

**BOUNDEDNESS OF SUBLINEAR OPERATORS
GENERATED BY CALDERÓN-ZYGMUND OPERATORS
ON GENERALIZED WEIGHTED MORREY SPACES**

BY

TURHAN KARAMAN, VAGIF S. GULIYEV* and AYHAN SERBETCI

Abstract. In this paper we study the boundedness for a large class of sublinear operators T generated by Calderón-Zygmund operators on generalized weighted Morrey spaces $M_{p,\varphi}(w)$ with the weight function $w(x)$ belonging to Muckenhoupt's class A_p . We find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operator T from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to the weak space $WM_{1,\varphi_2}(w)$. In all cases the conditions for the boundedness are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) , which do not assume any assumption on monotonicity of φ_1, φ_2 in r . Conditions of these theorems are satisfied by many important operators in analysis, in particular pseudo-differential operators, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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1. Introduction

Let \mathbb{R}^n be the n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$ with norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ be the open ball centered at x of radius r and ${}^c B(x, r)$ denote its complement and $|B(x, r)|$ is the Lebesgue measure of the ball $B(x, r)$.

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The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy$$

for all $f \in L_1^{loc}(\mathbb{R}^n)$. Let K be a Calderón-Zygmund singular integral operator, briefly a Calderón-Zygmund operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $f \in L_1^{loc}(\mathbb{R}^n)$ represented by

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y)f(y) dy, \quad x \notin \text{supp } f.$$

Here $k(x,y)$ is a continuous function away from the diagonal which provides the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \leq 1$ such that $|k(x,y)| \leq c_1|x-y|^{-n}$, for all $x,y \in \mathbb{R}^n, x \neq y$, and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq c_1 \left(\frac{|x-x'|}{|x-y|} \right)^\varepsilon |x-y|^{-n}$$

whenever $2|x-x'| \leq |x-y|$. It is well known that maximal operator and Calderón-Zygmund operator play an important role in harmonic analysis (see [11, 24, 34, 35, 37]).

The classical Morrey space was originally introduced by MORREY in [28] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [28, 30].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all classes of functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \leq p < \infty$ and $0 \leq \lambda \leq n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on \mathbb{R}^n .

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x, r))$ denotes the weak L_p -space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(B(x, r))} &\equiv \|f\chi_{B(x, r)}\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0} t |\{y \in B(x, r) : |f(y)| > t\}|^{1/p} \\ &= \sup_{0 < t \leq |B(x, r)|} t^{1/p} \left(f\chi_{B(x, r)} \right)^*(t) < \infty. \end{aligned}$$

Here g^* denotes the non-increasing rearrangement of a function g .

CHIARENZA and FRASCA [7] studied the boundedness of the maximal operator M in Morrey spaces. Their results can be summarized as follows:

Theorem 1.1. *Let $1 \leq p < \infty$ and $0 \leq \lambda < n$. Then for $p > 1$ the maximal operator M is bounded on $M_{p, \lambda}$, and for $p = 1$ M is bounded from $M_{1, \lambda}$ to $WM_{1, \lambda}$.*

FAZIO and RAGUSA [10] studied the boundedness of the Calderón-Zygmund operators in Morrey spaces, and their results imply the following statement for Calderón-Zygmund operators K .

Theorem 1.2. *Let $1 \leq p < \infty$, $0 < \lambda < n$. Then for $1 < p < \infty$ Calderón-Zygmund operator K is bounded on $M_{p, \lambda}$, and for $p = 1$ K is bounded from $M_{1, \lambda}$ to $WM_{1, \lambda}$.*

Note that Theorem 1.2 was proved by PEETRE [30] in the case of the classical Calderón-Zygmund singular integral operators.

Suppose that T represents a sublinear operator, which provides that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$

$$(1.1) \quad |Tf(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^n} dy,$$

where c_0 is independent of f and x . We point out that the condition (1.1) was first introduced by SORIA and WEISS in [32]. The condition (1.1) is satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operator, Carleson's maximal operator, Hardy-Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on (see [25], [32] for details).

