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Abstract. In this paper we study the boundedness of the parametric Marcinkiewicz integral operator μ_Ω^ρ on generalized Orlicz-Morrey spaces $M_{\Phi,\varphi}$. We find the sufficient conditions on the pair $(\varphi_1, \varphi_2, \Phi)$ which ensure the boundedness of the operators μ_Ω^ρ from one generalized Orlicz-Morrey space M_{Φ,φ_1} to another M_{Φ,φ_2}. As an application of the above result, the boundedness of the Marcinkiewicz operator associated with Schrödinger operator μ_j^L on generalized Orlicz-Morrey spaces is also obtained.

Keywords. Parametric Marcinkiewicz integrals, generalized Orlicz-Morrey spaces.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35.

1 Introduction

Suppose that S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and the following property

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$.

In 1960, Hörmander [9] defined the parametric Marcinkiewicz integral operator of higher dimension as follows.

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

where $0 < \rho < n$. It is well-known that the operator $\mu_\Omega^1 \equiv \mu_\Omega$ is just introduced by Stein in [16].

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A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces where the "Morrey-type measuring"of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one. Such spaces were first introduced and studied by Nakai [11]. Then another kind of generalized Orlicz-Morrey spaces were introduced by Sawano $et\,al.$ [13]. Our definition of generalized Orlicz-Morrey spaces introduced in [2] and used here is different from that of the papers [11] and [13].

Boundedness of classical operators of harmonic analysis on generalized Orlicz-Morrey spaces were recently studied in various papers, see for example [2,7,8,12]. In the present work, we shall prove the boundedness of the Marcinkiewicz operator μ_Ω^ρ from one generalized Orlicz-Morrey space M_{Φ,φ_1} to another M_{Φ,φ_2}.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Recall that a function $\Phi : [0, +\infty) \to [0, \infty)$ is called a Young function if it is a convex increasing function satisfying $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0, \infty)$ and $\Phi(t) \to \infty$ as $t \to \infty$.

For a Young function Φ, its inverse Φ^{-1} is defined by setting, for all $t \in (0, \infty)$

$$\Phi^{-1}(t) := \inf\{s \in (0, \infty) : \Phi(s) > t\}.$$

Recall that the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, is $\Phi(2r) \le k\Phi(r)$, and the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, is $\Phi(r) \le \frac{1}{2k}\Phi(kr)$, $r \ge 0$, where $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

The function

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \in [0, \infty)\}, \quad r \in [0, \infty)$$

complementary to a Young function Φ, is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$.

We will also use the numerical characteristics

$$a_\Phi := \inf_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

of Young functions.

Remark 2.1 It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \le b_\Phi < \infty$, see [10].

Orlicz space everywhere in the sequel is defined by a Young function Φ via the norm

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\right\}.$$

The space $L_\Phi^{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

The following generalized version of Hölder's inequality holds:

$$\|fg\|_{L_1} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}.$$

As is well known, Morrey spaces are widely used to investigate the local behavior of solutions to second order elliptic partial differential equations. Recall that the classical Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ are defined by

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\rm loc}(\mathbb{R}^n) : \|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $0 \le \lambda \le n$, $1 \le p < \infty$.

The spaces $M_{p,\varphi}(\mathbb{R}^n)$ defined by the norm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}$$

with a function $\varphi(x,r)$ positive on $\mathbb{R}^n \times (0,\infty)$ are known as generalized Morrey spaces.

Generalized Orlicz-Morrey Spaces which unify the generalized Morrey and Orlicz spaces are defined as follows.

Definition 2.1 *([2])(Generalized Orlicz-Morrey Space) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. The generalized Orlicz-Morrey space $M_{\Phi,\varphi}(\mathbb{R}^n)$ is the space of functions $f \in L^{\rm loc}_{\Phi}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \Phi^{-1}(|B(x,r)|^{-1}) \|f\|_{L_\Phi(B(x,r))}.$$

According to this definition, we recover the generalized Morrey space $M_{p,\varphi}$ under the choice $\Phi(r) = r^p$, $1 < p < \infty$ and Orlicz space under the choice $\varphi(x,r) = \Phi^{-1}(|B(x,r)|^{-1})$.

3 Marcinkiewicz operator in the spaces $M_{\Phi,\varphi}$

The following result concerning the boundedness of parametric Marcinkiewicz integral operator μ^{ρ}_{Ω} on L^p is known.

