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Oscillatory of Third-Order Neutral Differential Equations with Continuously Distributed Mixed Arguments

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Abstract: It is the purpose of this paper to give oscillation criteria for the third-order neutral differential equation with continuously distributed mixed arguments

$$\left[r(t)\left([x(t)+\int_a^b p(t,\mu)x[\tau(t,\mu)]d\mu]''\right)^\gamma\right]' + \int_c^d q_1(t,\xi)f(x[\phi_1(t,\xi)])d\xi + \int_c^d q_2(t,\eta)g(x[\phi_2(t,\eta)])d\eta = 0,$$

where $\gamma > 0$ is a quotient of odd positive integers. By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates or converges to zero.

Keywords: Oscillation, third order neutral differential equation, continuously distributed mixed arguments

1 Introduction

We are concerned with the oscillatory behavior of third-order neutral differential equation with continuously distributed mixed arguments

$$\left[r(t)\left([x(t)+\int_a^b p(t,\mu)x[\tau(t,\mu)]d\mu]''\right)^\gamma\right]' + \int_c^d q_1(t,\xi)f(x[\phi_1(t,\xi)])d\xi + \int_c^d q_2(t,\eta)g(x[\phi_2(t,\eta)])d\eta = 0, \quad t \geq t_0. \tag{1}$$

Throughout this paper, we will assume the following hypotheses:

(H1) $r(t) \in C^1([t_0,\infty),(0,\infty))$,

$$r'(t) \geq 0 \textit{ and } \int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} dt = \infty, \tag{2}$$

(H2) $p(t,\mu) \in C([t_0,\infty)\times[a,b],\mathbb{R})$, $0 \leq p(t) \equiv \int_a^b p(t,\mu)d\mu \leq P < 1$,

(H3) $\tau(t,\mu) \in C([t_0,\infty)\times[a,b],\mathbb{R})$ is not a decreasing function for μ and such that

$\tau(t,\mu) \leq t$ and $\lim_{t\to\infty}\min_{\mu\in[a,b]}\tau(t,\mu) = \infty$,

(H4) $\phi_1(t,\xi) \in C([t_0,\infty)\times[c,d],\mathbb{R})$ is not a decreasing function for ξ and such that

$\phi_1(t,\xi) \leq t$ and $\lim_{t\to\infty}\min_{\xi\in[c,d]}\phi_1(t,\xi) = \infty$,

(H5) $f(x) \in C(\mathbb{R},\mathbb{R})$, $\frac{f(u)}{u^\gamma} \geq \delta > 0$, $u \neq 0$, $g(x) \in C(\mathbb{R},\mathbb{R})$ and $f(x), g(x)$ has the same

properties,

(H6) $\phi_2(t,\eta) \in C([t_0,\infty)\times[c,d],\mathbb{R})$ is not an increasing function for η and such that

$\phi_2(t,\eta) \geq t$ and $\lim_{t\to\infty}\max_{\eta\in[c,d]}\phi_2(t,\eta) = \infty$,

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(H7) $q_1(t,\xi), q_2(t,\eta) \in C([t_0,\infty)\times[c,d],(0,\infty))$.

Define the function by

$$z(t) = x(t) + \int_a^b p(t,\mu)x[\tau(t,\mu)]d\mu. \quad (3)$$

Furthermore, the equation (1) is being like following:

$$\left[r(t)\left([z(t)]''\right)^{\gamma}\right]' + \int_c^d q_1(t,\xi)f(x[\phi_1(t,\xi)])d\xi$$

$$+\int_c^d q_2(t,\eta)g(x[\phi_2(t,\eta)])d\eta = 0. \quad (4)$$

A solution $x(t)$ of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory.

In recent years, there has been much research activity concerning the oscillation theory and
applications of differential equations[1,2,12,13].
Especially, the study content of oscillatory criteria of second order differential equations is very rich. In contrast, the study of oscillatory criteria of third-order differential equations is relatively less, but most of them are about delay equation; there are few results dealing with the oscillation of the solutions of third order neutral differential equations with continuously distributed delay in [2-11].

