}^n_+)$ on $WL_\Phi(\mathbb{R}^n_+)$, then*

$$\|\widetilde{T} f\|_{WL_\Phi(\mathcal{B}^+(x^0,r))} \le \frac{C}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}^+(x^0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}, \tag{6.6}$$

where the constants are independent of x^0, r and f.

Proof. (i) Denote $\mathcal{B}^+_r = \mathcal{B}^+(x^0, r)$, $\mathcal{B}^+_t = \mathcal{B}^+(x^0, t)$ and for any $f \in L^{\rm loc}_\Phi(\mathbb{R}^n_+)$ write $f = f_1 + f_2$ with $f_1 = f\chi_{2\mathcal{B}^+_r}$ and $f_2 = f\chi_{(2\mathcal{B}^+_r)^c}$. Because of the (Φ,Φ)-boundedness of the operator $\widetilde{T}$ (see Theorem 5.2) and $f_1 \in L_\Phi(\mathbb{R}^n_+)$ we have

$$\|\widetilde{T} f_1\|_{L_\Phi(\mathcal{B}^+_r)} \le \|\widetilde{T} f_1\|_{L_\Phi(\mathbb{R}^n_+)} \le C\|f_1\|_{L_\Phi(\mathbb{R}^n_+)} = C\|f\|_{L_\Phi(2\mathcal{B}^+_r)}.$$

It is easy to see that for arbitrary points $x \in \mathcal{B}^+_r$ and $y \in (2\mathcal{B}^+_r)^c$ it holds

$$\frac{1}{2}|x^0 - y| \le |\tilde{x} - y| \le \frac{3}{2}|x^0 - y|. \tag{6.7}$$

Applying (6.3) and the Fubuni theorem to $\widetilde{T}f_2$ we obtain

$$
\begin{aligned}
|\widetilde{T}f_2(x)| &\le C\int_{\mathbb{R}^n_+}\frac{|f_2(y)|}{|\tilde{x}-y|^n}\,dy\\
&\le C\int_{(2\mathcal{B}_r^+)^c}\frac{|f(y)|}{|x^0-y|^n}dy \le C\int_{(2\mathcal{B}_r^+)^c}|f(y)|\int_{|x^0-y|}^\infty\frac{dt}{t^{n+1}}\\
&\le C\int_{2r}^\infty\Big(\int_{2r\le|x^0-y|<t}|f(y)|dy\Big)\frac{dt}{t^{n+1}}\\
&\le C\int_{2r}^\infty\Big(\int_{\mathcal{B}_t^+}|f(y)|dy\Big)\frac{dt}{t^{n+1}}.
\end{aligned}
$$

Applying Hölder's inequality (Lemma 2.9), we obtain

$$
\begin{aligned}
\int_{(2\mathcal{B}_r^+)^c}\frac{|f(y)|}{|x^0-y|^n}dy &\lesssim \int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}_t^+)}\|1\|_{L_{\tilde\Phi}(\mathcal{B}_t^+)}\frac{dt}{t^{n+1}}\\
&= \int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}_t^+)}\frac{1}{\widetilde{\Phi}^{-1}(|\mathcal{B}_t^+|^{-1})}\frac{dt}{t^{n+1}}\\
&\approx \int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}_t^+)}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.
\end{aligned}
\tag{6.8}
$$

Direct calculations give

$$
\|\widetilde{T}f_2\|_{L_\Phi(\mathcal{B}_r^+)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}_t^+)}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}
\tag{6.9}
$$

and the above estimate holds for all $f\in L_\Phi(\mathbb{R}^n_+)$ satisfying (6.4). Thus

$$
\|\widetilde{T}f\|_{L_\Phi(\mathcal{B}_r^+)} \lesssim \|f\|_{L_\Phi(2\mathcal{B}_r^+)} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}_t^+)}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.
\tag{6.10}
$$

On the other hand,

$$
\begin{aligned}
\|f\|_{L_\Phi(2\mathcal{B}_r)} &= \frac{C}{\Phi^{-1}\big(r^{-n}\big)}\|f\|_{L_\Phi(2\mathcal{B}_r)}\int_{2r}^\infty \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\
&\le \frac{C}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty \|f\|_{L_\Phi(\mathcal{B}_t^+)}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}
\end{aligned}
\tag{6.11}
$$

which together with (6.10) gives (6.5).

(ii) Let now $f\in L_\Phi(\mathbb{R}^n_+)$, the weak (Φ,Φ)-boundedness of $\widetilde{T}$ (see Theorem 5.2) implies

$$
\|\widetilde{T}f_1\|_{WL_\Phi(\mathcal{B}_r^+)} \le \|\widetilde{T}f_1\|_{WL_\Phi(\mathbb{R}^n_+)} \le C\|f_1\|_{L_\Phi(\mathbb{R}^n_+)} = C\|f\|_{L_\Phi(2\mathcal{B}_r^+)}.
$$

Estimate (6.6) follows by (6.8). □

Theorem 6.4. *Let Φ any Young function, $\varphi_1,\varphi_2:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ be measurable functions satisfying (4.2) and $\widetilde{T}$ be a sublinear operator satisfying (6.3).*

(i) *If $\widetilde{T}$ bounded in $L_\Phi(\mathbb{R}^n_+)$ then it is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n_+)$ in $M_{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and*

$$
\|\widetilde{T}f\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n_+)}.
\tag{6.12}
$$

(ii) *If* $\widetilde{T}$ *bounded from* $L_\Phi(\mathbb{R}^n_+)$ *to* $WL_\Phi(\mathbb{R}^n_+)$ *then it is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n_+)$ *to* $WM_{\Phi,\varphi_2}(\mathbb{R}^n_+)$ *and*

$$\|\widetilde{T}f\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|f\|_{WM_{\Phi,\varphi_1}(\mathbb{R}^n_+)}$$

with constants independent of f*.*

Proof. Let $\widetilde{T}$ be bounded in $L_\Phi(\mathbb{R}^n_+)$. Then by Lemma 6.3 we have

$$\|\widetilde{T}f\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n_+)} \lesssim \sup_{x^0,\, r>0} \varphi_2(x^0,r)^{-1} \int_r^\infty \|f\|_{L_\Phi(\mathcal{B}^+(x^0,t))} \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

Applying the Theorem 6.1 to the above integral with

$$\begin{gathered} w(r) = \Phi^{-1}\big(r^{-n}\big), \quad v_2(x^0,r) = \varphi_2(x^0,r)^{-1}, \\ v_1(x^0,r) = \varphi_1(x^0,r)^{-1}\Phi^{-1}\big(r^{-n}\big), \quad g(x^0,r) = \|f\|_{L_\Phi(\mathcal{B}^+(x^0,r))}, \\ H_w^* g(x^0,r) = \int_r^\infty \|f\|_{L_\Phi(\mathcal{B}^+(x^0,t))} w(t)dt, \end{gathered}$$

where condition (6.2) is equivalent to (4.2), we obtain

$$\|\widetilde{T}f\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n_+)} \lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1(x^0,r)^{-1}\Phi^{-1}\big(r^{-n}\big)\|f\|_{L_\Phi(\mathcal{B}^+(x^0,r))} = \|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n_+)}.$$

