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Rough Fractional Multilinear Integral Operators on Generalized Weighted Morrey Spaces

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Abstract. In this paper, we study the boundedness of fractional multilinear integral operators with rough kernels $T^{A,m}_{\Omega,\alpha}$ on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1, φ_2) with $w \in A_p(\mathbb{R}^n)$ which ensures the boundedness of the operators $T^{A,m}_{\Omega,\alpha}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. In all cases the conditions for the boundedness of the operator $T^{A,m}_{\Omega,\alpha}$ is given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x, r)$, $\varphi_2(x, r)$ in r.

Key Words and Phrases: fractional multilinear integral, rough kernel, BMO; generalized weighted Morrey space.
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1. Introduction and results

The classical Morrey spaces were originally introduced by Morrey in [20] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7, 8, 11, 20, 23]. Mizuhara [19] introduced generalized Morrey spaces. Later, Guliyev [11] defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [18] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [12] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [12] Guliyev also studied the boundedness of the classical operators and its commutators in the spaces $M_{p,\varphi}(w)$ (see also Guliyev et al. [15, 16, 17]).

Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$ $(s > 1)$ is homogeneous of degree zero on $\mathbb{R}^n$ with zero means value on $\mathbb{S}^{n-1}$, A is a function defined on $\mathbb{R}^n$. Following [3], the rough fractional multilinear integral operator $T^{A,m}_{\Omega,\alpha}$, is defined by

$$T^{A,m}_{\Omega,\alpha}(f)(x) = \int_{\mathbb{R}^n} \frac{R_m(A; x, y)}{|x-y|^{n-\alpha+m-1}} \Omega(x-y) f(y) dy, \tag{1}$$

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where $0 < \alpha < n$, and $R_m(A;x,y)$ is the m-th remainder of Taylor series of A at x about y. More precisely

$$R_m(A;x,y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma. \tag{2}$$

When $m = 1$, then $T^A_{\Omega,\alpha} \equiv T^{A,1}_{\Omega,\alpha}$ is just the commutator of the fractional integral $T_{\Omega,\alpha} f(x)$ with a function A:

$$T^A_{\Omega,\alpha}(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (A(x) - A(y)) f(y) dy.$$

The weighted (L_p, L_q)-boundedness of such a commutator is given by Ding and Lu in [4]. When $m \geq 2$, $T^A_{\Omega,\alpha}$ is a non-trivial generalization of the above commutator. Wu and Yang in [27] proved the following results.

Theorem 1. *Let $m \geq 2$, $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$, $\omega^{s'} \in A(p/s', q/s')$ and A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$. Then there exists a constant C, independent of A and f, such that*

$$\|T^{A,m}_{\Omega,\alpha} f\|_{L_{q,\omega^q}(\mathbb{R}^n)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_{p,\omega^p}(\mathbb{R}^n)}. \tag{3}$$

Here, and in the sequel, we always denote by p' the conjugate index of any $p > 1$, that is $1/p + 1/p' = 1$, and by C a constant which is independent of the main parameters and may vary from line to line.

We define the generalized weighted Morrey spaces as follows.

Definition 1. *Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, i.e. the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y) dy\right)^{\frac{1}{p}},$$

and χ_B denotes the characteristic function of f in the set of B.

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy \right)^{\frac{1}{p}}.$$

The commutators are useful in many nondivergence elliptic equations with discontinuous coefficients, [5, 6, 7, 13, 14]. In the recent development of commutators, Pérez and Trujillo-González [24] generalized the multilinear commutators and proved the weighted Lebesgue estimates. Moreover, they showed that some classical integral operators and corresponding commutators are bounded in weighted Morrey spaces. Ye and Zhu in [26] obtained the boundedness of the multilinear commutators in weighted Morrey spaces $L_{p,\kappa}(w)$ for $1 < p < \infty$ and $0 < \kappa < 1$, where the symbol $\vec{b}$ belongs to bounded mean oscillation $(BMO)^n$. Furthermore, they established the weighted weak type estimate for these operators in weighted Morrey spaces of $L_{p,\kappa}(w)$ for $p = 1$ and $0 < \kappa < 1$.

