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Parametric Marcinkiewicz integral operator and its higher order commutators on generalized weighted Morrey spaces

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Abstract. *In this paper, we study the boundedness of parametric Marcinkiewicz integral operator and its higher order commutator with rough kernels on generalized weighted Morrey spaces.*

Keywords. Parametric Marcinkiewicz integrals, generalized weighted Morrey spaces, higher order commutator.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35.

1 Introduction

Suppose that S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

(iii) $\Omega \in L^1(S^{n-1})$.

The parametric Marcinkiewicz integral is defined by

$$\mu^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

where $0 < \rho < n$. When $\rho = 1$, we simply denote it by $\mu(f)$. It is well-known that the operator $\mu(f)$ is defined by Stein in [13].

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For $m \in \mathbb{N}$, $b \in BMO(\mathbb{R}^n)$, the higher-order commutator of parametric Marcinkiewicz integral is defined as follows

$$\mu^{\rho}_{b^m}(f)(x) = \left(\int_0^{\infty} \left| \frac{1}{t^{\rho}} \int_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} (b(x)-b(y))^m f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ were introduced by Morrey [10] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Mizuhara [9] introduced generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ (see, also [11,4]); Komori and Shirai [8] defined the weighted Morrey spaces $L^{p,\kappa}(\omega)$; Guliyev [3] gave a concept of generalized weighted Morrey space $\mathcal{M}^p_{\varphi}(\mathbb{R}^n, w)$ which could be viewed as extension of both $L^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}(\omega)$.

Let $1 \le p < \infty$ and let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and let w be a non-negative measurable function on $\mathbb{R}^n$. Following [3], we denote the generalized weighted Morrey space $\mathcal{M}^p_{\varphi}(\mathbb{R}^n, w)$, the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}^p_{\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L^p(B(x,r),w)},$$

where

$$\|f\|_{L^p(B(x,r),w)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy \right)^{\frac{1}{p}}.$$

Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x.

In this paper, we consider the boundedness of parametric Marcinkiewicz integral operator and its higher order commutator with rough kernels on generalized weighted Morrey spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Background materials

Even though the A_p class is well-known, for completeness, we offer the definition of A_p weight functions.

Definition 2.1 *For, $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if*

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{p'}{p}} dx \right)^{\frac{p'}{p}} < \infty, \tag{2.1}$$

where the supremum is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. A locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B w(y)dy \le Cw(x), \qquad a.e\ \ x \in B$$

for some constant $C > 0$. We define $A_{\infty} = \bigcup_{p\ge1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)dx$.
For any $w \in A_p$, by (2.1) we have

$$\left(w^{-\frac{p'}{p}}(B)\right)^{1/p'} = \|w^{-\frac{1}{p}}\|_{L^{p'}(B)} \le C|B|\,(w(B))^{-1/p} \tag{2.2}$$

and

$$w^{1-p'} \in A_{p'}. \tag{2.3}$$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 2.2 *Suppose that* $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|dy,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \;:\; \|b\|_* < 1\}.$$

The following results concerning the boundedness of Marcinkiewicz integral and its higher-order commutator on weighted L^p space are known.

Theorem 2.1 *[12] Suppose that* $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Then, for every* $q' < p < \infty$ *and* $w \in A_{p/q'}$, *there is a constant* C *independent of* f *such that*

$$\|\mu^\rho(f)\|_{L^p(\mathbb{R}^n,w)} \le C\|f\|_{L^p(\mathbb{R}^n,w)}.$$

Theorem 2.2 *[12] Suppose that* $b \in BMO(\mathbb{R}^n)$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Then, for every* $q' < p < \infty$ *and* $w \in A_{p/q'}$, *there is a constant* C *independent of* f *such that*

$$\|\mu^\rho_{b^m}(f)\|_{L^p(\mathbb{R}^n,w)} \le C\|f\|_{L^p(\mathbb{R}^n,w)}.$$

In the next sections where we prove our main estimates, we use the following lemma.