The case $p=1$ is treated in the same manner using (6.6) and (6.2),

$$\begin{aligned} \|\widetilde{T}f\|_{WM_{1,\varphi_2}(\mathbb{R}^n_+)} &\lesssim \sup_{x^0,\, r>0} \varphi_2(x^0,r)^{-1} \int_r^\infty \|f\|_{L_\Phi(\mathcal{B}^+(x^0,t))} \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \\ &= \sup_{x^0,\, r>0} \varphi_1(x^0,r)^{-1}\Phi^{-1}\big(r^{-n}\big)\|f\|_{L_\Phi(\mathcal{B}^+(x^0,r))} \\ &= \|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n_+)}. \end{aligned}$$

□

7. Commutators of sublinear operators generated by nonsingular integrals in the space $M_{\Phi,\varphi}(\mathbb{R}^n_+)$

For a function $a \in BMO$ and sublinear operator $\widetilde{T}$ satisfying (6.3) define the commutator $\widetilde{T}_a = [a,\widetilde{T}]f = a\widetilde{T}f - \widetilde{T}(af)$. Suppose that for any $f \in L_1(\mathbb{R}^n_+)$ with compact support and $x \notin \operatorname{supp} f$, it holds

$$|\widetilde{T}_a f(x)| \le C \int_{\mathbb{R}^n_+} |a(x)-a(y)|\frac{|f(y)|}{|\tilde{x}-y|^n}\,dy, \tag{7.1}$$

with a constant independent of f and x. Suppose in addition that $\widetilde{T}_a$ is bounded in $L_\Phi(\mathbb{R}^n_+)$ satisfying $\|\widetilde{T}_a f\|_{L_\Phi(\mathbb{R}^n_+)} \le C\|a\|_*\|f\|_{L_\Phi(\mathbb{R}^n_+)}$. Our aim is to show boundedness of $\widetilde{T}_a$ in $M_{\Phi,\varphi}(\mathbb{R}^n_+)$. For this goal we recall some well known properties of the BMO functions.

Lemma 7.1 (John-Nirenberg lemma [31])**.** *Let* $a \in BMO$ *and* $p \in (1,\infty)$*. Then for any ball* $\mathcal{B}$ *it holds*

$$\Big(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |a(y)-a_{\mathcal{B}}|^p dy\Big)^{1/p} \le C(p)\|a\|_*. \tag{7.2}$$

Definition 7.2. A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $t \in [1,\infty)$(resp. $t \in [0,1]$) and $s \in [0,\infty)$,

$$\Phi(st) \le Ct^p \Phi(s).$$

Remark 7.3. We know that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ (see [29]).

Before proving the main theorems, we need the following lemma.

Lemma 7.4 ([30])**.** *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$|b_{\mathcal{B}_r} - b_{\mathcal{B}_t}| \le C\|b\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t,$$

where C is independent of b, x, r, and t.

In the following lemma which was proved in [24] we provide a generalization of the property (7.2), from L_p-norms to Orlicz norms.

Lemma 7.5. *Let $b \in BMO$ and Φ be a Young function. Let Φ is lower type p_0 and upper type p_1 with $1 \le p_0 \le p_1 < \infty$, then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-n}\big)\|b(\cdot) - b_{B(x,r)}\|_{L_\Phi(B(x,r))}.$$

For the variable exponent Lebesgue space $L_{p(\cdot)}$ Lemma 7.5 was proved in [28]. For a Young function Φ, let

$$a_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \quad b_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Remark 7.6. It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \le b_\Phi < \infty$ (See, for example [33]).

Remark 7.7. Remarks 7.6 and Remark 7.3 show that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $1 < a_\Phi \le b_\Phi < \infty$.

To estimate the commutator we shall employ the same idea which we used in the proof of Lemma 6.3.

Lemma 7.8. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $a \in BMO$ and $\widetilde{T}_a$ be a bounded operator in $L_\Phi(\mathbb{R}^n_+)$ satisfying* (7.1)*. Suppose that for all $f \in L^{\rm loc}_\Phi(\mathbb{R}^n_+)$ and $r > 0$ holds*

$$\int_1^\infty \Big(1 + \ln \frac{t}{r}\Big)\|f\|_{L_\Phi(\mathcal{B}^+_t(x^0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} < \infty. \tag{7.3}$$

Then

$$\|\widetilde{T}_a f\|_{L_\Phi(\mathcal{B}^+_r)} \lesssim \frac{\|a\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \Big(1 + \ln \frac{t}{r}\Big)\|f\|_{L_\Phi(\mathcal{B}^+(x^0,t))}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

Proof. Decompose f as $f = f\chi_{2\mathcal{B}^+_r} + f\chi_{(2\mathcal{B}^+_r)^c} = f_1 + f_2$. From the boundedness of $\widetilde{T}_a$ in $L_\Phi(\mathbb{R}^n_+)$ it follows that

$$\|\widetilde{T}_a f_1\|_{L_\Phi(\mathcal{B}^+_r)} \le \|\widetilde{T}_a f_1\|_{L_\Phi(\mathbb{R}^n_+)} \lesssim \|a\|_*\|f_1\|_{L_\Phi(\mathbb{R}^n_+)} = \|a\|_*\|f\|_{L_\Phi(2\mathcal{B}^+_r)}.$$

On the other hand, because of (6.7), we can write

$$
\begin{aligned}
\|\widetilde{T}_a f_2\|_{L_\Phi(\mathcal{B}_r^+)} &\lesssim \Big(\int_{\mathcal{B}_r^+}\Big(\int_{(2\mathcal{B}_r^+)^c}\frac{|a(x)-a(y)||f(y)|}{|x^0-y|^n}dy\Big)^p dx\Big)^{1/p}\\
&\lesssim \Big(\int_{\mathcal{B}_r^+}\Big(\int_{(2\mathcal{B}_r^+)^c}\frac{|a(y)-a_{\mathcal{B}_r^+}||f(y)|}{|x^0-y|^n}dy\Big)^p dx\Big)^{1/p}\\
&\quad + \Big(\int_{\mathcal{B}_r^+}\Big(\int_{(2\mathcal{B}_r^+)^c}\frac{|a(x)-a_{\mathcal{B}_r^+}||f(y)|}{|x^0-y|^n}dy\Big)^p dx\Big)^{1/p} = I_1 + I_2.
\end{aligned}
$$

We estimate I_1 as follows

$$
\begin{aligned}
I_1 &\lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{(2\mathcal{B}_r^+)^c}\frac{|a(y)-a_{\mathcal{B}_r^+}||f(y)|}{|x^0-y|^n}\,dy\\
&= \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{(2\mathcal{B}_r^+)^c}|a(y)-a_{\mathcal{B}_r^+}||f(y)|\int_{|x^0-y|}^{\infty}\frac{dt}{t^{n+1}}\,dy\\
&= \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\int_{2r\le|x^0-y|\le t}|a(y)-a_{\mathcal{B}_r^+}|\,|f(y)|dy\,\frac{dt}{t^{n+1}}\\
&\lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\int_{\mathcal{B}_t^+}|a(y)-a_{\mathcal{B}_r^+}||f(y)|dy\,\frac{dt}{t^{n+1}}.
\end{aligned}
$$