It has been proved by many authors that most of the operators which are bounded on a weighted Lebesgue space are also bounded in an appropriate weighted Morrey space (see [2, 25]). As far as we know, there is no research regarding boundedness of the fractional multilinear integral operator on Morrey space. In this paper, we are going to prove that these results are valid for the rough fractional multilinear integral operator $T^A_{\Omega,\alpha}$ on generalized weighted Morrey space. Our main results can be formulated as follows.

Theorem 2. *Let $0 < \alpha < n$, $1 \le s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$, $\omega^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)(\omega^p(B(x,\tau)))^{\frac{1}{p}}}{(\omega^q(B(x,t)))^{\frac{1}{q}}} \frac{dt}{t} \le C_0\, \varphi_2(x,r), \tag{4}$$

where C_0 does not depend on x and r. If A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$, then the operator $T^{A,m}_{\Omega,\alpha}$ is bounded from $M_{p,\varphi_1}(\omega^p)$ to $M_{q,\varphi_2}(\omega^q)$. Moreover, there is a constant $C > 0$ independent of f such that

$$\|T^{A,m}_{\Omega,\alpha} f\|_{M_{q,\varphi_2}(\omega^q)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{M_{p,\varphi_1}(\omega^p)}.$$

In the case $\omega \equiv 1$ we get the following corollary proved in [1].

Corollary 1. *[1] Let $0 < \alpha < n$, $1 \le s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \le C_0\, \varphi_2(x,r), \tag{5}$$

where C_0 does not depend on x and r. If A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$, then the operator $T^{A,m}_{\Omega,\alpha}$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$. Moreover, there is a *constant $C > 0$ independent of f such that*

$$\|T^{A,m}_{\Omega,\alpha} f\|_{M_{q,\varphi_2}} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{M_{p,\varphi_1}}.$$

2. Some preliminaries

We begin with some properties of A_p weights which play a great role in the proofs of our main results.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x)dx$. For any given weight function ω on $\mathbb{R}^n$, $\Omega_0 \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_{p,\omega}(\Omega_0)$ the space of all functions f satisfying

$$\|f\|_{L_{p,w}(\Omega)} = \left(\int_{\Omega_0} |f(x)|^p w(x)dx \right)^{\frac{1}{p}} < \infty.$$

A weight w is said to belong to A_p for $1 < p < \infty$, if there exists a constant

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'}dx \right)^{p-1} \le C, \tag{6}$$

where s' is the dual of s, such that $\frac{1}{s} + \frac{1}{s'} = 1$. The class A_1 is defined by

$$\frac{1}{|B|} \int_B \omega(y)dy \le C \cdot \operatorname*{ess\,inf}_{x\in B} \omega(x) \quad \text{for every ball} \quad B \subset \mathbb{R}^n. \tag{7}$$

A weight ω is said to belong to $A_\infty(\mathbb{R}^n)$ if there are positive numbers C and δ such that

$$\frac{\omega(E)}{\omega(B)} \le C \Big(\frac{|E|}{|B|} \Big)^\delta$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty = \bigcup_{1 \le p < \infty} A_p.$$

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L_p-boundedness of Hardy-Littlewood maximal function in [21].

We also need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [22] to study the weighted boundedness of fractional integral operators.

Given $1 \le p \le q < \infty$, we will say that $\omega \in A_{p,q}$ if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$, the inequality

$$\Big(\frac{1}{|B|} \int_B \omega(y)^{-p'}dy \Big)^{1/p'} \Big(\frac{1}{|B|} \int_B \omega(y)^q dy \Big)^{1/q} \le C, \tag{8}$$

holds when $1 < p < \infty$, and for every ball $B \subset \mathbb{R}^n$ the inequality

$$\Big(\frac{1}{|B|}\int_B \omega(y)^q dy\Big)^{1/q} \le C \operatorname*{ess\,inf}_{x\in B} \omega(x), \tag{9}$$

holds when $p = 1$.

By (8), we have

$$\Big(\int_B \omega(y)^{-p'} dy\Big)^{1/p'} \Big(\int_B \omega(y)^q dy\Big)^{1/q} \le C|B|^{1/p'+1/q}. \tag{10}$$

We summarize some properties about weights $A_{p,q}$ (see [9, 22]).