Lemma 2.1 *[3]*
i) Let $w \in A_\infty$ *and* $b \in BMO(\mathbb{R}^n)$. *Let also* $1 \le p < \infty$, $x \in \mathbb{R}^n$, $m > 0$ *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{mp} w(y)dy\right)^{1/p} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)^m \|b\|_*^m,$$

where C *is independent of* $f,\ w,\ x,\ r_1,\ r_2$ *and* $b_{B(x,r_2),w} = \frac{1}{w(B(x,r_2))}\int_{B(x,r)} b(y)w(y)dy$.

ii) Let $w \in A_p$ *and* $b \in BMO(\mathbb{R}^n)$. *Let also* $1 < p < \infty$, $x \in \mathbb{R}^n$, $m > 0$ *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{mp'} w^{1-p'}(y)dy\right)^{1/p'}$$
$$\le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)^m \|b\|_*^m,$$

where C *is independent of* $f,\ w,\ x,\ r_1,\ r_2$.

3 Local Guliyev estimates

Inspiring by the ideas of [3] (see, also [7]) and [2] we prove the following local estimates for the operators μ^ρ and $\mu^\rho_{b^m}$.

Lemma 3.1 *Suppose that* $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Then, for every* $q' < p < \infty$ *and* $w \in A_{p/q'}$, *there is a constant* C *independent of* f *such that*

$$\|\mu^\rho(f)\|_{L^p(B(x_0,r),w)}$$
$$\leq C\,(w(B(x_0,r)))^{1/p}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t),w)}\,(w(B(x_0,t)))^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{3.1}$$

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ and $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{^c(2B)}(y), \quad r > 0,$$

and have

$$\|\mu^\rho(f)\|_{L^p(B,w)} \leq \|\mu^\rho\,(f_1)\,\|_{L^p(B,w)} + \|\mu^\rho\,(f_2)\,\|_{L^p(B,w)}.$$

Since $f_1 \in L^p(\mathbb{R}^n,w)$ and from the boundedness of μ^ρ in $L^p(\mathbb{R}^n,w)$ (Theorem 2.1) it follows that

$$\|\mu^\rho\,(f_1)\,\|_{L^p(B,w)} \leq \|\mu^\rho\,(f_1)\,\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f_1\|_{L^p(\mathbb{R}^n,w)} = \|f\|_{L^p(2B,w)}.$$

By using Hölder's inequality at (2.1), we have

$$|B| \lesssim (w(B))^{1/p}\,\|w^{-\frac{1}{p}}\|_{L^{p'}(B)}.$$

Then, for $q' < p < \infty$,

$$\begin{aligned}
\|\mu^\rho\,(f_1)\,\|_{L^p(B,w)} &\lesssim |B|\|f\|_{L^p(2B,w)}\int_{2r}^{\infty}\frac{dt}{t^{n+1}} \\
&\lesssim |B|\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t),w)}\,\frac{dt}{t^{n+1}} \\
&\lesssim (w(B))^{1/p}\,\|w^{-\frac{1}{p}}\|_{L^{p'}(B)}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t),w)}\,\frac{dt}{t^{n+1}} \\
&\lesssim (w(B))^{1/p}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t),w)}\,\|w^{-\frac{1}{p}}\|_{L^{p'}(B(x_0,t))}\,\frac{dt}{t^{n+1}}.
\end{aligned}$$

By (2.2), we get

$$\|\mu^\rho\,(f_1)\,\|_{L^p(B,w)} \lesssim w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t),w)}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{3.2}$$

Note that, using spherical coordinates we have

$$\|\Omega(x-\cdot)\|_{L^q(B(x_0,t))} \lesssim \|\Omega\|_{L^q(S^{n-1})}\,|B(0,t+|x-x_0|)|^{\frac{1}{q}}. \tag{3.3}$$

It's clear that $x \in B$, $y \in^c (2B)$ implies $\frac{1}{2}|x_0 - y| \leq |x-y| \leq \frac{3}{2}|x_0 - y|$. Then by the Minkowski inequality, we get

$$\begin{aligned}
|\mu^\rho(f_2)(x)| &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |f_2(y)| \left(\int_{|x-y|}^{\infty} \frac{dt}{t^{1+2\rho}} \right)^{1/2} dy \\
&\lesssim \int_{^c(2B)} \frac{|f(y)||\Omega(x-y)|}{|x-y|^n} dy \\
&\lesssim \int_{^c(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^n} dy.
\end{aligned} \tag{3.4}$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{^c(2B)} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy &\approx \int_{^c(2B)} |\Omega(x-y)|\,|f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx \int_{2r}^{\infty} \int_{2r \leq |x_0-y| < t} |\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned} \tag{3.5}$$