In recent years, B. Baculíková, J. Džurina [12], are studied asymptotic properties of the couple of third order neutral differential equations

$$\left[a(t)\left([x(t)\mp p(t)x(\delta(t))]''\right)^{\gamma}\right]' + q(t)x^{\gamma}(\tau(t)) = 0,$$

where a(t), q(t), p(t) are positive functions, $\gamma > 0$ is quotient of odd positive integers and $\tau(t)\le t$, $\delta(t)\le t$.

Zhang, Gao, Yu [13] are concerned with oscillatory behavior of third order neutral differential equation with continuously distributed delay

$$\left[r(t)[x(t)+\int_a^b p(t,\mu)x[\tau(t,\mu)]d\mu]''\right]'$$

$$+\int_c^d q(t,\xi)f(x[\sigma(t,\xi)])d\xi = 0.$$

Our results improve the results established in [13] because of $\gamma > 0$ is quotient of odd positive integers and $\phi_2(t,\eta)\ge t$.

2 Several lemmas

Lemma 2.1. Let $x(t)$ be a positive solution of (1), $z(t)$ is defined as in (3). Then $z(t)$ has only one of the following two properties:

(I) $z(t) > 0, z'(t) > 0, z''(t) > 0$,
(II) $z(t) > 0, z'(t) < 0, z''(t) > 0$,
where $t \ge t_1$, t_1 sufficiently large.

Proof. Let $x(t)$ be a positive solution of (1) on $[t_0,\infty)$. We see that $z(t) > x(t) > 0$, and

$$\left[r(t)([z(t)]'')^{\gamma}\right]' = -\int_c^d q_1(t,\xi)f(x[\phi_1(t,\xi)])d\xi$$

$$-\int_c^d q_2(t,\eta)g(x[\phi_2(t,\eta)])d\eta < 0.$$

Then $r(t)([z(t)]'')^{\gamma}$ is a decreasing function and therefore eventually of one sign, so $z''(t)$ is either eventually positive or eventually negative on $t\ge t_1\ge t_0$. We assert that $z''(t) > 0$ on $t\ge t_1\ge t_0$. Otherwise, assume that $z''(t) < 0$, then there exists a constant $M > 0$, such that

$$r(t)(z''(t))^{\gamma} \le -M < 0.$$

By integrating the last inequality from t_1 to t, we obtain

$$z'(t) \le z'(t_1) - M^{\frac{1}{\gamma}}\int_{t_1}^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}ds.$$

Let $t\to\infty$. Then from (H1), we have $z'(t)\to-\infty$, and therefore eventually $z'(t) < 0$.
Since $z''(t) < 0$ and $z'(t) < 0$, we have $z(t) < 0$, which contradicts our assumption $z(t) > 0$. Therefore, z(t) has only one of the two properties (I) and (II).
This completes the proof.

Lemma 2.2. Let x(t) be a positive solution of (1), correspondingly $z(t)$ has the property (II). If

$$\int_{t_0}^{\infty}\int_v^{\infty}\left[\frac{1}{r(u)}\int_u^{\infty}(q_3(s)+q_4(s))ds\right]^{\frac{1}{\gamma}}dudv = \infty, \quad (5)$$

then $\lim_{t\to\infty}x(t) = \lim_{t\to\infty}z(t) = 0$, where $q_3(t) = K^{\gamma}\delta\int_c^d q_1(t,\xi)d\xi, q_4(t) = K^{\gamma}\delta\int_c^d q_2(t,\eta)d\eta$.