Applying Hölder's inequality, Lemma 7.1 and (7.4), we obtain

$$
\begin{aligned}
I_1 &\lesssim \Big(\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\int_{\mathcal{B}_t^+}|a(y)-a_{\mathcal{B}_t^+}||f(y)|dy\,\frac{dt}{t^{n+1}}\\
&\quad + \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}|a_{\mathcal{B}_r^+}-a_{\mathcal{B}_t^+}|\int_{\mathcal{B}_t^+}|f(y)|dy\,\frac{dt}{t^{n+1}}\Big)\\
&\lesssim \Big(\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\Big\|a(\cdot)-a_{\mathcal{B}_t^+}\Big\|_{L_{\widetilde{\Phi}}(\mathcal{B}_t^+)}\|f\|_{L_\Phi(\mathcal{B}_t^+)}\,\frac{dt}{t^{n+1}}\\
&\quad + \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}|a_{\mathcal{B}_r^+}-a_{\mathcal{B}_t^+}|\|f\|_{L_\Phi(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\Big)\\
&\lesssim \|a\|_*\,\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.
\end{aligned}
$$

To estimate I_2 note that

$$
I_2 = \|a(\cdot)-a_{\mathcal{B}_r^+}\|_{L_\Phi(\mathcal{B}_r^+)}\int_{(2\mathcal{B}_r^+)^c}\frac{|f(y)|}{|x^0-y|^n}dy.
$$

By Lemma 7.1 and (6.8) we obtain

$$
I_2 \lesssim \frac{\|a\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{(2\mathcal{B}_r^+)^c}\frac{|f(y)|}{|x^0-y|^n}dy \lesssim \frac{\|a\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\|f\|_{L_\Phi(\mathcal{B}_t^+)}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.
$$

Summing I_1 and I_2 we obtain that for all $p\in(1,\infty)$,

$$
\|\widetilde{T}_a f_2\|_{L_\Phi(\mathcal{B}_r^+)} \lesssim \frac{\|a\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.
$$

Finally,

$$
\|\widetilde{T}_a f\|_{L_\Phi(\mathcal{B}_r^+)} \lesssim \|a\|_*\,\|f\|_{L_\Phi(2\mathcal{B}_r^+)} + \frac{\|a\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}
$$

and the statement follows by (6.11). □

Theorem 7.9. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $a \in BMO$ and $\varphi_1, \varphi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying (4.4). Suppose $\widetilde{T}_a$ be a sublinear operator bounded on $L_\Phi(\mathbb{R}^n_+)$ and satisfying (7.1). Then $\widetilde{T}_a$ is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|\widetilde{T}_a f\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|a\|_* \|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n_+)} \tag{7.4}$$

with a constant independent of f.

The statement of the theorem follows by Lemma 7.8 and Theorem 6.1 in the same manner as the proof of Theorem 6.4.

8. Singular and nonsingular integral operators in the spaces $M_{\Phi,\varphi}$

In this section we deal with Calderón-Zygmund type integrals and their commutators with BMO functions. We start with the definition of the corresponding kernel.

Definition 8.1. A measurable function $\mathcal{K}(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called a variable Calderón-Zygmund kernel if:

- (i) $\mathcal{K}(x,\cdot)$ is a Calderón-Zygmund kernel for almost all $x \in \mathbb{R}^n$:
  - (a) $\mathcal{K}(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$,
  - (b) $\mathcal{K}(x,\mu\xi) = \mu^{-n}\mathcal{K}(x,\xi)$ for all $\mu > 0$,
  - (c) $\int_{\mathbb{S}^{n-1}} \mathcal{K}(x,\xi)d\sigma_\xi = 0$, $\int_{\mathbb{S}^{n-1}} |\mathcal{K}(x,\xi)|d\sigma_\xi < +\infty$,
- (ii) $\max_{|\beta|\le 2n} \|D^\beta_\xi \mathcal{K}(x,\xi)\|_{L_\infty(\mathbb{R}^n\times\mathbb{S}^{n-1})} = M < \infty$ independently of x.

The singular integrals

$$\begin{aligned}
\mathfrak{K}f(x) &= \text{P.V.} \int_{\mathbb{R}^n} \mathcal{K}(x, x-y)f(y)dy,\\
\mathfrak{C}[a,f](x) &= \text{P.V.} \int_{\mathbb{R}^n} \mathcal{K}(x,x-y)f(y)[a(x)-a(y)]dy\\
&= a(x)\mathfrak{K}f(x) - \mathfrak{K}(af)(x)
\end{aligned}$$

are bounded in $L_\Phi(\mathbb{R}^n)$ (see [39]), moreover

$$|\mathcal{K}(x,\xi)| \le |\xi|^{-n}\big|\mathcal{K}\big(x, \frac{\xi}{|\xi|}\big)\big| \le M|\xi|^{-n}$$

which implies

$$|\mathfrak{K}f(x)| \le C\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad |\mathfrak{C}[a,f](x)| \le C\int_{\mathbb{R}^n} \frac{|a(x)-a(y)||f(y)|}{|x-y|^n}\,dy$$

and hence the validity of all results from § 4. Let us note that any measurable function $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the condition (4.4) satisfies also (4.2) with $\varphi_1 \equiv \varphi_2 \equiv \varphi$. Hence the following results hold as a simple application of the estimates from § 4.

Theorem 8.2. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable function such that for all $x \in \mathbb{R}^n$ and $r > 0$*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\Big(\operatorname{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Big)\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \le C\varphi(x,r). \tag{8.1}$$

Then for any $f \in M_{\Phi,\varphi}(\mathbb{R}^n)$ and $a \in BMO$ there exist constants depending on n, p, φ and the kernel such that

$$\|\mathfrak{K}f\|_{M_{\Phi,\varphi}(\mathbb{R}^n)} \le C\|f\|_{M_{\Phi,\varphi}(\mathbb{R}^n)}, \quad \|\mathfrak{C}[a,f]\|_{M_{\Phi,\varphi}(\mathbb{R}^n)} \le C\|a\|_*\|f\|_{M_{\Phi,\varphi}(\mathbb{R}^n)}.$$

The above theorem follows from (6.12) and (7.4).

Example 8.3. The weight $\varphi(r) = r^\beta\,\Phi^{-1}\big(r^{-n}\big)$, $0 < \beta < n$ satisfies condition (8.1).

Example 8.4. The weight $\varphi(r) = r^\beta\,\Phi^{-1}\big(r^{-n}\big)\ln^m(e+r)$, $m \ge 1$, $0 < \beta < n$ satisfies condition (8.1) and the space $M_{\Phi,\varphi}$ does not coincide with any Morrey space.

Since we aim at studying regularity properties of the solution of the Dirichlet problem (3.1) we need of some additional local results.