In this study, we prove the boundedness of the sublinear operator T satisfying condition (1.1) generated by Calderón-Zygmund operator from

one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1 < p < \infty$, and from the space $M_{1,\varphi_1}(w)$ to the weak space $WM_{1,\varphi_2}(w)$. Finally, we apply this result to several particular operators such as pseudo-differential operators, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C , where C is independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$ and say that A and B are equivalent.

2. Weighted Morrey spaces

A weight function is a locally integrable function on \mathbb{R}^n which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E , we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E . Given a weight w , we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball B , we have $w(2B) \leq Dw(B)$. We denote $w \in \Delta_2$, for short, when w satisfies doubling condition.

If w is a weight function, we denote by $L_p(w) \equiv L_p(\mathbb{R}^n, w)$ the weighted Lebesgue space defined by the norm

$$\|f\|_{L_{p,w}} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty, \quad \text{when } 1 \leq p < \infty$$

and by $\|f\|_{L_{\infty,w}} = \text{ess sup}_{x \in \mathbb{R}^n} |w(x)f(x)|$ when $p = \infty$.

We recall that a weight function w is in the Muckenhoupt's class A_p , $1 < p < \infty$, if

$$\begin{aligned} [w]_{A_p} &:= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} < \infty, \end{aligned}$$

where the sup is taken with respect to all balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B by Hölder's inequality the following holds

$$(2.1) \quad [w]_{A_p}^{1/p} = |B|^{-1} \|w\|_{L_1(B)}^{1/p} \|w^{-1/p}\|_{L_{p'}(B)} \geq 1.$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ we define $A_\infty = \bigcup_{1 \leq p < \infty} A_p$.

Lemma 2.1 ([12]). *Let $w \in A_p$, $1 \leq p < \infty$, then*

(1) $w \in \Delta_2$. *Moreover, for all $\lambda > 1$ we have $w(\lambda B) \leq \lambda^{np}[w]_{A_p} w(B)$,*
and

(2) *there exists $C > 0$ and $\delta > 0$ such that for any ball B and a measurable set $S \subset B$, $\frac{w(S)}{w(B)} \leq C \left(\frac{|S|}{|B|} \right)^\delta$.*

Definition 2.1 ([19]). Let $1 \leq p < \infty$, $0 < \kappa < 1$ and w be a weight function. We denote by $L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}^n, w)$ the weighted Morrey space, the space of all classes of locally integrable functions f with the norm $\|f\|_{L_{p,\kappa}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} w(B(x, r))^{-\frac{\kappa}{p}} \|f\|_{L_p(B(x, r))} < \infty$.

By $WL_{p,\kappa}(w) \equiv WL_{p,\kappa}(\mathbb{R}^n, w)$ we denote the weak weighted Morrey space, the space of all locally integrable functions f with the norm $\|f\|_{WL_{p,\kappa}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} w(B(x, r))^{-\frac{\kappa}{p}} \|f\|_{WL_p(B(x, r))} < \infty$.

Remark 2.1. Alternatively, we could define the weighted Morrey spaces with cubes instead of balls. Hence we shall use these two definitions of weighted Morrey spaces appropriate to calculation.

Remark 2.2. (1) If $w \equiv 1$ and $\kappa = \lambda/n$ with $0 < \lambda < n$, then $L_{p,\lambda/n}(1) = M_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey spaces.

(2) If $\kappa = 0$, then $L_{p,0}(w) = L_p(w)$ is the weighted Lebesgue spaces.

The following theorem was proved in [19].

Theorem 2.3. *Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Then the operators M and K are bounded on $L_{p,\kappa}(w)$ for $p > 1$ and from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.*

3. Generalized Morrey spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 3.2. Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$, $1 \leq p < \infty$, the generalized Morrey space, the space of all classes of functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite norm $\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x, r))}$. Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space, the space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x, r))} < \infty.$$

In [13]-[16], [21], [27] and [29] there were obtained sufficient conditions on φ_1 and φ_2 for the boundedness of the maximal operator M and Calderón-Zygmund operator K from M_{p,φ_1} to M_{p,φ_2} , $1 < p < \infty$ (see also [2]-[5]). In [29] the following condition was imposed on $\varphi(x, r)$:

$$(3.1) \quad c^{-1}\varphi(x, r) \leq \varphi(x, t) \leq c\varphi(x, r)$$

whenever $r \leq t \leq 2r$, where $c (\geq 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition:

$$(3.2) \quad \int_r^\infty \varphi(x, t)^p \frac{dt}{t} \leq C \varphi(x, r)^p,$$

for the sublinear operator T satisfying the condition (1.1), where $C (> 0)$ does not depend on r and $x \in \mathbb{R}^n$.