Proof. Let $x(t)$ be a positive solution of (1). Since $z(t) > 0$ and $z'(t) < 0$, then there
exists finite $\lim_{t\to\infty}z(t) = I$. We assert that $I = 0$. Assume that $I > 0$, then we have $I+\varepsilon > z(t) > I$ for all $\varepsilon > 0$. Choosing $\varepsilon < \frac{I(1-P)}{P}$, we obtain

$$x(t) = z(t) - \int_a^b p(t,\mu)[x(\tau(t,\mu))]d\mu$$
$$> I - \int_a^b p(t,\mu)[x(\tau(t,\mu))]d\mu$$
$$\geq I - p(t)[z(\tau(t,a))]$$
$$\geq I - P(I+\varepsilon) > Kz(t), \quad (6)$$

where $K = \frac{I-P(1+\varepsilon)}{I+\varepsilon} > 0$. Using $(H5)$ and (6), we find from (1) that

$$[r(t)(z''(t))^\gamma]' = -\int_c^d q_1(t,\xi)f(x[\phi_1(t,\xi)])d\xi$$
$$-\int_c^d q_2(t,\eta)g(x[\phi_2(t,\eta)])d\eta$$
$$\leq -\int_c^d q_1(t,\xi)(x[\phi_1(t,\xi)])^\gamma \delta d\xi$$
$$-\int_c^d q_2(t,\eta)(x[\phi_2(t,\eta)])^\gamma \delta d\eta$$
$$\leq -K^\gamma \delta \int_c^d q_1(t,\xi)(z[\phi_1(t,\xi)])^\gamma d\xi$$
$$-K^\gamma \delta \int_c^d q_2(t,\eta)(z[\phi_2(t,\eta)])^\gamma d\eta.$$

Note that z(t) has property (II), (H4) and (H6), we have

$$[r(t)(z''(t))^\gamma]' \leq -K^\gamma.\delta.(z[\phi_1(t,d)])^\gamma \int_c^d q_1(t,\xi)d\xi$$
$$-K^\gamma.\delta.(z[\phi_2(t,c)])^\gamma \int_c^d q_2(t,\eta)d\eta$$
$$= -q_3(t)(z(\phi_3(t)))^\gamma - q_4(t)(z(\phi_4(t)))^\gamma \quad (7)$$

where $\phi_3(t) = \phi_1(t,d), \phi_4(t) = \phi_2(t,c)$.
Integrating inequality (7) from t to ∞, we obtain

$$r(t)(z''(t))^\gamma \geq \int_t^\infty (q_3(s)(z(\phi_3(s)))^\gamma + q_4(s)(z(\phi_4(s)))^\gamma)ds.$$

Using $(z(\phi_3(s)))^\gamma \geq I^\gamma, (z(\phi_4(s)))^\gamma \geq I^\gamma$, we obtain

$$z''(t) \geq \frac{I}{r^{\frac{1}{\gamma}}}\left[\int_t^\infty (q_3(s)+q_4(s))\right]^{\frac{1}{\gamma}} ds. \quad (8)$$

Integrating inequality (8) from t to ∞, we have

$$-z'(t) \geq I\int_t^\infty \left[\frac{1}{r(u)}\int_u^\infty (q_3(s)+q_4(s))ds\right]^{\frac{1}{\gamma}} du.$$

Integrating the last inequality from t_1 to ∞, we obtain

$$z(t_1) \geq I\int_{t_1}^\infty \int_v^\infty \left[\frac{1}{r(u)}\int_u^\infty (q_3(s)+q_4(s))ds\right]^{\frac{1}{\gamma}} dudv.$$

This contradicts (5). Then $I = 0$; moreover the inequality $0 \leq x(t) \leq z(t)$ implies $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Lemma 2.3 [12, Lemma 3]. Assume that $u(t) > 0$, $u'(t) \geq 0$, $u''(t) \leq 0$ on (t_0,∞). Then for each $\ell \in (0,1)$ there exists a $T_\ell \geq t_0$ such that
$$\frac{u(\tau(t))}{\tau(t)} \geq \ell\frac{u(t)}{t} \quad for\ t \geq T_\ell.$$

Proof. It follows from the mean value theorem and the monotone properties of $u'(t)$ that
$u(t) - u(\tau(t)) \leq u'(\tau(t))(t-\tau(t))$ or

$$\frac{u(t)}{u(\tau(t))} \leq 1 + \frac{u'(\tau(t))}{u(\tau(t))}(t-\tau(t)). \quad (9)$$