Applying Hölder's inequality and (3.3), we get

$$\begin{aligned}
&\int_{^c(2B)} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n} dy \\
&\lesssim \|\Omega\|_{L^q(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L^{q'}(B(x_0,t))} |B(0, t+|x-x_0|)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}.
\end{aligned} \tag{3.6}$$

Note that for $t > 2r$ and $|x - x_0| < r$ we have $t + |x - x_0| < t + r < \frac{3}{2}t$. Since $q' < p < \infty$, $v = \frac{p}{q'} > 1$ and $w \in A_v$, from the Hölder's inequality and (2.2) we get that

$$\begin{aligned}
\int_{^c(2B)} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)} \, \|w^{-\frac{1}{v}}\|_{L^{v'}(B(x_0,t))}^{\frac{1}{q'}} \, t^{\frac{n}{q}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)} \, w(B(x_0,t))^{-\frac{1}{p}} \, |B(x_0,t)|^{\frac{1}{q'}} \, t^{\frac{n}{q}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)} w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}.
\end{aligned}$$

Therefore

$$|\mu^\rho(f_2)(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}. \tag{3.7}$$

Moreover, for all $p \in (1,\infty)$, the inequality

$$\|\mu^\rho(f_2)\|_{L^p(B,w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t} \tag{3.8}$$

holds. Combining (3.2) and (3.8), the proof of Lemma 3.1 is completed.

Lemma 3.2 *Suppose that $b \in BMO(\mathbb{R}^n)$, $m \in \mathbb{N}$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ satisfying satisfying* (1.1)-(1.2) *and $0 < \rho < n$. Then, for every $q' < p < \infty$ and $w \in A_{p/q'}$, there is a constant C independent of f such that*

$$\|\mu^{\rho}_{b^m}(f)\|_{L^p(B(x_0,r),w)} \\ \le C\|b\|_*^m \left(w(B(x_0,r))\right)^{1/p} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^m \frac{\|f\|_{L^p(B(x_0,t),w)}}{(w(B(x_0,t)))^{\frac{1}{p}}}\frac{dt}{t}. \tag{3.9}$$

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{c_{(2B)}}(y), \quad r > 0,$$

and have

$$\|\mu^{\rho}_{b^m}(f)\|_{L^p(B,w)} \le \|\mu^{\rho}_{b^m}(f_1)\|_{L^p(B,w)} + \|\mu^{\rho}_{b^m}(f_2)\|_{L^p(B,w)}.$$

Since $f_1 \in L^p(\mathbb{R}^n,w)$ and from the boundedness of $\mu^{\rho}_{b^m}$ in $L^p(\mathbb{R}^n,w)$ (Theorem 2.2) it follows that

$$\|\mu^{\rho}_{b^m}(f_1)\|_{L^p(B,w)} \le \|\mu^{\rho}_{b^m}(f_1)\|_{L^p(\mathbb{R}^n,w)} \le C\|b\|_*^m\|f_1\|_{L^p(\mathbb{R}^n,w)} = C\|b\|_*^m\|f\|_{L^p(2B,w)}.$$

As the proof of (3.2), we get

$$\|\mu^{\rho}_{b^m}(f_1)\|_{L^p(B,w)} \le C\|b\|_*^m w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}. \tag{3.10}$$

We now turn to deal with the term $\|\mu^{\rho}_{b^m}(f_2)\|_{L^p(B,w)}$. For any given $x \in B$, we have

$$\begin{aligned}|\mu^{\rho}_{b^m}(f_2)| &\le C|(b(x)-b_{B,w})^m||\mu^{\rho}(f_2)(x)| + C|\mu^{\rho}((b-b_{B,w})^m f_2)(x)| \\ &= I_1 + I_2.\end{aligned}$$