In [9] the following statement containing the result in [27, 29] was proved for a sublinear operator T satisfying the condition (1.1).

Theorem 3.4. *Let $\varphi(x, r)$ satisfies the conditions (3.1)-(3.2). Let T be a sublinear operator satisfying the condition (1.1), and bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Then the operator T is bounded on $M_{p,\varphi}$.*

The following statement, containing results obtained in [27], [29] was proved in [13] (see also [14, 15]).

Theorem 3.5. *Let $1 \leq p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$(3.3) \quad \int_r^\infty \varphi_1(x, t) \frac{dt}{t} \leq C \varphi_2(x, r),$$

where C does not depend on x and r . Then the maximal operator M and Calderón-Zygmund operator K are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} .

The following statements, containing results Theorems 3.4 and 3.5 was proved in [17].

Theorem 3.6. *Let $1 \leq p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$(3.4) \quad \int_r^\infty \frac{\text{ess inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C \varphi_2(x, r),$$

where C does not depend on x and r . Let T be a sublinear operator satisfying the condition (1.1) bounded on $L_p(\mathbb{R}^n)$ for $p > 1$ and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Then the operator T is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} . Moreover, for $p > 1$, $\|Tf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}$, and for $p = 1$, $\|Tf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}$.

4. Sublinear operators generated by Calderón-Zygmund operators in the generalized weighted Morrey spaces $M_{p,\varphi}(w)$

We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

Definition 4.3. Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and w be a non-negative measurable function on \mathbb{R}^n . We denote by $M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^n, w)$, $1 \leq p < \infty$, the generalized weighted Morrey space, the space of all classes of functions $f \in L_{p,w}^{\text{loc}}(\mathbb{R}^n)$ with finite norm $\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x, r))}$. We denote by $WM_{p,\varphi}(w) \equiv WM_{p,\varphi}(\mathbb{R}^n, w)$ the weak generalized weighted Morrey space of all functions $f \in WL_{p,w}^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{WL_{p,w}(B(x, r))} < \infty.$$

Remark 4.3. (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $w \equiv 1$ and $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(1) = M_{p,\lambda}$ is the classical Morrey space and $WM_{p,\varphi}(1) = WM_{p,\lambda}$ is the weak Morrey space.

(4) If $\varphi(x, r) \equiv w(B(x, r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_p(w)$ is the weighted Lebesgue space.

In this section we are going to use the following result on the boundedness of the Hardy operator $(Hg)(t) := \frac{1}{t} \int_0^t g(r) d\mu(r)$, $0 < t < \infty$, where μ be a non-negative Borel measure on $(0, \infty)$.

Theorem 4.7. *The inequality $\text{ess sup}_{t>0} w(t)Hg(t) \leq c \text{ess sup}_{t>0} v(t)g(t)$ holds for all non-negative and non-increasing g on $(0, \infty)$ if and only if*

$$A := \sup_{t>0} \frac{w(t)}{t} \int_0^t \frac{d\mu(r)}{\text{ess sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

Note that, in the case $d\mu(t) = dt$ Theorem 4.7 was proved in [6]. In the general case Theorem 4.7 can be proved analogously.