Theorem 3.1 *[15] Suppose that $1 < p, q < \infty$, $\Omega \in L^q(S^{n-1})$ and $0 < \rho < n$. Then, there is a constant C independent of f such that*

$$\|\mu^{\rho}_{\Omega}(f)\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)}.$$

The following interpolation result is from [4].

Lemma 3.1 *Let T be a sublinear operator of weak type (p,p) for any $p \in (1,\infty)$. Then T is bounded on $L^{\Phi}(\mathbb{R}^n)$, where Φ is a Young function satisfying that $1 < a_\Phi \le b_\Phi < \infty$.*

As a consequence of Lemma 3.1 and Theorem 3.1, we get the following result.

Corollary 3.1 *Let Φ be a Young function, $0 < \rho < n$ and $\Omega \in L^q(S^{n-1})$ $(q > 1)$. If $1 < a_\Phi \le b_\Phi < \infty$, then μ^{ρ}_{Ω} is bounded on $L^{\Phi}(\mathbb{R}^n)$.*

We will use the following statements on the boundedness of the weighted Hardy operator

$$H^*_w g(r) := \int_r^\infty g(s) w(s) ds, \quad r \in (0,\infty),$$

where w is a weight.

The following theorem was proved in [6].

Theorem 3.2 *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{r>0} v_2(r) H_w^* g(r) \leq C \sup_{r>0} v_1(r) g(r) \tag{3.1}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty.$$

Moreover, the value $C=B$ is the best constant for (3.1).

We also use the following lemma to prove our main estimates.

Lemma 3.2 *For a Young function Φ and all balls B, the following inequality is valid*

$$\|f\|_{L_1(B)} \leq 2|B| \Phi^{-1}\left(|B|^{-1}\right) \|f\|_{L_\Phi(B)}.$$

Proof. The proof follows from Hölder's inequality and the well known facts

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \quad \text{for} \quad r \geq 0 \tag{3.2}$$

and $\|\chi_B\|_{L_\Phi} = \frac{1}{\Phi^{-1}(|B|^{-1})}$.

The following lemma was a generalization of the [1, Lemma 3.2] for Orlicz spaces.

Lemma 3.3 *Let Φ be a Young function and $\Omega \in L^\infty(S^{n-1})$. If $1 < a_\Phi \leq b_\Phi < \infty$, then the inequality*

$$\|\mu_\Omega^\rho(f)\|_{L_\Phi(B(x_0,r))} \lesssim \frac{1}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^\infty \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t},$$

holds for any ball $B(x_0,r)$, $0<\rho<n$, and for all $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r>0,$$

and have

$$\|\mu_\Omega^\rho(f)\|_{L_\Phi(B)} \leq \|\mu_\Omega^\rho(f_1)\|_{L_\Phi(B)} + \|\mu_\Omega^\rho(f_2)\|_{L_\Phi(B)}.$$

Since $L^\infty(S^{n-1}) \subsetneqq L^q(S^{n-1})$, from the boundedness of μ_Ω^ρ in $L_\Phi(\mathbb{R}^n)$ provided by Corollary 3.1 it follows that

$$\|\mu_\Omega^\rho(f_1)\|_{L_\Phi(B)} \leq \|\mu_\Omega^\rho(f_1)\|_{L_\Phi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|f\|_{L_\Phi(2B)}.$$

It's clear that $x \in B$, $y \in \complement(2B)$ implies $\frac{1}{2}|x_0-y| \leq |x-y| \leq \frac{3}{2}|x_0-y|$. Then by the Minkowski inequality and conditions on Ω, we get

$$\begin{aligned} \mu_\Omega^\rho(f_2)(x) &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |f_2(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^{1+2\rho}}\right)^{1/2} dy \\ &\lesssim \int_{\complement_{(2B)}} \frac{|f(y)|}{|x-y|^n} dy \lesssim \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy. \end{aligned} \tag{3.3}$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n}dy &\approx \int_{\complement_{(2B)}} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}dy \\ &= \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |f(y)|dy\frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|dy\frac{dt}{t^{n+1}}.\end{aligned}$$

By Lemma 3.2, we get

$$\int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n}dy \lesssim \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}. \tag{3.4}$$

Moreover

$$\|\mu_\Omega^\rho(f_2)\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}. \tag{3.5}$$

is valid. Thus

$$\|\mu_\Omega^\rho(f)\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(2B)} + \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}.$$