Corollary 8.5. *Let $\Omega \subset \mathbb{R}^n$, $\partial\Omega \in C^{1,1}$, $a \in BMO(\Omega)$ and $f \in M_{\Phi,\varphi}(\Omega)$ with Φ and φ as in Theorem 8.2. Then*

$$\|\mathfrak{K}f\|_{M_{\Phi,\varphi}(\Omega)} \le C\|f\|_{M_{\Phi,\varphi}(\Omega)} \quad \|\mathfrak{C}[a,f]\|_{M_{\Phi,\varphi}(\Omega)} \le C\|a\|_*\|f\|_{M_{\Phi,\varphi}(\Omega)} \tag{8.2}$$

with $C = C(n, p, \varphi, \Omega, \mathcal{K})$.

Corollary 8.6. *Let Φ and φ be as in Theorem 8.2 and $a \in VMO$ with VMO-modulus γ_a. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ such that for any ball $\mathcal{B}_r$ with a radius $r \in (0, \rho_0)$ and all $f \in M_{\Phi,\varphi}(\mathcal{B}_r)$ holds*

$$\|\mathfrak{C}[a,f]\|_{M_{\Phi,\varphi}(\mathcal{B}_r^+)} \le C\varepsilon\|f\|_{M_{\Phi,\varphi}(\mathcal{B}_r^+)}, \tag{8.3}$$

with $C = C(n, p, \varphi, \Omega, \mathcal{K})$.

To obtain the above estimates it suffices to extend $\mathcal{K}(x,\cdot)$ and $f(\cdot)$ as zero outside Ω (see [9, Theorem 2.11] for details). Recall that the extension of a keeps its BMO norm or VMO-modulus according to [1, 32].

For any $x, y \in \mathbb{R}^n_+$, $\tilde{x} = (x', -x_n)$ define the *generalized reflection* $\mathcal{T}(x;y)$ as

$$\mathcal{T}(x;y) = x - 2x_n\frac{\mathbf{a}^n(y)}{a^{nn}(y)} \quad \mathcal{T}(x) = \mathcal{T}(x;x) : \mathbb{R}^n_+ \to \mathbb{R}^n_-,$$

where $\mathbf{a}^n$ is the last row of the coefficients matrix $\mathbf{a}$. Then there exist positive constants C_1, C_2 depending on n and Λ, such that

$$C_1|\widetilde{x} - y| \le |\mathcal{T}(x) - y| \le C_2|\widetilde{x} - y|, \quad \forall x, y \in \mathbb{R}^n_+.$$

For any $f \in M_{\Phi,\varphi}(\mathbb{R}^n_+)$ and $a \in BMO$ consider the nonsingular integral operators

$$\widetilde{\mathfrak{K}}f(x) = \int_{\mathbb{R}^n_+} \mathcal{K}(x, \mathcal{T}(x) - y)f(y)dy, \quad \widetilde{\mathfrak{C}}[a,f](x) = a(x)\mathfrak{K}f(x) - \mathfrak{K}(af)(x).$$

The kernel $\mathcal{K}(x, \mathcal{T}(x) - y) : \mathbb{R}^n \times \mathbb{R}^n_+ \to \mathbb{R}$ is not singular and verifies the conditions (i)(b) and (ii) from Definition 8.1. Moreover

$$|\mathcal{K}(x, \mathcal{T}(x) - y)| \le M|\mathcal{T}(x) - y|^{-n} \le C|\tilde{x} - y|^{-n},$$

which implies

$$|\widetilde{\mathfrak{K}}f(x)| \le C\int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|}\,dy, \quad |\widetilde{\mathfrak{C}}[a,f](x)| \le C\int_{\mathbb{R}^n_+} |a(x) - a(y)|\frac{|f(y)|}{|\tilde{x} - y|}\,dy.$$

The following estimates are simple consequence of the results in § 6 and § 7.

Theorem 8.7. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $a \in BMO(\mathbb{R}^n_+)$ and φ be measurable function satisfying* (8.1)*. Then the operators $\widetilde{\mathfrak{K}}f$ and $\widetilde{\mathfrak{C}}[a,f]$ are continuous in $M_{\Phi,\varphi}$ and for all $f \in M_{\Phi,\varphi}(\mathbb{R}^n_+)$ holds*

$$\begin{aligned} \|\widetilde{\mathfrak{K}}f\|_{M_{\Phi,\varphi}(\mathbb{R}^n_+)} &\le C\|f\|_{M_{\Phi,\varphi}(\mathbb{R}^n_+)}, \\ \|\widetilde{\mathfrak{C}}[a,f]\|_{M_{\Phi,\varphi}(\mathbb{R}^n_+)} &\le C\|a\|_*\|f\|_{M_{\Phi,\varphi}(\mathbb{R}^n_+)} \end{aligned} \tag{8.4}$$

with a constant dependent on known quantities only.

Corollary 8.8. *Let Φ and φ be as in Theorem 8.7 and $a \in VMO$ with a VMO-modulus γ_a. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ such that for any ball $\mathcal{B}^+_r$ with a radius $r \in (0, \rho_0)$ and all $f \in M_{\Phi,\varphi}(\mathcal{B}^+_r)$ holds*

$$\|\widetilde{\mathfrak{C}}[a,f]\|_{M_{\Phi,\varphi}(\mathcal{B}^+_r)} \le C\varepsilon\|f\|_{M_{\Phi,\varphi}(\mathcal{B}^+_r)}, \tag{8.5}$$

where C is independent of ε, f and r.

The proof of the above corollary is as that of [9, Theorem 2.13].

9. Dirichlet problem

We consider the Dirichlet problem for second order linear equations

$$\begin{gathered} \mathcal{L}u := a^{ij}(x)D_{ij}u = f(x) \quad \text{a.a. } x \in \Omega, \\ u \in\ W_{2,\Phi,\varphi}(\Omega) \cap W^0_{1,\Phi}(\Omega) \end{gathered} \tag{9.1}$$

subject to the following conditions:

(H1) Uniform ellipticity of $\mathcal{L}$: there exists a constant $\Lambda > 0$, such that

$$\begin{gathered} \Lambda^{-1}|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{a.a. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n \\ a^{ij}(x) = a^{ji}(x) \quad 1 \le i,j \le n. \end{gathered}$$

This assumption implies immediately essential boundedness of the coefficients $a^{ij} \in L_\infty(\Omega)$.

(H2) Regularity of the data: $a^{ij} \in VMO(\Omega)$ and $f \in M_{\Phi,\varphi}(\Omega)$ with $1 < p < \infty$ and $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ measurable.