By (3.7), we have

$$I_1 \lesssim |(b(x)-b_{B,w})^m| \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}.$$

Then from Lemma 2.1 we get

$$\|I_1\|_{L^p(B,w)} \lesssim \|b\|_*^m w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}.$$

When $\Omega \in L^q(S^{n-1})$, it follows from (3.4), (3.5) and (3.6) that

$$I_2 \lesssim \int_{2r}^{\infty} \|(b-b_{B,w})^m f\|_{L^{q'}(B(x_0,t))} t^{\frac{n}{q}} \frac{dt}{t^{n+1}}.$$

Set $v = \frac{p}{q'} > 1$. Since $w \in A_v$, from (2.3), we know $w^{1-v'} \in A_{v'}$. By Hölder's inequality

$$I_2 \lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)} \|(b-b_{B,w})^m\|_{L^{v'q'}(B(x_0,t),w^{1-v'})}\, t^{\frac{n}{q}} \frac{dt}{t^{n+1}}.$$

Since $w^{1-v'} \in A_{v'}$, from (2.2), we know

$$\Big(w^{1-v'}(B(x_0,t))\Big)^{\frac{1}{v'q'}} \le Ct^{\frac{n}{q'}}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}. \tag{3.11}$$

Using (3.11) and Lemma 2.1, we obtain

$$\begin{aligned}\|(b-b_{B,w})^m\|_{L^{v'q'}(B(x_0,t),w^{1-v'})} &= \left(\int_{B(x_0,t)} |b(y)-b_{B,w}|^{mv'q'} w^{1-v'}(y)dy\right)^{\frac{1}{v'q'}} \\ &\lesssim \|b\|_*^m \Big(1+\ln\frac{t}{r}\Big)^m \Big(w^{1-v'}(B(x_0,t))\Big)^{\frac{1}{v'q'}} \\ &\lesssim \|b\|_*^m \Big(1+\ln\frac{t}{r}\Big)^m t^{\frac{n}{q'}} \left(w(B(x_0,t))\right)^{-\frac{1}{p}}.\end{aligned}$$

Hence

$$I_2 \lesssim \|b\|_*^m \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^m \|f\|_{L^p(B(x_0,t),w)} \left(w(B(x_0,t))\right)^{-\frac{1}{p}} \frac{dt}{t}.$$

Therefore,

$$\|I_2\|_{L^p(B,w)} \lesssim \|b\|_*^m w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \tag{3.12}$$

Combining (3.10) and (3.12), the proof of Lemma 3.2 is completed.

Remark 3.1 For the case $\rho = 1$ and $m = 1$ the local estimate (3.9) was proved in [6, Lemma 5.2]. But there are some gaps in that proof. We also fill the gaps of proof of [6, Lemma 5.2] in the proof of Lemma 3.2.

4 Main results

Theorem 4.1 *Let* $1 < p < \infty$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Let also* $w \in A_{p/q'}$ *with* $q' < p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{w\big(B(x,t)\big)^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{4.1}$$

where C does not depend on x and r. Then the operator μ^ρ is bounded from $\mathcal{M}^p_{\varphi_1}(\mathbb{R}^n, w)$ to $\mathcal{M}^p_{\varphi_2}(\mathbb{R}^n, w)$.

Proof. The proof follows from [5, Theorem 3.1] and Lemma 3.1. We can also give the following alternative proof for Theorem 4.1.

Since $f \in \mathcal{M}^p_{\varphi_1}(\mathbb{R}^n, w)$ and the fact $\|f\|_{L^p(B(x,t),w)}$ is a non-decreasing function of t, we get

$$\begin{aligned}\frac{\|f\|_{L^p(B(x,t),w)}}{\operatorname*{ess\,inf}_{0<t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} &\le \operatorname*{ess\,sup}_{0<t<s<\infty} \frac{\|f\|_{L^p(B(x,t),w)}}{\varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} \\ &\le \sup_{s>0,\ x\in\mathbb{R}^n} \frac{\|f\|_{L^p(w,B(x,s))}}{\varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} \\ &\le \|f\|_{\mathcal{M}^p_{\varphi_1}(\mathbb{R}^n,w)}.\end{aligned}$$