Using the mean value theorem once more, we see that

$$u(\tau(t)) \geq u(\tau(t)) - u(t_0) \geq u'(\tau(t))(\tau(t)-t_0).$$

So for each $\ell \in (0,1)$ there is a $T_\ell \geq t_0$ such that

$$\frac{u(\tau(t))}{u'(\tau(t))} \geq \ell\tau(t), t \geq T_\ell. \quad (10)$$

Combining (9) with (10), we get

$$\frac{u(t)}{u(\tau(t))} \leq 1 + \frac{1}{\ell\tau(t)}(t-\tau(t)) \leq \frac{t}{\ell\tau(t)}$$

and the proof is complete.

Lemma 2.4 [12, Lemma 4]. Assume that $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$ on (T_ℓ,∞). Then
$$\frac{z(t)}{z'(t)} \geq \frac{t-T_\ell}{2} \quad for\ t \geq T_\ell.$$

Proof. Set

$$Z(t) := (t-T_\ell)z(t) - \frac{(t-T_\ell)^2}{2}z'(t).$$

Then $Z(T_\ell) = 0$, and

$$Z'(t) = z(t) - \frac{(t-T_\ell)^2}{2}z''(t).$$

We shall prove that $Z(t) > 0$. By Taylor's theorem, since $z''(t)$ is non-increasing, we have

$$z(t) \geq z(T_\ell) + (t-T_\ell)z'(T_\ell) + \frac{(t-T_\ell)^2}{2}z''(t).$$

This implies

$$Z'(t) = z(t) - \frac{(t-T_\ell)^2}{2}z''(t) \geq z(T_\ell) + (t-T_\ell)z'(T_\ell) > 0.$$

Since $Z(T_\ell) = 0$, one gets $Z(t) > 0$ for $t \geq T_\ell$ which implies the desired inequality.

3 Main results

In this section we give some new oscillation criteria for (1).

Theorem 3.1. Assume that (2) and (5) hold and there exists a positive function $\rho \in C^1([t_0,\infty),(0,\infty))$. Let $x(t)$ be a solution of (1). If

$$\limsup_{t\to\infty}\int_T^t \left[P_\ell(s)-\frac{r(s)(\rho'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(s)}\right]ds=\infty, \tag{11}$$

where

$$P_\ell(t)=q_5(t)\rho(t)\ell^{\gamma}\left(\frac{\phi_5(t)}{t}\right)^{\gamma}\left(\frac{\phi_5(t)-T_\ell}{2}\right)^{\gamma}$$

$$+q_6(t)\rho(t)\ell^{\gamma}\left(\frac{\phi_6(t)}{t}\right)^{\gamma}\left(\frac{\phi_6(t)-T_\ell}{2}\right)^{\gamma} \tag{12}$$

and $q_5(t)=\delta(1-P)^{\gamma}\int_c^d q_1(t,\xi)d\xi$, $\phi_5(t)=\phi_1(t,c)$, $q_6(t)=\delta(1-P)^{\gamma}\int_c^d q_2(t,\eta)d\eta$, $\phi_6(t)=\phi_2(t,d)$, then every solution of equation (1) is either oscillatory or tends to zero.

Proof. Assume (1) has a nonoscillatory solution $x(t)$. We may assume without loss of generality that $x(t)>0, t\geq t_1; x(\tau(t,\mu))>0, (t,\mu)\in[t_1,\infty)\times[a,b]$ and $x(\phi_1(t,\xi))>0$, $(t,\xi)\in[t_1,\infty)\times[c,d], x(\phi_2(t,\eta))>0$, $(t,\eta)\in[t_1,\infty)\times[c,d]$ for all $t_1\in[t_0,\infty)$. $z(t)$ is defined as in (3). By Lemma 2.1, we have that $z(t)$ has the property (I) or the property (II).