Theorem 9.1 (Interior estimate)**.** *Let $u \in W^{\rm loc}_{2,\Phi}(\Omega)$ and $\mathcal{L}$ be a linear uniformly elliptic operator with VMO coefficients such that $\mathcal{L}u \in M^{\rm loc}_{\Phi,\varphi}(\Omega)$ with $\Phi \in \Delta_2 \cap \nabla_2$ and φ satisfying* (8.1)*. Then $D_{ij}u \in M_{\Phi,\varphi}(\Omega')$ for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and*

$$\|D^2u\|_{M_{\Phi,\varphi}(\Omega')} \le C\big(\|u\|_{M_{\Phi,\varphi}(\Omega'')} + \|\mathcal{L}u\|_{M_{\Phi,\varphi}(\Omega'')}\big), \tag{9.2}$$

where the constant depends on known quantities and $\operatorname{dist}(\Omega', \partial\Omega'')$.

Proof. Take an arbitrary point $x \in \operatorname{supp} u$ and a ball $\mathcal{B}_r(x) \subset \Omega'$, choose a point $x_0 \in \mathcal{B}_r(x)$ and fix the coefficients of $\mathcal{L}$ in x_0. Consider the constant coefficients operator $\mathcal{L}_0 = a^{ij}(x_0)D_{ij}$. From the classical theory we know that a solution $v \in C^\infty_0(\mathcal{B}_r(x))$ of $\mathcal{L}_0 v = (\mathcal{L}_0 - \mathcal{L})v + \mathcal{L}v$ can be presented as Newtonian type potential

$$v(x) = \int_{\mathcal{B}_r} \Gamma^0(x-y)[(\mathcal{L}_0 - \mathcal{L})v(y) + \mathcal{L}v(y)]dy,$$

where $\Gamma^0(x-y)=\Gamma(x_0,x-y)$ is the fundamental solution of $\mathcal{L}_0$. Taking $D_{ij}v$ and unfreezing the coefficients we obtain for all $i,j=1,\dots,n$ (cf. [9])

$$\begin{aligned} D_{ij}v(x) &= \mathrm{P.V.}\int_{\mathcal{B}_r}\Gamma_{ij}(x,x-y)[\mathcal{L}v(y)+\big(a^{hk}(x)-a^{hk}(y)\big)D_{hk}v(y)]dy \\ &\quad + \mathcal{L}v(x)\int_{\mathbb{S}^n}\Gamma_j(x,y)y_i d\sigma_y \qquad (9.3)\\ &= \mathfrak{K}_{ij}\mathcal{L}v(x)+\mathfrak{C}_{ij}[a^{hk},D_{hk}v](x)+\mathcal{L}v(x)\int_{\mathbb{S}^{n-1}}\Gamma_j(x;y)y_i d\sigma_y. \end{aligned}$$

Here $\Gamma_{ij}(x,\xi)$ stand for the derivatives $D_{\xi_i\xi_j}\Gamma(x,\xi)$. The known properties of the fundamental solution imply that $\Gamma_{ij}(x,\xi)$ are variable Calderón-Zygmund kernels in the sense of Definition 8.1. The representation formula (9.3) still holds for any $v\in W_{2,p}(\mathcal{B}_r)\cap W^0_{1,p}(\mathcal{B}_r)$ because of the approximation properties of the Sobolev functions with C_0^∞ functions. In view of (8.2), (8.3) and (9.3) for each $\varepsilon>0$ there exists $r_0(\varepsilon)$ such that for any $r<r_0(\varepsilon)$ it holds

$$\|D^2v\|_{\Phi,\varphi;r}\le C\left(\varepsilon\|D^2v\|_{\Phi,\varphi;r}+\|\mathcal{L}v\|_{\Phi,\varphi;r}\right),\quad \|\cdot\|_{\Phi,\varphi;r}:=\|\cdot\|_{M_{\Phi,\varphi}(\mathcal{B}_r^+)}.$$

Choosing ε (and hence also r!) small enough we can move the norm of D^2v on the left-hand side that gives

$$\|D^2v\|_{\Phi,\varphi;r}\le C\|\mathcal{L}v\|_{\Phi,\varphi;r}\,. \qquad (9.4)$$

Define a cut-off function $\eta(x)$ such that for $\theta\in(0,1)$, $\theta'=\theta(3-\theta)/2>\theta$ and $s=0,1,2$ we have

$$\eta(x)=\begin{cases}1 & x\in\mathcal{B}_{\theta r}\\ 0 & x\notin\mathcal{B}_{\theta' r}\end{cases}\qquad \eta(x)\in C_0^\infty(\mathcal{B}_r),\quad |D^s\eta|\le C[\theta(1-\theta)r]^{-s}.$$

Applying (9.4) to $v(x)=\eta(x)u(x)\in W_{2,\Phi,\varphi}(\mathcal{B}_r)\cap W^0_{1,\Phi}(\mathcal{B}_r)$ we obtain

$$\begin{aligned}\|D^2u\|_{\Phi,\varphi;\theta r}&\le C\|\mathcal{L}v\|_{\Phi,\varphi;\theta' r}\\ &\le C\Big(\|\mathcal{L}u\|_{\Phi,\varphi;\theta' r}+\frac{\|Du\|_{\Phi,\varphi;\theta' r}}{\theta(1-\theta)r}+\frac{\|u\|_{\Phi,\varphi;\theta' r}}{[\theta(1-\theta)r]^2}\Big).\end{aligned}$$

Define the weighted semi-norm

$$\Theta_s=\sup_{0<\theta<1}\big[\theta(1-\theta)r\big]^s\|D^su\|_{\Phi,\varphi;\theta r},\quad s=0,1,2.$$

Because of the choice of θ' we have $\theta(1-\theta)\le 2\theta'(1-\theta')$. Thus, after standard transformations and taking the supremum with respect to $\theta\in(0,1)$ the last inequality rewrites as

$$\Theta_2\le C\left(r^2\|\mathcal{L}u\|_{\Phi,\varphi;r}+\Theta_1+\Theta_0\right)\,. \qquad (9.5)$$

Lemma 9.2 (Interpolation inequality). *There exists a constant C independent of r such that*

$$\Theta_1\le\varepsilon\Theta_2+\frac{C}{\varepsilon}\Theta_0\quad \textit{for any } \varepsilon\in(0,2).$$

Proof. By simple scaling arguments we obtain in $M_{\Phi,\varphi}(\mathbb{R}^n)$ an interpolation inequality analogous to [17, Theorem 7.28]

$$\|Du\|_{\Phi,\varphi;r}\le\delta\|D^2u\|_{\Phi,\varphi;r}+\frac{C}{\delta}\|u\|_{\Phi,\varphi;r},\quad \delta\in(0,r)\,.$$

We can always find some $\theta_0 \in (0,1)$ such that

$$\begin{aligned}\Theta_1 &\le 2[\theta_0(1-\theta_0)r]\|Du\|_{\Phi,\varphi;\theta_0 r}\\ &\le 2[\theta_0(1-\theta_0)r]\Big(\delta\|D^2u\|_{\Phi,\varphi;\theta_0 r}+\frac{C}{\delta}\|u\|_{\Phi,\varphi;\theta_0 r}\Big).\end{aligned}$$

The assertion follows choosing $\delta = \frac{\varepsilon}{2}[\theta_0(1-\theta_0)r] < \theta_0 r$ for any $\varepsilon \in (0,2)$. □