Lemma 4.2. *Let $1 \leq p < \infty$, $w \in A_p$, and let T be a sublinear operator satisfying the condition (1.1), and bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$.*

Then, for $1 < p < \infty$ the inequality

$$\|Tf\|_{L_{p,w}(B(x_0,r))} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{p,w}^{\text{loc}}(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$(4.1) \quad \|Tf\|_{WL_{1,w}(B(x_0,r))} \lesssim w(B) \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))} \|w^{-1}\|_{L_{\infty}(B(x_0,t))} \frac{dt}{t^{n+1}},$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{1,w}^{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $p \in (1, \infty)$ and $w \in A_p$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r , $2B = B(x_0, 2r)$. We represent f as

$$(4.2) \quad f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\mathbb{C}_{(2B)}}(y), \quad r > 0.$$

Then we have $\|Tf\|_{L_{p,w}(B)} \leq \|Tf_1\|_{L_{p,w}(B)} + \|Tf_2\|_{L_{p,w}(B)}$. Since $f_1 \in L_p(w)$, $Tf_1 \in L_p(w)$ and from the boundedness of T in $L_p(w)$ it follows that $\|Tf_1\|_{L_{p,w}(B)} \leq \|Tf_1\|_{L_{p,w}} \leq C\|f_1\|_{L_{p,w}} = C\|f\|_{L_{p,w}(2B)}$, where the constant $C > 0$ is independent of f .

It is clear that $x \in B$, $y \in \mathbb{C}_{(2B)}$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|Tf_2(x)| \leq 2^n c_0 \int_{\mathbb{C}_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned} \int_{\mathbb{C}_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy &\approx \int_{\mathbb{C}_{(2B)}} |f(y)| \int_{|x_0 - y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\approx \int_{2r}^{\infty} \int_{2r \leq |x_0 - y| < t} |f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned}$$

Applying Hölder's inequality, we get

$$(4.3) \quad \int_{\mathbb{C}_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}.$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$(4.4) \quad \|Tf_2\|_{L_{p,w}(B)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

is valid. Thus

$$\|Tf\|_{L_{p,w}(B)} \lesssim \|f\|_{L_{p,w}(2B)} + w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}.$$

On the other hand,

$$(4.5) \quad \begin{aligned} \|f\|_{L_{p,w}(2B)} &\approx |B| \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\ &\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\leq w(B)^{\frac{1}{p}} \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}. \end{aligned}$$

Thus

$$\|Tf\|_{L_{p,w}(B)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}.$$

Let $p = 1$. From the weak $(1, 1)$ boundedness of T and (4.5) it follows that:

$$(4.6) \quad \begin{aligned} \|Tf_1\|_{WL_{1,w}(B)} &\leq \|Tf_1\|_{WL_1(w)} \lesssim \|f_1\|_{L_{1,w}} = \|f\|_{L_{1,w}(2B)} \\ &\lesssim w(B) \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))} \|w^{-1}\|_{L_{\infty}(B(x_0,t))} \frac{dt}{t^{n+1}}. \end{aligned}$$

Then by (4.4) and (4.6) we get the inequality (4.1). \square

Theorem 4.8. *Let $1 \leq p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition*

$$(4.7) \quad \int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) w(B(x, s))^\frac{1}{p}}{t^{n+1}} \|w^{-1/p}\|_{L_{p'}(B(x, t))} dt \leq C \varphi_2(x, r),$$

where C does not depend on x and r . Let T be a sublinear operator satisfying the condition (1.1), bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$. Then the operator T is bounded from $M_{p, \varphi_1}(w)$ to $M_{p, \varphi_2}(w)$ for $p > 1$ and from $M_{1, \varphi_1}(w)$ to $WM_{1, \varphi_2}(w)$. Moreover, for $p > 1$