When $z(t)$ has the property (I), we obtain

$$\begin{aligned}x(t)&=z(t)-\int_a^b p(t,\mu)x[\tau(t,\mu)]d\mu\\ &\geq z(t)-\int_a^b p(t,\mu)z[\tau(t,\mu)]d\mu\\ &\geq z(t)-z[\tau(t,b)]\int_a^b p(t,\mu)d\mu\\ &\geq\left(1-\int_a^b p(t,\mu)d\mu\right)z(t)\\ &\geq(1-P)z(t).\end{aligned} \tag{13}$$

Using (H4), (H5) and (H6), we have

$$[r(t)([z(t)]'')^{\gamma}]'=-\int_c^d q_1(t,\xi)f(x[\phi_1(t,\xi)])d\xi$$

$$-\int_c^d q_2(t,\eta)g(x[\phi_2(t,\eta)])d\eta$$

$$\leq-\delta(1-P)^{\gamma}\int_c^d q_1(t,\xi)z^{\gamma}(\phi_1(t,\xi))d\xi$$

$$-\delta(1-P)^{\gamma}\int_c^d q_2(t,\eta)z^{\gamma}(\phi_2(t,\eta))d\eta$$

$$\leq-\delta(1-P)^{\gamma}z^{\gamma}(\phi_1(t,c))\int_c^d q_1(t,\xi)d\xi$$

$$-\delta(1-P)^{\gamma}z^{\gamma}(\phi_2(t,d))\int_c^d q_2(t,\eta)d\eta$$

$$\leq-q_5(t)z^{\gamma}(\phi_5(t))-q_6(t)z^{\gamma}(\phi_6(t)). \tag{14}$$

Furthermore

$$[r(t)([z(t)]'')^{\gamma}]'\leq 0.$$

The last inequality together with $r'(t)\geq 0$ gives $z'''(t)\leq 0$. So there exists a $T\geq t_0$ such that z(t) satisfies $z(\tau(t,\mu))>0$, $z(\phi_1(t,\xi))>0$, $z(\phi_2(t,\eta))>0$, $z'(t)>0$, $z''(t)>0$, $z'''(t)\leq 0$, for $t\in[T,\infty)$.

Define the function $w(t)$ by Riccati substitution

$$w(t)=\rho(t)\frac{r(t)([z(t)]'')^{\gamma}}{(z'(t))^{\gamma}}. \tag{15}$$

We see that $w(t)$ is positive and satisfies

$$\begin{aligned}w'(t)&=\left(\frac{\rho(t)}{(z'(t))^{\gamma}}\right)'r(t)([z(t)]'')^{\gamma}+(r(t)([z(t)]'')^{\gamma})'\frac{\rho(t)}{(z'(t))^{\gamma}}\\ &=\frac{\rho'(t)([z(t)]'')^{\gamma}r(t)}{(z'(t))^{\gamma}}-\frac{([z(t)]')^{\gamma})'\rho(t)r(t)[z(t)]'')^{\gamma}}{([z(t)]')^{2\gamma}}\\ &+(r(t)([z(t)]'')^{\gamma})'\frac{\rho(t)}{(z'(t))^{\gamma}}.\end{aligned}$$

From definition of $w(t)$ and (14), we have

$$\begin{aligned}w'(t)&\leq\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\gamma(z'(t))^{\gamma-1}[z(t)]''\rho(t)r(t)([z(t)]'')^{\gamma}}{([z(t)]')^{2\gamma}}\\ &-[q_5(t)z^{\gamma}(\phi_5(t))+q_6(t))z^{\gamma}(\phi_6(t))]\frac{\rho(t)}{(z'(t))^{\gamma}}\\ &\leq\frac{\rho'(t)}{\rho(t)}w(t)-\gamma\rho(t)\frac{r(t)([z(t)]'')^{\gamma+1}}{(z'(t))^{\gamma+1}}\\ &-[q_5(t)z^{\gamma}(\phi_5(t))+q_6(t))z^{\gamma}(\phi_6(t))]\frac{\rho(t)}{(z'(t))^{\gamma}}\\ &\leq\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\gamma}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^{\frac{\gamma+1}{\gamma}}\\ &-[q_5(t)z^{\gamma}(\phi_5(t))+q_6(t))z^{\gamma}(\phi_6(t))]\frac{\rho(t)}{(z'(t))^{\gamma}}.\end{aligned} \tag{16}$$