Interpolating Θ_1 in (9.5), we obtain

$$\frac{r^2}{4}\|D^2u\|_{\Phi,\varphi;r/2} \le \Theta_2 \le C\left(r^2\|\mathcal{L}u\|_{\Phi,\varphi;r}+\|u\|_{\Phi,\varphi;r}\right)$$

and hence the Caccioppoli-type estimate

$$\|D^2u\|_{\Phi,\varphi;r/2} \le C\Big(\|\mathcal{L}u\|_{\Phi,\varphi;r}+\frac{1}{r^2}\|u\|_{\Phi,\varphi;r}\Big). \tag{9.6}$$

Let $\mathbf{v} = \{v_{ij}\}_{ij=1}^n \in [L_{\Phi,\omega}(\mathcal{B}_r)]^{n^2}$ be arbitrary function matrix. Define the operators

$$\mathcal{S}_{ijhk}(v_{hk})(x) = \mathfrak{C}_{ij}[a^{hk}, v_{hk}](x) \quad i,j,h,k = 1,\dots,n.$$

Because of the VMO properties of a^{ij}'s we can choose r so small that

$$\sum_{i,j,h,k=1}^{n}\|\mathcal{S}_{ijhk}\| < 1. \tag{9.7}$$

Now for a given $u \in W_{2,\Phi}(\mathcal{B}_r)\cap W_{1,\Phi}^0(\mathcal{B}_r)$ with $\mathcal{L}u \in M_{\Phi,\varphi}(\mathcal{B}_r)$ define

$$\mathcal{H}_{ij}(x) = \mathfrak{K}_{ij}\mathcal{L}u(x) + \mathcal{L}u(x)\int_{\mathbb{S}^{n-1}}\Gamma_j(x;y)y_i d\sigma_y$$

and (8.2) implies $\mathcal{H}_{ij} \in M_{\Phi,\varphi}(\mathcal{B}_r)$. Define the operator $\mathcal{W}$ by the setting

$$\mathcal{W}\mathbf{v} = \Big\{\sum_{h,k=1}^{n}\big(\mathcal{S}_{ijhk}v_{hk}+\mathcal{H}_{ij}(x)\big)\Big\}_{ij=1}^{n} : \big[M_{\Phi,\varphi}(\mathcal{B}_r)\big]^{n^2} \to \big[M_{\Phi,\varphi}(\mathcal{B}_r)\big]^{n^2}.$$

By (9.7) the operator $\mathcal{W}$ is a contraction mapping and there exists a unique fixed point $\widetilde{\mathbf{v}} = \{\widetilde{v}_{ij}\}_{ij=1}^n \in [M_{\Phi,\varphi}(\mathcal{B}_r)]^{n^2}$ of $\mathcal{W}$ such that $\mathcal{W}\widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}$. On the other hand it follows from the representation formula (9.3) that also $D^2u = \{D_{ij}u\}_{ij=1}^n$ is a fixed point of $\mathcal{W}$. Hence $D^2u \equiv \widetilde{\mathbf{v}}$, that is $D_{ij}u \in M_{\Phi,\varphi}(\mathcal{B}_r)$ and in addition (9.6) holds. The interior estimate (9.2) follows from (9.6) by a finite covering of Ω' with balls $\mathcal{B}_{r/2}$, $r < \operatorname{dist}(\Omega', \partial\Omega'')$. □

To prove a local boundary estimate for the norm of $D_{ij}u$ we define the space $W_{2,\Phi}^{\gamma_0}(\mathcal{B}_r^+)$ as a closure of $C_{\gamma_0} = \{u \in C_0^\infty(\mathcal{B}(x^0,r)) : u(x) = 0 \text{ for } x_n \le 0\}$ with respect to the norm of $W_{2,p}$.

Theorem 9.3 (Boundary estimate). *Let $u \in W_{2,\Phi}^{\gamma_0}(\mathcal{B}_r^+)$ and suppose that $\mathcal{L}u \in M_{\Phi,\varphi}(\mathcal{B}_r^+)$ with $\Phi \in \Delta_2 \cap \nabla_2$ and φ satisfying* (8.1). *Then $D_{ij}u \in M_{\Phi,\varphi}(\mathcal{B}_r^+)$ and for each $\varepsilon > 0$ there exists $r_0(\varepsilon)$ such that*

$$\|D_{ij}u\|_{\Phi,\varphi;\mathcal{B}_r^+} \le C\|\mathcal{L}u\|_{\Phi,\varphi;\mathcal{B}_r^+}, \quad \forall r \in (0,r_0). \tag{9.8}$$

Proof. For $u \in W^{\gamma_0}_{2,\Phi}(\mathcal{B}^+_r)$ the boundary representation formula holds (see [10])

$$\begin{aligned} D_{ij}u(x) = \mathrm{P.V.}\int_{\mathcal{B}^+_r} \Gamma_{ij}(x, x-y)\mathcal{L}u(y)dy \\ + \mathrm{P.V.}\int_{\mathcal{B}^+_r} \Gamma_{ij}(x, x-y)\big[a^{hk}(x) - a^{hk}(y)\big] D_{hk}u(y)dy \\ + \mathcal{L}u(x)\int_{\mathbb{S}^{n-1}} \Gamma_j(x,y)y_i d\sigma_y + I_{ij}(x), \quad \forall i,j = 1,\dots,n, \end{aligned} \tag{9.9}$$

where we have set

$$\begin{aligned} I_{ij}(x) &= \int_{\mathcal{B}^+_r} \Gamma_{ij}(x, \mathcal{T}(x)-y)\mathcal{L}u(y)dy \\ &\quad + \int_{\mathcal{B}^+_r} \Gamma_{ij}(x, \mathcal{T}(x)-y)\big[a^{hk}(x) - a^{hk}(y)\big] D_{hk}u(y)dy \\ &\forall i,j = 1,\dots,n-1, \\ I_{in}(x) &= I_{ni}(x) \\ &= \int_{\mathcal{B}^+_r} \Gamma_{il}(x, \mathcal{T}(x)-y)(D_n\mathcal{T}(x))^l \\ &\quad \times \big\{\big[a^{hk}(x) - a^{hk}(y)\big] D_{hk}u(y) + \mathcal{L}u(y)\big\}\, dy, \quad \forall i = 1,\dots,n-1, \\ I_{nn}(x) &= \int_{\mathcal{B}^+_r} \Gamma_{ls}(x, \mathcal{T}(x)-y)(D_n\mathcal{T}(x))^l (D_n\mathcal{T}(x))^s \\ &\quad \times \big\{\big[a^{hk}(x) - a^{hk}(y)\big] D_{hk}u(y) + \mathcal{L}u(y)\big\}\, dy, \end{aligned}$$

where $D_n\mathcal{T}(x) = \big((D_n\mathcal{T}(x))^1, \dots, (D_n\mathcal{T}(x))^n\big) = \mathcal{T}(e_n, x)$. Applying estimates (8.4) and (8.5), taking into account the VMO properties of the coefficients a^{ij}'s, it is possible to choose r_0 so small that

$$\|D_{ij}u\|_{p,\varphi;\mathcal{B}^+_r} \le C\|\mathcal{L}u\|_{p,\varphi;\mathcal{B}^+_r} \quad \text{for each } r < r_0.$$