$$\|Tf\|_{M_{p, \varphi_2}(w)} \lesssim \|f\|_{M_{p, \varphi_1}(w)},$$

and for $p = 1$

$$\|Tf\|_{WM_{1, \varphi_2}(w)} \lesssim \|f\|_{M_{1, \varphi_1}(w)}.$$

Proof. By Lemma 4.2 and Theorem 4.7 we have for $p > 1$

$$\begin{aligned} & \|Tf\|_{M_{p, \varphi_2}(w)} \\ & \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \int_r^\infty \|f\|_{L_{p, w}(B(x, t))} \|w^{-1/p}\|_{L_{p'}(B(x, t))} \frac{dt}{t^{n+1}} \\ & \approx \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \int_0^{r^{-n}} \|f\|_{L_{p, w}(B(x, t^{-\frac{1}{n}}))} \|w^{-1/p}\|_{L_{p'}(B(x, t^{-\frac{1}{n}}))} dt \\ & = \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r^{-\frac{1}{n}})^{-1} r \frac{1}{r} \int_0^r \|f\|_{L_{p, w}(B(x, t^{-\frac{1}{n}}))} \|w^{-1/p}\|_{L_{p'}(B(x, t^{-\frac{1}{n}}))} dt \\ & \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x, r^{-\frac{1}{n}})^{-1} w(B(x, r^{-\frac{1}{n}}))^{-\frac{1}{p}} \|f\|_{L_{p, w}(B(x, r^{-\frac{1}{n}}))} \\ & = \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{L_{p, w}(B(x, r))} \\ & = \|f\|_{M_{p, \varphi_1}(w)} \end{aligned}$$

and for $p = 1$

$$\begin{aligned} & \|Tf\|_{WM_{1, \varphi_2}(w)} \\ & \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \int_r^\infty \|f\|_{L_{1, w}(B(x, t))} \|w^{-1}\|_{L_\infty(B(x, t))} \frac{dt}{t^{n+1}} \\ & \approx \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \int_0^{r^{-n}} \|f\|_{L_{1, w}(B(x, t^{-\frac{1}{n}}))} \|w^{-1}\|_{L_\infty(B(x, t^{-\frac{1}{n}}))} dt \\ & = \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r^{-\frac{1}{n}})^{-1} \int_0^r \|f\|_{L_{1, w}(B(x, t^{-\frac{1}{n}}))} \|w^{-1}\|_{L_\infty(B(x, t^{-\frac{1}{n}}))} dt \end{aligned}$$

$$\begin{aligned} &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x, r^{-\frac{1}{n}})^{-1} w(B(x, r^{-\frac{1}{n}}))^{-1} \|f\|_{L_{1,w}(B(x, r^{-\frac{1}{n}}))} \\ &= \|f\|_{M_{1,\varphi_1}(w)}. \end{aligned}$$

□

Corollary 4.1. *Let $1 \leq p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition (4.7). Then the operators M and K are bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and bounded from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.*

Note that in the case $w \equiv 1$ Corollary 4.1 was proved in [1].

In the case $\varphi_1(x, r) = \varphi_2(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, from Theorem 4.8 we get the following new result.

Corollary 4.2. *Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Let also T be a sublinear operator satisfying the condition (1.1), bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$. Then the operator T is bounded on the weighted Morrey spaces $L_{p,\kappa}(w)$ for $p > 1$ and from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.*

Proof. Let $1 \leq p < \infty$, $w \in A_p$ and $0 < \kappa < 1$. Then the pair $(w(B(x, r))^{\frac{\kappa-1}{p}}, w(B(x, r))^{\frac{\kappa-1}{p}})$ satisfies the condition (4.7). Indeed,

$$\begin{aligned} &\int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} w(B(x, s))^{\frac{\kappa}{p}}}{t^{n+1}} \|w^{-1/p}\|_{L_{p'}(B(x,t))} dt \\ &= \int_r^\infty \frac{w(B(x, t))^{\frac{\kappa}{p}}}{t^{n+1}} \|w^{-1/p}\|_{L_{p'}(B(x,t))} dt \\ &\leq [w]_{A_p}^{1/p} \int_r^\infty w(B(x, t))^{\frac{\kappa-1}{p}} \frac{dt}{t} \\ &\leq C w(B(x, r))^{\frac{\kappa-1}{p}}, \end{aligned}$$

where the last inequality follows from Lemma 13 in [4]. Then we get the proof. □

Note that from Corollary 4.2 we get Theorem 2.3.

5. Some applications

In this section, we will apply Theorem 4.8 to several particular operators such as the pseudo-differential operators, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

5.1. Pseudo-differential operators

Pseudo-differential operators are generalizations of differential operators and singular integrals. Let m be real number, $0 \leq \delta < 1$ and $0 \leq \rho < 1$. Following [18, 36], a symbol in $S_{\rho,\delta}^m$ is a smooth function $\sigma(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β the following estimate holds:

$$\left| D_x^\alpha D_\xi^\beta \sigma(x, \xi) \right| \leq C_{\alpha\beta} (1 + |\xi|)^{m - \rho|\beta| + \delta|\alpha|},$$

where $C_{\alpha\beta} > 0$ is independent of x and ξ . A symbol in $S_{\rho,\delta}^{-\infty}$ is one which satisfies the above estimates for each real number m .