Lemma 1. *Given $1 \le p \le q < \infty$, we have.*

(i) $\omega \in A_{p,q}$ if and only if $\omega^q \in A_{1+q/p'}$;
(ii) $\omega \in A_{p,q}$ if and only if $\omega^{-p'} \in A_{1+p'/q}$;
(iii) $\omega \in A_{p,p}$ if and only if $\omega^p \in A_p$;
(iV) If $p_1 < p_2$ and $q_2 > q_1$, then $A_{p_1,q_1} \subset A_{p_2,q_2}$.

In this paper, we need the following statement on the boundedness of the Hardy type operator

$$(H_1 g)(t) := \frac{1}{t}\int_0^t \ln\Big(e + \frac{t}{r}\Big) g(r) d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0, \infty)$.

The following lemma was proved in [12].

Lemma 2. *Let b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2)}|^p dy\right)^{\frac{1}{p}} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_*,$$

where $C > 0$ is independent of f, x, r_1 and r_2.

The following lemma was proved by Guliyev in [12].

Lemma 3 ([12])**.** *i) Let $w \in A_\infty$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w(B(x,r_1))}\int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^p w(y) dy\Big)^{\frac{1}{p}} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_*,$$

where $C > 0$ is independent of f, x, r_1 and r_2.

ii) Let $w \in A_p$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then

$$\Big(\frac{1}{w^{1-p'}(B(x,r_1))}\int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{p'} w(y)^{1-p'} dy\Big)^{\frac{1}{p'}} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_*,$$

where $C > 0$ is independent of f, x, r_1 and r_2.

Below we present some conclusions about $R_m(A; x, y)$.

Lemma 4 ([22])**.** *Suppose b is a function on $\mathbb{R}^n$ with the m-th derivatives in $L_q(\mathbb{R}^n)$, $q > n$. Then*

$$|R_m(b; x, y)| \le C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{B(x, 5\sqrt{n}|x-y|)} \int_{B(x,5\sqrt{n}|x-y|)} |D^\gamma b(z)|dz \right)^{1/q}.$$

The following property is valid.

Lemma 5. *Let $x \in B(x_0, r)$, $y \in B(x_0, 2^{j+1}r) \backslash B(x_0, 2^j r)$. Assume that A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$. Then there exists a constant C, independent of A, such that*

$$\begin{aligned} &|R_m(A; x, y)| \\ &\le C|x-y|^{m-1} \left(j \sum_{|\gamma|=m-1} \|D^\gamma A\|_* + \sum_{|\gamma|=m-1} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}| \right). \end{aligned} \tag{11}$$

Proof. For fixed $x \in \mathbb{R}^n$, let

$$\bar{A}(x) = A(x) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} (D^\gamma A)_{B(x,5\sqrt{n}|x-y|)} x^\gamma.$$

Then

$$\begin{aligned} &|R_m(A; x, y)| = |R_m(\bar{A}; x, y)| \le \\ &\le |R_{m-1}(\bar{A}; x, y)| + \sum_{|\gamma|=m-1} \frac{1}{\gamma!} |(D^\gamma \bar{A}(y))||x-y|^{m-1}. \end{aligned} \tag{12}$$

From Lemma 4 we have

$$|R_{m-1}(\bar{A}; x, y)| \le C|x-y|^{m-1} \sum_{|\gamma|=m-1} \|D^\gamma A\|_*. \tag{13}$$

If $x \in B(x_0, r)$, $y \in B(x_0, 2^{j+1}r) \backslash B(x_0, 2^j r)$, then $2^{j-1}r \le |x-y| \le 2^{j+2}r$. Thus, we have

$$B(x_0, 2^{j-1}r) \subset B(x, 5\sqrt{n}|x-y|) \subset 100\sqrt{n}B(x_0, 2^j r).$$

Then

$$\frac{|100\sqrt{n}B(x_0, 2^j r)|}{|B(x, 5\sqrt{n}|x-y|)|} \le \frac{|100\sqrt{n}B(x_0, 2^j r)|}{|B(x_0, 2^{j-1}r)|} \le C.$$

Hence

$$|(D^\gamma A)_{B(x,5\sqrt{n}|x-y|)} - (D^\gamma A)_{B(x_0,2^j r)}| \le$$

$$\leq \frac{1}{|B(x,5\sqrt{n}|x-y|)|}\int_{B(x,5\sqrt{n}|x-y|)}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,2^jr)}|dy \leq$$
$$\leq \frac{1}{|100\sqrt{n}B(x_0,2^jr)|}\int_{100\sqrt{n}B(x_0,2^jr)}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,2^jr)}|dy \leq$$
$$\leq C\|D^\gamma A\|_*.$$