For an arbitrary function matrix $\mathbf{w} = \{w_{ij}\}^n_{ij=1} \in [M_{\Phi,\varphi}(\mathcal{B}^+_r)]^{n^2}$ define

$$\begin{gathered} \mathcal{S}_{ijhk}(w_{hk})(x) = \mathfrak{C}_{ij}[a^{hk}, w_{hk}](x), \quad i,j,h,l = 1,\dots,n, \\ \widetilde{\mathcal{S}}_{ijhk}(w_{hk})(x) = \widetilde{\mathfrak{C}}_{ij}[a^{hk}, w_{hk}](x), \quad i,j = 1,\dots,n-1;\ h,k = 1,\dots,n, \\ \widetilde{\mathcal{S}}_{inhk}(w_{hk})(x) = \widetilde{\mathfrak{C}}_{il}[a^{hk}, w_{hk}](D_n\mathcal{T}(x))^l, \quad i,h,k = 1,\dots,n, \\ \widetilde{\mathcal{S}}_{nnhk}(w_{hk})(x) = \widetilde{\mathfrak{C}}_{ls}[a^{hk}, w_{hk}](x)(D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s, \quad h,k = 1,\dots,n. \end{gathered}$$

Because of (8.3) and (8.5) we can take r so small that

$$\sum_{i,j,h,k=1}^{n} \|\mathcal{S}_{ijhk} + \widetilde{\mathcal{S}}_{ijhk}\| < 1. \tag{9.10}$$

Now, given $u \in W^{\gamma_0}_{2,p}(\mathcal{B}^+_r)$ with $\mathcal{L}u \in M_{\Phi,\varphi}(\mathcal{B}^+_r)$ we set

$$\begin{aligned} \widetilde{\mathcal{H}}_{ij}(x) = \mathfrak{K}_{ij}\mathcal{L}u(x) + \widetilde{\mathfrak{K}}_{ij}\mathcal{L}u(x) + \widetilde{\mathfrak{K}}_{il}\mathcal{L}u(x)(D_n\mathcal{T}(x))^l \\ + \widetilde{\mathfrak{K}}_{ls}\mathcal{L}u(x)(D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s + \mathcal{L}u(x)\int_{\mathbb{S}^{n-1}} \Gamma_j(x,y)y_i d\sigma_y \end{aligned}$$

and the Theorems 8.2 and 8.7 imply $\widetilde{\mathcal{H}}_{ij} \in M_{\Phi,\varphi}(\mathcal{B}_r^+)$. Define the operator

$$\mathcal{U}\mathbf{w} = \Big\{ \sum_{h,k=1}^{n} \big(\mathcal{S}_{ijhk}(w_{hk}) + \widetilde{\mathcal{S}}_{ijhk}(w_{hk})\big)\big) + \widetilde{\mathcal{H}}_{ij}(x) \Big\}_{ij=1}^{n}.$$

By (9.10) it is a contraction mapping in $\big[M_{\Phi,\varphi}(\mathcal{B}_r^+)\big]^{n^2}$ and there is unique fixed point $\widetilde{\mathbf{w}} = \{\widetilde{w}_{ij}\}_{ij=1}^n$ such that $\mathcal{U}\widetilde{\mathbf{w}} = \widetilde{\mathbf{w}}$. On the other hand, it follows from the representation formula (9.9) that also $D^2u = \{D_{ij}u\}_{ij=1}^n$ is a fixed point of $\mathcal{U}$. Hence $D^2u \equiv \widetilde{\mathbf{w}}$, $D_{ij}u \in M_{\Phi,\varphi}(\mathcal{B}_r^+)$ and the estimate (9.8) holds. □

Theorem 9.4. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $\mathcal{L}$ be uniformly elliptic operator satisfying conditions H_1) and H_2). Then for any function $f \in M_{\Phi,\varphi}(\Omega)$ the unique solution of the problem* (9.1) *has second derivatives in $M_{\Phi,\varphi}(\Omega)$. Moreover*

$$\|D^2u\|_{M_{\Phi,\varphi}(\Omega)} \le C\big(\|u\|_{M_{\Phi,\varphi}(\Omega)} + \|f\|_{M_{\Phi,\varphi}(\Omega)}\big) \tag{9.11}$$

and the constant C depends on known quantities only.

Proof. Since $M_{\Phi,\varphi}(\Omega) \subset L_\Phi(\Omega)$, problem (9.1) is uniquely solvable in the Sobolev space $W_{2,\Phi}(\Omega) \cap W^0_{1,\Phi}(\Omega)$ according to [10]. By local flattering of the boundary, covering with semi-balls, taking a partition of unity subordinated to that covering and applying of estimate (9.8) we obtain a boundary a priori estimate that unified with (9.2) ensures validity of (9.11). □

Acknowledgments. The authors are grateful to Professor Vicentiu Radulescu for his valuable comments.

V. S. Guliyev and M. Omarova were partially supported by the 1st Azerbaijan-Russia Joint Grant Competition (Agreement number No. 18-51-06005), and by a grant from the Presidium of Azerbaijan National Academy of Science 2015.

References

[1] P. Acquistapace; *On BMO regularity for linear elliptic systems*, Ann. Mat. Pura. Appl., **161** (1992), 231–270.

[2] A. Akbulut, V. S. Guliyev, R. Mustafayev; *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (1) (2012), 27-43.

[3] C. Bennett, R. Sharpley; *Interpolation of operators*, Academic Press, Boston, 1988.

[4] G. Bonanno, G. Molica Bisci, V. Radulescu; *Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz-Sobolev spaces*, C. R. Math. Acad. Sci. Paris, **349** (5-6) (2011), 263-268.

[5] G. Bonanno, G. Molica Bisci, V. D. Radulescu; *Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz-Sobolev spaces*, Nonlinear Anal., **75** (12) (2012), 4441-4456.

[6] V. Burenkov, A. Gogatishvili, V. S. Guliyev, R. Mustafayev; *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ., **55** (8-10) (2010), 739-758.

[7] S. Campanato; *Sistemi ellitici in forma divergenza. Regolaritá all'interno*, Quaderni di Scuola Norm. Sup. Pisa (1980).

[8] F. Chiarenza, M. Frasca; *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat., **7** (1987), 273-279.

[9] F. Chiarenza, M. Frasca, P. Longo; *Interior $W^{2,p}$-estimates for nondivergence ellipic equations with discontinuous coefficients*, Ricerche Mat., **40** (1991), 149-168.

[10] F. Chiarenza, M. Frasca, P. Longo; *$W^{2,p}$-solvability of Dirichlet problem for nondivergence ellipic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841-853.

[11] A. Eroglu; *Fractional oscillatory integral operators and their commutators on generalized Orlicz-Morrey spaces of the third kind*, Eurasian Math. J., **8** (2017), no. 3, 36-47.