On the other hand, by (3.2) we get

$$\begin{aligned}\Phi^{-1}\big(|B|^{-1}\big) &\approx \Phi^{-1}\big(|B|^{-1}\big)r^n\int_{2r}^{\infty}\frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}\end{aligned}$$

and then

$$\|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}. \tag{3.6}$$

Thus

$$\|\mu_\Omega^\rho(f)\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(|B|^{-1}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t}.$$

Theorem 3.3 *Let $0<\rho<n$, Φ any Young function, φ_1, φ_2 and Φ satisfy the condition*

$$\int_r^{\infty}\Big(\operatorname*{ess\,sup}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(|B(x_0,s)|^{-1}\big)}\Big)\Phi^{-1}\big(|B(x_0,t)|^{-1}\big)\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{3.7}$$

where C does not depend on x and r. Let also $\Omega\in L^\infty(S^{n-1})$. If Φ satisfy the condition $1<a_\Phi\le b_\Phi<\infty$ then the operator μ_Ω^ρ is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof. By Lemma 3.3 and Theorem 3.2 we have

$$\begin{aligned}\|\mu_\Omega^\rho(f)\|_{M_{\Phi,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\varphi_2(x,r)^{-1}\int_r^{\infty}\Phi^{-1}\big(|B(x,t)|^{-1}\big)\|f\|_{L_\Phi(B(x,t))}\frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\varphi_1(x,r)^{-1}\Phi^{-1}\big(|B(x,r)|^{-1}\big)\|f\|_{L_\Phi(B(x,r))} \\ &= \|f\|_{M_{\Phi,\varphi_1}(\mathbb{R}^n)}.\end{aligned}$$

4 Applications

The study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [14] considered L_p estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R_j^L = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. Then, Dziubanński and Zienkiewicz [3] introduced the Hardy type space $H_L^1(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$.

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions μ_j associated with the Schrödinger operator L by

$$\mu_j^L f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} K_j^L(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $K_j^L(x,y) = \widetilde{K_j^L}(x,y)|x-y|$ and $\widetilde{K_j^L}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In particular, when $V = 0$, $K_j^\Delta(x,y) = \widetilde{K_j^\Delta}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K_j^\Delta}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} \Delta^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In this paper, we write $K_j(x,y) = K_j^\Delta(x,y)$ and

$$\mu_j f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} K_j(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Obviously, μ_j are classical Marcinkiewicz functions. Therefore, it will be an interesting thing to study the property of μ_j^L. In this section, we show that Marcinkiewicz integrals associated with Schrödinger operators are bounded from one generalized Orlicz-Morrey space M_{Φ,φ_1} to another M_{Φ,φ_2}.

Note that a nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y) dy \right)^{1/q} \le C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V(y) dy \right) \tag{4.1}$$

holds for every ball $x \in \mathbb{R}^n$ and $r > 0$, see [14]. It is worth pointing out that the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only n and the constant C in (4.1), such that $V \in B_{q+\varepsilon}$. We always assume that $0 \ne V \in B_n$.

Lemma 4.1 *Let $V \in B_n$ and Φ be a Young function satisfying the condition $1 < a_\Phi \le b_\Phi < \infty$, then the inequality*

$$\|\mu_j^L(f)\|_{L_\Phi(B(x_0,r))} \lesssim \frac{1}{\Phi^{-1}\big(|B(x_0,r)|^{-1}\big)} \int_{2r}^\infty \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(|B(x_0,t)|^{-1}\big) \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. The validity of the following inequality was proved in [5]

$$\mu_j^L f(x) \le \mu_j f(x) + CMf(x), \;\; a.e. \;\; x \in \mathbb{R}^n,$$

where M denotes the well-known Hardy-Littlewood maximal operator. Statements of the Lemma 4.1 for the operators M and μ_j was proved in [2, Lemma 4.4] and Lemma 3.3, respectively. Then we get that the statements of the Lemma 4.1 also true for the operators μ_j^L, $j = 1, \ldots, n$.

Theorem 4.1 *Let $V \in B_n$, Φ be a Young function and (φ_1, φ_2) satisfies the condition* (3.7). *If $1 < a_\Phi \le b_\Phi < \infty$, then the operator μ_j^L is bounded from M_{Φ,φ_1} to M_{Φ,φ_2}.*

Proof. The statement of Theorem 4.1 follows by Lemma 4.1 and Theorem 3.2 in the same manner as in the proof of Theorem 3.3.

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