Note that

$$|(D^\gamma A)_{B(x_0,2^jr)}-(D^\gamma A)_{B(x_0,r)}| \leq$$
$$\leq \sum_{k=1}^{j}|(D^\gamma A)_{B(x_0,2^kr)}-(D^\gamma A)_{B(x_0,2^{k-1}r)}| \leq$$
$$\leq 2^n j\|D^\gamma A\|_*.$$

Then

$$|(D^\gamma A)_{B(x,5\sqrt{n}|x-y|)}-(D^\gamma A)_{B(x_0,r)}| \leq$$
$$\leq |(D^\gamma A)_{B(x,5\sqrt{n}|x-y|)}-(D^\gamma A)_{B(x_0,2^jr)}|+|(D^\gamma A)_{B(x_0,2^jr)}-(D^\gamma A)_{B(x_0,r)}| \leq$$
$$\leq C_j\|D^\gamma A\|_*.$$

Thus

$$|D^\gamma \bar{A}(y)| = |D^\gamma A(y)-(D^\gamma A)_{B(x,5\sqrt{n}|x-y|)}| \leq$$
$$\leq |D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|+|(D^\gamma A)_{B(x,5\sqrt{n}|x-y|)}-(D^\gamma A)_{B(x_0,r)}| \leq$$
$$\leq |D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|+C_j\|D^\gamma A\|_*. \tag{14}$$

Combining (12), (13) and (14), we get the validity of (11).

3. A local weighted estimates

In the following theorem we get the local weighted estimate (see, for example, [10, 11] in the case $w=1$, $m=1$ and [12] in the case $w\in A_p$, $m=1$) for the operator $T^{A,m}_{\Omega,\alpha}$.

Theorem 3. *Let $1\leq s'<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Suppose that $\omega^{s'}\in A_{\frac{p}{s'},\frac{q}{s'}}$ and A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$. Then for any $r>0$ there is a constant C independent of f such that*

$$\|T^{A,m}_{\Omega,\alpha}(f)\|_{L_{q,\omega^q}(B(x_0,r))} \leq \tag{15}$$
$$\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*(\omega^q(B(x_0,r)))^{\frac{1}{q}}\int_{2r}^{\infty}\|f\|_{L_{p,\omega^p}(B(x_0,t))}(\omega^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}.$$

Proof. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)_{\chi_{B(x_0,2r)}(y)}$, $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0, 2r)$. Then

$$\|T^{A,m}_{\Omega,\alpha}(f)\|_{L_{q,\omega^q}(B(x_0,r))} \le \|T^{A,m}_{\Omega,\alpha}(f_1)\|_{L_{q,\omega^q}(B(x_0,r))} + \|T^{A,m}_{\Omega,\alpha}(f_2)\|_{L_{q,\omega^q}(B(x_0,r))}.$$

Since $f_1 \in L_{p,\omega^p}(\mathbb{R}^n)$, by the boundedness of $T^A_{\Omega,\alpha}$ from $L_{p,\omega^p}(\mathbb{R}^n)$ to $L_{q,\omega^q}(\mathbb{R}^n)$ (Theorem 1) we get

$$\begin{aligned}
&\|T^{A,m}_{\Omega,\alpha}(f_1)\|_{L_{q,\omega^q}(B(x_0,r))} \le \|T^{A,m}_{\Omega,\alpha}(f_1)\|_{L_{q,\omega^q}(\mathbb{R}^n)} \le \\
&\le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f_1\|_{L_{p,\omega^p}(\mathbb{R}^n)} = \\
&= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_{p,\omega^p}(B(x_0,2r))}.
\end{aligned}$$

Note that $q > p > 1$ and $\frac{s'p}{p'(p-s')} \ge 1$, then by Holder's inequality

$$\begin{aligned}
1 &\le \left(\frac{1}{|B|}\int_B w(y)^p dy\right)^{\frac{1}{p}} \left(\frac{1}{|B|}\int_B w(y)^{-p'} dy\right)^{\frac{1}{p'}} \le \\
&\le \left(\frac{1}{|B|}\int_B w(y)^q dy\right)^{\frac{1}{q}} \left(\frac{1}{|B|}\int_B w(y)^{-\frac{s'p}{p-s'}} dy\right)^{\frac{p-s'}{s'p}}.
\end{aligned}$$