[12] A. Eroglu, J. V. Azizov; *A note on the fractional integral operators in generalized Morrey spaces on the Heisenberg group*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **37** (2017), no. 1, Mathematics, 86-91.

[13] A. Eroglu, V.S. Guliyev, J.V. Azizov; *Characterizations of fractional integral operators in generalized Morrey spaces on Carnot groups*, Math. Notes, **102** (2017), no. 5-6, 722-734

[14] F. Deringoz, V.S. Guliyev, S. Samko; *Boundedness of maximal and singular operators on generalized Orlicz-Morrey spaces*, Operator Theory, Operator Algebras and Applications. Series: Operator Theory: Advances and Applications, vol. **242** (2014), 139-158.

[15] G. Di Fazio, D.K. Palagachev, M.A. Ragusa; *Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients*, J. Funct. Anal. **166** (2) (1999), 179-196.

[16] D. Fan, S. Lu, D. Yang; *Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients*, Georgian Math. J., **5** (1998), 425-440.

[17] G. Gilbert, N. S. Trudinger; *Elliptic partial differential equations of second order*, 2nd ed. Springer-Verlag, 1983.

[18] V. S. Guliyev; *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian)

[19] V.S. Guliyev; *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl., Art. ID 503948, (2009), 20 pp.

[20] V. S. Guliyev, J. J. Hasanov, S. G. Samko; *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces*, Math. Scand., **107** (2) (2010), 285-304.

[21] V. S. Guliyev, S. S. Aliyev, T. Karaman, P. Shukurov; *Boundedness of sublinear operators and commutators on generalized Morrey spaces*, Integral Equations and Operator Theory, **71** (3) (2011), 327-355.

[22] V. S. Guliyev; *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math., **3** (2) (2013), 79-94.

[23] V. S. Guliyev; *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.). **193** (2) (2013), 211-227.

[24] V. S. Guliyev, F. Deringoz; *On the Riesz potential and its commutators on generalized Orlicz-Morrey spaces*, J. Funct. Spaces Appl., vol. **2014**, Article ID 617414, (2014), 11 pages.

[25] V. S. Guliyev, L. Softova; *Global regularity in generalized Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients*, Potential Anal. **38** (4) (2013), 843-862.

[26] V. S. Guliyev, L. Softova; *Generalized Morrey regularity for parabolic equations with discontinuity data*, Proc. Edinb. Math. Soc., **58** (2) (2015), no. 1, 199-218.

[27] J. J. Hasanov; *Φ-admissible sublinear singular operators and generalized Orlicz-Morrey spaces*, J. Funct. Spaces, Art. ID 505237, (2014), 7 pp.

[28] M. Izuki, Y. Sawano; *Variable Lebesgue norm estimates for BMO functions*, Czechoslovak Math. J., **62** (137) (2012), no. 3, 717-727.

[29] V. Kokilashvili, M. M. Krbec; *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, Singapore, 1991.

[30] S. Janson; *Mean oscillation and commutators of singular integral operators*, Ark. Mat., **16** (1978), 263-270.

[31] F. John, L. Nirenberg; *On functions of bounded mean osillation*, Comm. Pure Appl. Math., **14** (1961), 415-426.

[32] P. W. Jones; *Extension theorems for BMO*, Indiana Univ. Math. J. 29 (1980), 41-66.

[33] M. A. Krasnoselskii, Ya. B. Rutickii; *Convex functions and Orlicz spaces*, English translation P. Noordhoff Ltd., Groningen, 1961.

[34] PeiDe Liu, YouLiang Hou, and MaoFa Wang; *Weak Orlicz space and its applications to the martingale theory*, Science China Mathematics **53** (4) (2010), 905-916.

[35] M. Mihailescu, V. Radulescu, D. Repovs; *On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting*, J. Math. Pures Appl., (9) **93** (2010), no. 2, 132-148.

[36] T. Mizuhara; *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Anal., Proc. Conf., Sendai/Jap. 1990, ICM-90 Satell. Conf. Proc., (1991) 183-189.
[37] C. B. Morrey; *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), 126-166.
[38] E. Nakai; *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr., **166** (1994), 95-103.
[39] E. Nakai; *Calderón-Zygmund operators on Orlicz-Morrey spaces and modular inequalities*, Banach and function spaces II, 393-410, Yokohama Publ., Yokohama, 2008.
[40] D. Palagachev, L. Softova; *Fine regularity for elliptic systems with discontinuous ingredients*, J. Arch. Math., **86** (2) (2006), 145-153.
[41] J. Peetre; *On the theory of* $L_{p,\lambda}$, J. Funct. Anal. **4** (1969), 71-87.
[42] L. E. Persson, N. Samko; *Weighted Hardy and potential operators in the generalized Morrey spaces*, J. Math. Anal. Appl., **377** (2) (2011), 792-806.
[43] L. C. Piccinini; *Inclusioni tra spazi di Morrey*, Boll. Unione Mat. Ital., IV. Ser. **2** (1969), 95-99.
[44] V. D. Radulescu, D. D. Repovs; *Partial differential equations with variable exponents. Variational methods and qualitative analysis*, Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2015.
[45] M. M. Rao, Z. D. Ren; *Theory of Orlicz Spaces*, M. Dekker, Inc., New York, 1991.
[46] D. Sarason; *On functions of vanishes mean oscillation*, Trans. Amer. Math. Soc., **207** (1975), 391-405.
[47] Y. Sawano; *A Handbook of Harmonic Analysis*, Tokyo, 2011.
[48] L. G. Softova; *Singular integrals and commutators in generalized Morrey spaces*, Acta Math. Sin., Engl. Ser., **22** (2006), 757-766.
[49] L. G. Softova; *The Dirichlet problem for elliptic equations with VMO coefficients in generalized Morrey spaces*, Advances in harmonic analysis and operator theory, 371-386, Oper. Theory Adv. Appl., 229, Birkhäuser/Springer Basel AG, Basel, 2013.
[50] G. Weiss; *A note on Orlicz spaces*, Portugal Math., **15** (1956), 35-47.
[51] J. Xiao; *A new perspective on the Riesz potential*, Adv. Nonlinear Anal., **6** (2017), no. 3, 317-326.

Vagif S. Guliyev
Ahi Evran University, Department of Mathematics, 40100 Kirsehir, Turkey.
S.M. Nikol'skii Institute of Mathematics at RUDN University, Moscow, 117198, Russia.
Institute of Mathematics and Mechanics, Az 1141 Baku, Azerbaijan
E-mail address: vagif@guliyev.com

Aysel A. Ahmadli
Dumlupinar University, Department of Mathematics, 40100 Kytahya, Turkey
E-mail address: aysel.ahmadli@gmail.com

Mehriban N. Omarova
Baku State University, AZ1141 Baku, Azerbaijan.
Institute of Mathematics and Mechanics, Az 1141 Baku, Azerbaijan
E-mail address: mehribanomarova@yahoo.com

Lubomira G. Softova
Department of Mathematics, University of Salerno, Fisciano, Italy
E-mail address: lsoftova@unisa.it