From Lemma 2.3 with $u(t)=z'(t)$, we have for $\ell\in(0,1)$

$$\frac{1}{z'(t)}\geq\ell\frac{\tau(t)}{t}\frac{1}{z'(\tau(t))},\ t\geq T_\ell$$

which with (16) gives

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\gamma}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^{\frac{\gamma+1}{\gamma}} - q_5(t)\rho(t)\ell^\gamma\left(\frac{\phi_5(t)}{t}\right)^\gamma \frac{z^\gamma(\phi_5(t))}{(z'(\phi_5(t)))^\gamma} - q_6(t)\rho(t)\ell^\gamma\left(\frac{\phi_6(t)}{t}\right)^\gamma \frac{z^\gamma(\phi_6(t))}{(z'(\phi_6(t)))^\gamma}.$$

Using the fact from Lemma 2.4 that $z(t) \geq \frac{t-T_\ell}{2}z'(t)$, we have

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\gamma}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^{\frac{\gamma+1}{\gamma}} - q_5(t)\rho(t)\ell^\gamma\left(\frac{\phi_5(t)}{t}\right)^\gamma\left(\frac{\phi_5(t)-T_\ell}{2}\right)^\gamma - q_6(t)\rho(t)\ell^\gamma\left(\frac{\phi_6(t)}{t}\right)^\gamma\left(\frac{\phi_6(t)-T_\ell}{2}\right)^\gamma. \tag{17}$$

From (12), we obtain

$$w'(t) \leq -P_\ell(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\gamma}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^\lambda \tag{18}$$

where $\lambda := \frac{\gamma+1}{\gamma}$. Define $A \geq 0$ and $B \geq 0$ by $A^\lambda := \frac{\gamma}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^\lambda$, $B^{\lambda-1} := \frac{\rho'(t)r^{\frac{1}{\gamma+1}}(t)(\gamma)^{\frac{1}{\gamma+1}}}{(\gamma+1)\rho^{\frac{1}{\lambda}}(t)}$.
Then using the inequality[14]

$$\lambda AB^{\lambda-1} - A^\lambda \leq (\lambda-1)B^\lambda \tag{19}$$

which yields

$$\frac{\rho'(t)}{\rho(t)}w(t) - \frac{\gamma}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^\lambda \leq \frac{r(t)(\rho'(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(t)}.$$

From this last inequality and (18), we find

$$w'(t) \leq -P_\ell(t) + \frac{r(t)(\rho'(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(t)}.$$

Integrating both sides from T to t , we get

$$\int_T^t \left[P_\ell(s) - \frac{r(s)(\rho'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(s)}\right]ds \leq w(T) - w(t) \leq w(T),$$

which contradicts to assumption (11). If $z(t)$ has the property (II). Since (5) holds, then the conditions in Lemma 2.2 are satisfied. Hence $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Theorem 3.1.

Remark 3.1. From Theorem 3.1, we can obtain different conditions for oscillation of equation (1) with different choices of $\rho(t)$.

Theorem 3.2. Let $D \equiv (t,s) : t \geq s \geq t_0$; $D_0 \equiv (t,s) : t > s \geq t_0$. A function $H \in C(D,R)$ is said to belong to X class ($H \in X$) if it satisfies $H(t,t) = 0$, $t \geq t_0$; $H(t,s) > 0$, $(t,s) \in D_0$; $\frac{\partial H(t,s)}{\partial s} \leq 0$, there exist $\rho \in C^1([t_0,\infty),(0,\infty))$ and $h \in C(D_0,R)$ such that

$$\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)}H(t,s) = -\frac{h(t,s)}{\rho(s)}(H(t,s))^{\frac{\gamma}{\gamma+1}}. \tag{20}$$

Assume that (2) and (5) hold and there exist $\rho \in C^1([t_0,\infty),(0,\infty))$ and ($H \in X$) such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_2)}\int_{t_2}^t \left[H(t,s)P_\ell(s) - \frac{(h_-(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}\rho^\gamma}\right]ds = \infty, \tag{21}$$

where $h_-(t) := \max\{0,-h(t)\}$. Then every solution of equation (1) is either oscillatory or tends to zero.