This means

$$r^{\frac{n}{s'}-\alpha} \le (\omega^q(B(x_0,r)))^{\frac{1}{q}} \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))}.$$

Then

$$\begin{aligned}
&\|f\|_{L^p(\omega^p,B(x_0,2r))} \le C r^{\frac{n}{s'}-\alpha} \|f\|_{L_{p,\omega^p}(B(x_0,2r))} \int_{2r}^\infty t^{\alpha-\frac{n}{s'}-1}\, dt \le \\
&\le C(\omega^q(B(x_0,r)))^{\frac{1}{q}} \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))} \int_{2r}^\infty \|f\|_{L_{p,\omega^p}(B(x_0,t))} t^{\alpha-\frac{n}{s'}-1}\, dt \le \\
&\le C(\omega^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^\infty \|f\|_{L_{p,\omega^p}(B(x_0,t))} \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,t))} t^{\alpha-\frac{n}{s'}-1}\, dt.
\end{aligned}$$

Since $\omega^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$, by (10), for all $r > 0$ we get

$$(\omega^q(B(x_0,r)))^{\frac{1}{q}} \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))} \le C r^{\frac{n}{s'}-\alpha}. \tag{16}$$

Then

$$\|T^{A,m}_{\Omega,\alpha}(f_1)\|_{L_{q,\omega^q}(B(x_0,r))} \le$$

$$\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* (\omega^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega^p}(B(x_0,t))} (\omega^q(B(x_0,t)))^{-\frac{1}{q}} \frac{dt}{t}. \qquad (17)$$

Let $\triangle_i = (B(x_0, 2^{j+1}r)) \setminus (B(x_0, 2^j r))$, and let $x \in B(x_0, r)$. By Lemma 5,

$$|T^{A,m}_{\Omega,\alpha}(f_2)(x)| \leq \left| \int_{(B(x_0,2r))^c} \frac{R_m(A;x,y)}{|x-y|^{n-\alpha+m-1}} \Omega(x-y) f(y) dy \right| \leq$$
$$\leq C \sum_{j=1}^{\infty} \int_{\triangle_i} \frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}} \times$$
$$\times \left(j \sum_{|\gamma|=m-1} \|D^\gamma A\|_* + \sum_{|\gamma|=m-1} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}| \right) dy \leq$$
$$\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^{\infty} j \int_{\triangle_i} \frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}} dy +$$
$$+ C \sum_{|\gamma|=m-1} \sum_{j=1}^{\infty} \int_{\triangle_i} \frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}| dy =$$
$$= I_1 + I_2. \qquad (18)$$

By Holder's inequalities

$$\int_{\triangle_i} \frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}} \leq \Big(\int_{\triangle_i} |\Omega(x-y)|^s dy \Big)^{\frac{1}{s}} \Big(\int_{\triangle_i} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}} dy \Big)^{\frac{1}{s'}}.$$

If $x \in B(x_0, s)$ and $y \in \triangle_i$, then by direct calculation we can see that $2^{j-1}r \leq |y-x| < 2^{j+1}r$. Hence

$$\Big(\int_{\triangle_i} |\Omega(x-y)|^s dy \Big)^{\frac{1}{s}} \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} |B(x_0, 2^{j+1}r)|^{\frac{1}{s}}. \qquad (19)$$

We also note that if $x \in B(x_0, r)$, $y \in B(x_0, 2r)^c$, then $|y-x| \approx |y-x_0|$. Consequently

$$\Big(\int_{\triangle_i} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}} dy \Big)^{\frac{1}{s'}} \leq \frac{1}{|B(x_0, 2^{j+1}r)|^{1-\alpha/n}} \Big(\int_{B(x_0,2^{j+1}r)} |f(y)|^{s'} dy \Big)^{\frac{1}{s'}}. \qquad (20)$$

Then

$$I_1 \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^{\infty} j (2^{j+1}r)^{\alpha - \frac{n}{s'}} \Big(\int_{B(x_0,2^{j+1}r)} |f(y)|^{s'} dy \Big)^{\frac{1}{s'}}. \qquad (21)$$