Since (φ_1, φ_2) satisfies (4.1), we have

$$\int_r^\infty \|f\|_{L^p(B(x,t),w)} \, (w(B(x,t)))^{-\frac{1}{p}} \, \frac{dt}{t}$$
$$\leq \int_r^\infty \frac{\|f\|_{L^p(B(x,t),w)}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{(w(B(x,t)))^{\frac{1}{p}}} \frac{dt}{t}$$
$$\leq \|f\|_{\mathcal{M}^p_{\varphi_1}(\mathbb{R}^n,w)} \int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{(w(B(x,t)))^{\frac{1}{p}}} \frac{dt}{t}$$
$$\lesssim \|f\|_{\mathcal{M}^p_{\varphi_1}(\mathbb{R}^n,w)} \varphi_2(x,r).$$

Then by (3.1) we get

$$\|\mu^\rho\|_{\mathcal{M}^p_{\varphi_2}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r)^{-1/p}) \|\mu^\rho\|_{L^p(B(x,r),w)}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L^p(B(x,t),w)} \, (w(B(x,t)))^{-\frac{1}{p}} \, \frac{dt}{t}$$
$$\lesssim \|f\|_{\mathcal{M}^p_{\varphi_1}(\mathbb{R}^n,w)}.$$

Theorem 4.2 *Let* $1 < p < \infty$*,* $b \in BMO(\mathbb{R}^n)$*,* $m \in \mathbb{N}$*,* $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$*. Let also* $w \in A_{p/q'}$ *with* $q' < p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)^m \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{w\big(B(x,t)\big)^{\frac{1}{p}}} \frac{dt}{t} \leq C \, \varphi_2(x,r),$$

where C *does not depend on* x *and* r*. Then the operator* $\mu^\rho_{b^m}$ *is bounded from* $\mathcal{M}^p_{\varphi_1}(\mathbb{R}^n, w)$ *to* $\mathcal{M}^p_{\varphi_2}(\mathbb{R}^n, w)$*.*

Proof. The proof of Theorem 4.2 is similar to the proof of Theorem 4.1.

Remark 4.1 In the case $w = 1$ and $m = 1$, Theorems 4.1 and 4.2 are proved in [1] for $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})\,(0 < \alpha \leq 1)$, respectively. Since $\mathrm{Lip}_\alpha(S^{n-1})\,(0 < \alpha \leq 1) \subsetneqq L^q(S^{n-1})\,(q > 1)$, our results are better than the results of [1].

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References

1. Aliyev, S.S., Guliyev, V.S.: *Boundedness of parametric Marcinkiewicz integral operator and their commutators on generalized Morrey spaces*, Georgian Math. J. **19** (2), 195-208(2012).
2. Cui, R., Li, Z.: *Boundedness of Marcinkiewicz integrals and their commutators on generalized weighted Morrey spaces*, J. Funct. Spaces, **2015**, Art. ID 450145 (2015), 9 pp.

3. Guliyev, V.S.: *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3**(3), 33-61(2012).
4. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl., **2009**, Art. ID 503948 (2009), 20 pp.
5. Guliyev, V.S.: *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N.Y.) **193**(2), 211-227 (2013).
6. Guliyev, V.S., Akbulut, A., Hamzayev V., Kuzu, O.: *Commutators of Marcinkiewicz integrals associated with Schrodinger operator on generalized weighted Morrey spaces*, J. Math. Inequal. **10**(4), 947-970(2016).
7. Guliyev, V.S., Hamzayev, V.: *Rough singular integral operators and its commutators on generalized weighted Morrey spaces*, Math. Inequal. Appl. **19**(3), 863-881(2016).
8. Komori, Y., Shirai, S.: *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282**(2), 219-231(2009).
9. Mizuhara, T.: *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo 183-189 (1991).
10. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**(1), 126-166 (1938).
11. Nakai, E.: *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, (1994), 95-103.
12. Shi, X., Jiang, Y.: *Weighted boundedness of parametric Marcinkiewicz integral and higher order commutator*, Anal. Theory Appl., **25**(1), 25-39 (2009).
13. Stein, E.M.: *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc., **88**, 430-466 (1958).