Proof. Suppose that $x(t)$ is a non-oscillatory solution of (1) and $z(t)$ is defined as in (3). If case (I) of Lemma 2.1 holds then proceeding as in the proof of Theorem 3.1, we see that (18) holds for $t > t_0$. Multiply both sides of (18) by $H(t,s)$. $t \geq t_2 \geq t_0$ such that integrating from t_2 to t, we get

$$\begin{aligned}\int_{t_2}^t H(t,s)P_\ell(s)ds \leq& -\int_{t_2}^t H(t,s)w'(s)ds \\ &+ \int_{t_2}^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\ &- \int_{t_2}^t H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}w^\lambda(s)ds \\ &= H(t,t_2)w(t_2) + \int_{t_2}^t \frac{\partial H(t,s)}{\partial s}w(s)ds \\ &+ \int_{t_2}^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds \\ &- \int_{t_2}^t H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}w^\lambda(s)ds \\ &= H(t,t_2)w(t_2) \\ &+ \int_{t_2}^t \left[\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)}H(t,s)\right]w(s)ds \\ &- \int_{t_2}^t H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}w^\lambda(s)ds.\end{aligned}$$

Using (19), we obtain

$$\int_{t_2}^{t} H(t,s)P_\ell(s)ds = H(t,t_2)w(t_2) + \int_{t_2}^{t}\left[-\frac{h(t,s)}{\rho(s)}(H(t,s))^{\frac{1}{\lambda}}\right]w(s)ds - \int_{t_2}^{t} H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}w^{\lambda}(s)ds$$

$$\int_{t_2}^{t} H(t,s)P_\ell(s)ds = H(t,t_2)w(t_2) + \int_{t_2}^{t}\left[\frac{h_-(t,s)}{\rho(s)}(H(t,s))^{\frac{\gamma}{\gamma+1}}\right]w(s)ds - \int_{t_2}^{t} H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}w^{\lambda}(s)ds. \quad (22)$$

Therefore, as in Theorem 3.1, by letting $A^{\lambda} := H(t,s)\frac{\gamma}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^{\lambda}(t)$, $B^{\lambda-1} := \frac{h_-(t,s)(r(t)\gamma)^{\frac{1}{\gamma+1}}}{(\gamma+1)(\rho(t))^{\frac{1}{\lambda}}}$.

Then using the inequality

$$\lambda AB^{\lambda-1} - A^{\lambda} \le (\lambda-1)B^{\lambda}.$$

We have

$$\int_{t_2}^{t}\left[\frac{h_-(t,s)}{\rho(s)}(H(t,s))^{\frac{\gamma}{\gamma+1}}\right]w(s)ds - \int_{t_2}^{t} H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}w^{\lambda}(s)ds \le \int_{t_2}^{t}\frac{(h_-(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}\rho^{\gamma}}ds.$$

From this last inequality and (22), we find

$$\int_{t_2}^{t} H(t,s)P_\ell(s)ds \le H(t,t_2)w(t_2) + \int_{t_2}^{t}\frac{(h_-(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}\rho^{\gamma}}ds$$

and this implies that

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[H(t,s)P_\ell(s) - \frac{(h_-(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}\rho^{\gamma}}\right]ds \le w(t_2).$$

The last inequality contradicts (21). This completes the proof of Theorem 3.2.