Since $s' < p$, it follows from Holder's inequality that

$$\Big(\int_{B(x_0,2^{j+1}r)} |f(y)|^{s'} dy \Big)^{\frac{1}{s'}} \leq C \|f\|_{L_{p,\omega^p}(B(x_0,2^{j+1}r))} \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,2^{j+1}r))}. \qquad (22)$$

Then

$$I_1 \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^{\infty} j(2^{j+1}r)^{\alpha-\frac{n}{s'}} \Big(\int_{B(x_0,2^{j+1}s)} |f(y)|^{q'} dy \Big)^{\frac{1}{q'}} \le$$

$$\le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^{\infty} \Big(1+\ln \frac{2^{j+1}r}{r}\Big)(2^{j+1}r)^{\alpha-\frac{n}{s'}} \|f\|_{L_{p,\omega^p}(B(x_0,2^{j+1}r))} \times$$

$$\times \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,2^{j+1}r))} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}l} \Big(1+\ln \frac{t}{r}\Big) \times$$

$$\times \|f\|_{L_{p,\omega^p}(B(x_0,t))} \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,t))} t^{\alpha-\frac{n}{s'}-1} dt \le$$

$$\le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \int_{2r}^{\infty} \Big(1+\ln \frac{t}{r}\Big) \|f\|_{L_{p,\omega^p}(B(x_0,t))} \|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,t))} t^{\alpha-\frac{n}{s'}-1}\, dt.$$

From (16) we know

$$\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))} \le C r^{\frac{n}{s'}-\alpha} (\omega^q(B(x_0,r)))^{-\frac{1}{q}}. \tag{23}$$

Then

$$I_1 \le C \int_{2r}^{\infty} \Big(1+\ln \frac{t}{r}\Big) \|f\|_{L_{p,\omega^p}(B(x_0,t))} (\omega^q(B(x_0,t)))^{-\frac{1}{q}} \frac{dt}{t}. \tag{24}$$

On the other hand, by Holder's inequality and (19), (20), we have

$$\int_{\triangle_i} \frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}| dy \le$$

$$\le \Big(\int_{\triangle_i} |\Omega(x-y)|^s dy \Big)^{\frac{1}{s}} \Big(\int_{\triangle_i} \frac{|D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)} f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}} dy \Big)^{\frac{1}{s'}} \le$$

$$\le C \sum_{|\gamma|=m-1} \sum_{j=1}^{\infty} (2^{j+1}r)^{\alpha-\frac{n}{s'}} \left(\int_{B(x_0,2^{j+1}r)} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}|^{s'} |f(y)|^{s'} dy \right)^{\frac{1}{s'}}.$$

Applying Holder's inequality, we get

$$\left(\int_{B(x_0,2^{j+1}r)} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}|^{s'} |f(y)|^{s'} dy \right)^{\frac{1}{s'}} \le$$

$$\le C \|f\|_{L_{p,\omega^p}(B(x_0,2^{j+1}r))} \|(D^\gamma A(\cdot) - (D^\gamma A)_{B(x_0,r)}) \omega(\cdot)^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,2^{j+1}r))}.$$

Consequently

$$I_2 \le C \sum_{|\gamma|=m-1} \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} (2^{j+1}r)^{\alpha-\frac{n}{s'}} \|f\|_{L^p(\omega^p, B(x_0,t))} \times$$

$$\times \|(D^\gamma A(\cdot) - (D^\gamma A)_{B(x_0,r)})\omega(\cdot)^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,t))} dt \le$$
$$\le C \sum_{|\gamma|=m-1} \int_{2r}^{\infty} \|f\|_{L^p(\omega^p, B(x_0,t))} \times$$
$$\times \|(D^\gamma A(\cdot) - (D^\gamma A)_{B(x_0,r)})\omega(\cdot)^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,t))} t^{\alpha - \frac{n}{s'} - 1} dt.$$

By $\omega^{s'} \in A_{\frac{p}{s'}, \frac{q}{s'}}$ and (ii) of Lemma 1 we know $\omega^{-\frac{s'p}{p-s'}} \in A_{1+\frac{ps'}{(p-s')q}}$. Then it follows from Lemma 3 and the inequality (23) that

$$\|(D^\gamma A(\cdot) - (D^\gamma A)_{B(x_0,r)})\omega(\cdot)^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,t))} \le$$
$$\le \Big(\int_{B(x_0,t)} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}|^{\frac{ps'}{p-s'}} \omega^{-\frac{ps'}{p-s'}}(y) dy \Big)^{\frac{p-s'}{ps'}} \le$$
$$\le C\|D^\gamma A\|_* \Big(1 + \ln \frac{t}{r}\Big) (\omega^{-\frac{ps'}{p-s'}}(B(x_0,r)))^{\frac{p-s'}{ps'}} =$$
$$= C\|D^\gamma A\|_* \Big(1 + \ln \frac{t}{r}\Big) \|\omega^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,r))} \le$$
$$\le C\|D^\gamma A\|_* \Big(1 + \ln \frac{t}{r}\Big) r^{\frac{n}{s'} - \alpha} (\omega^q(B(x_0,r)))^{-\frac{1}{q}}.$$

Then

$$I_2 \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_{p,\omega^p}(B(x_0,t))} (\omega^q(B(x_0,t)))^{-\frac{1}{q}} \frac{dt}{t}. \qquad (25)$$

Combining the estimates for I_1 and I_2, we have

$$\sup_{x \in B(x_0,r)} |T^{A,m}_{\Omega,\alpha}(f_2)(x)| \le$$
$$\le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_{p,\omega^p}(B(x_0,t))} (\omega^q(B(x_0,t)))^{-\frac{1}{q}} \frac{dt}{t}.$$

Then we get

$$\|T^{A,m}_{\Omega,\alpha}(f_2)\|_{L^p(\omega, B(x_0,r))} \le$$
$$\le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* (\omega^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega^p}(B(x_0,t))} (\omega^q(B(x_0,t)))^{-\frac{1}{q}} \frac{dt}{t}. \qquad (26)$$

This completes the proof of Theorem 3.

4. Proof of Theorem 2

Since $f \in M_{p,\varphi_1}(\omega^p)$, then by the fact $\|f\|_{L^p(\omega^p,B(x_0,t))}$ is a non-decreasing function of t, we get

$$\frac{\|f\|_{L_{p,\omega^p}(B(x_0,t))}}{\operatorname*{ess\,inf}\limits_{0<t<\tau<\infty} \varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}} \leq$$
$$\leq \operatorname*{ess\,sup}_{0<t<\tau<\infty} \frac{\|f\|_{L_{p,\omega^p}(B(x_0,t))}}{\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}} \leq$$
$$\leq \sup_{\tau>0,x_0\in\mathbb{R}^n} \frac{\|f\|_{L_{p,\omega^p}(B(x_0,\tau))}}{\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}} \leq$$
$$\leq \|f\|_{M_{p,\varphi_1}(\omega^p)}.$$

Since (φ_1,φ_2) satisfies (5), we have

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,\omega^p}(B(x_0,t))}(\omega^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t} \leq$$
$$\leq \int_r^\infty \frac{\|f\|_{L_{p,\omega^p}(B(x_0,t))}}{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}{(\omega^q(B(x_0,t)))^{\frac{1}{q}}}\frac{dt}{t} \leq$$
$$\leq C\|f\|_{M_{p,\varphi_1}(\omega^p)}\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}{(\omega^q(B(x_0,t)))^{\frac{1}{q}}}\frac{dt}{t} \leq$$
$$\leq C\|f\|_{M_{p,\varphi_1}(\omega^p)}\varphi_2(x_0,t).$$

Then by (15) we get

$$\|T^{A,m}_{\Omega,\alpha}(f)\|_{M_{q,\varphi_2}(\omega^q)} \leq$$
$$\leq C \sup_{x_0\in\mathbb{R}^n,t>0}\frac{1}{\varphi_2(x_0,t)}\Big(\frac{1}{\omega^q(B(x_0,t))}\int_{B(x_0,t)}|T^{A,m}_{\Omega,\alpha}(f)(y)|^q\omega^q(y)dy\Big)^{1/q} \leq$$
$$\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|_* \sup_{x_0\in\mathbb{R}^n,t>0}\frac{1}{\varphi_2(x_0,t)}\int_r^\infty\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,\omega^p}(B(x_0,t))}(\omega^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t} \leq$$
$$\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\|f\|_{M_{p,\varphi_1}(\omega^p)}.$$

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