Example 3.1. Consider the following third order neutral differential equation

$$\left(t^2\left[\left(x(t)+\int_a^b e^{-t}x(\frac{t}{2})d\mu\right)''\right]^3\right)' + \int_c^d tf\left(x\left[\frac{t}{2}\right]\right)d\xi + \int_c^d tg(x[2t])d\eta = 0, \quad (23)$$

where $r(t)=t^2$, $\gamma=3$, $p(t,\mu)=e^{-t}$, $\tau(t,\mu)=\phi_1(t,\xi)=\frac{t}{2}$, $q_1(t,\xi)=t$, $\phi_2(t,\eta)=2t$, $q_2(t,\eta)=t$. It is clear that condition (2) and (5) hold. We obtain

$$\int_{t_0}^{\infty}\left(\frac{1}{t^2}\right)^{\frac{1}{3}}dt=\infty,$$

and

$$\int_{t_0}^{\infty}\int_v^{\infty}\left[\frac{1}{r(u)}\int_u^{\infty}(q_3(s)+q_4(s))ds\right]^{\frac{1}{\gamma}}dudv = \int_{t_0}^{\infty}\int_v^{\infty}\left[\frac{1}{u^2}\int_u^{\infty}2ksds\right]dudv=\infty,$$

where
$q_3(t)=K^{\gamma}\delta\int_c^d q_1(t,\xi)d\xi = K.\delta.t.(d-c)\cong k.t$ (k is a constant),
$q_4(t)=K^{\gamma}\delta\int_c^d q_1(t,\eta)d\eta = K.\delta.t.(d-c)\cong k.t$ (k is a constant).
Therefore, by Theorem 3.1, pick $\rho(t)=t$, we have

$$\limsup_{t\to\infty}\int_T^t\left[P_\ell(s)-\frac{r(s)(\rho'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(s)}\right]ds = \int_T^t\left[ks^2(s-2T_\ell)^3+\left(s-\frac{T_\ell}{2}\right)^3-\frac{s^2}{4^4.s^3}\right]ds=\infty,$$

where

$$\begin{aligned}P_\ell(t) &= q_5(t)\rho(t)\ell^{\gamma}\left(\frac{\phi_5(t)}{t}\right)^{\gamma}\left(\frac{\phi_5(t)-T_\ell}{2}\right)^{\gamma} + q_6(t)\rho(t)\ell^{\gamma}\left(\frac{\phi_6(t)}{t}\right)^{\gamma}\left(\frac{\phi_6(t)-T_\ell}{2}\right)^{\gamma}\\ &= kt.t.\left(\frac{1}{2}\right)^3(t-2T_\ell)^3 + kt.t.\left(\frac{1}{2}\right)^3\left(t-\frac{T_\ell}{2}\right)^3\\ &\cong kt^2\left[(t-2T_\ell)^3+\left(t-\frac{T_\ell}{2}\right)^3\right],\end{aligned}$$

$q_5(t)=\delta(1-P)^{\gamma}\int_c^d q_1(t,\xi)d\xi = 1.(1-\frac{1}{2})^3\int_c^d td\xi$
$\cong kt$ (k is a constant),
$q_6(t)=\delta(1-P)^{\gamma}\int_c^d q_2(t,\eta)d\eta = 1.(1-\frac{1}{2})^3\int_c^d td\eta$
$\cong kt$ (k is a constant).

Hence, by Theorem 3.1 every solution of (23) is oscillatory or tends to zero.

4 Conclusion

In this paper, we have investigated the oscillatory behavior of third-order neutral differential equation with continuously distributed mixed arguments

$$\left[r(t)\left([x(t)+\int_a^b p(t,\mu)x[\tau(t,\mu)]d\mu]''\right)^{\gamma}\right]'$$

$$+\int_c^d q_1(t,\xi)f(x[\phi_1(t,\xi)])d\xi$$

$$+\int_c^d q_2(t,\eta)g(x[\phi_2(t,\eta)])d\eta = 0.$$

We have used generalized Riccati transformation and integral averaging technique and have established some new sufficient conditions which ensure that every solution of equation (1) is either oscillatory or tends to zero.

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