The operator A given by

$$Af(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi$$

is called a pseudo-differential operator with symbol $\sigma(x, \xi) \in S_{\rho,\delta}^m$, where f is a Schwartz function and \hat{f} denotes the Fourier transform of f . As usual, $L_{\rho,\delta}^m$ will denote the class of pseudo-differential operators with symbols in $S_{\rho,\delta}^m$.

MILLER [26] showed the boundedness of the pseudo-differential operators of the Hörmander class $L_{1,0}^0$ on weighted Lebesgue spaces whenever the weight function belongs to Muckenhoupt’s class A_p , $1 < p < \infty$. In [8] it is shown that pseudo-differential operators in $L_{1,0}^0$ are Calderón-Zygmund operators, then from Corollary 4.1 we get the following new results.

Corollary 5.3. *Let $w \in A_p$, $1 \leq p < \infty$, and (φ_1, φ_2) satisfy the condition (4.7). If A is a pseudo-differential operator of the Hörmander class $L_{1,0}^0$, then the operator A is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and bounded from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.*

Corollary 5.4. *Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. If A is a pseudo-differential operator of the Hörmander class $L_{1,0}^0$, then the operator A is bounded on $L_{p,\kappa}(w)$ for $p > 1$ and from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.*

5.2. Littlewood-Paley operator

The Littlewood-Paley functions play an important role in classical harmonic analysis, for example in the study of non-tangential convergence of Fatou type and boundedness of Riesz transforms and multipliers [33, 34, 35, 37]. The Littlewood-Paley operator (see [22, 37]) is defined as follows.

Definition 5.4. Suppose that $\psi \in L_1(\mathbb{R}^n)$ satisfies

$$(5.1) \quad \int_{\mathbb{R}^n} \psi(x) dx = 0.$$

Then the generalized Littlewood-Paley g function g_ψ is defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $F_t(f) = \psi_t * f$ such that $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$.

The following theorem for the Littlewood-Paley operator g_ψ is valid (see [24], Theorem 5.2.2).

Theorem 5.9. *Suppose that $\psi \in L_1(\mathbb{R}^n)$ satisfies (5.1) and the following properties:*

$$(5.2) \quad |\psi(x)| \leq \frac{C}{(1 + |x|)^{n+1}},$$

$$(5.3) \quad |\nabla\psi(x)| \leq \frac{C}{(1 + |x|)^{n+2}},$$

where $C > 0$ are both independent of x . If $w \in A_p$, then g_ψ is bounded on $L_p(w)$ for all $1 < p < \infty$.

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$, then, for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ may be viewed as a mapping from $[0, \infty)$ to H , and it is clear that $g_\psi(f)(x) = \|F_t(f)(x)\|$. Indeed, by Minkowski inequality and the conditions on ψ , we get

$$\begin{aligned} g_\psi(f)(x) &\leq \int_{\mathbb{R}^n} |f(y)| \left(\int_0^\infty |\psi_t(x-y)|^2 \frac{dt}{t} \right)^{1/2} dy \\ &\leq C \int_{\mathbb{R}^n} |f(y)| \left(\int_0^\infty \frac{t^{-2n}}{(1 + |x-y|/t)^{2(n+1)}} \frac{dt}{t} \right)^{1/2} dy \\ &= C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy. \end{aligned}$$

Thus we get the following new results.

Corollary 5.5. *Let $1 < p < \infty$, $w \in A_p$, (φ_1, φ_2) satisfy the condition (4.7) and $\psi \in L_1(\mathbb{R}^n)$ satisfies (5.1)-(5.3). Then the Littlewood-Paley operator g_ψ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

Corollary 5.6. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$ and $\psi \in L_1(\mathbb{R}^n)$ satisfies (5.1)-(5.3). Then the Littlewood-Paley operator g_ψ is bounded on $L_{p,\kappa}(w)$.*

5.3. Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in \mathbb{R}^n equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is a homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(tx) = \Omega(x), \text{ for any } t > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1} , that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0.$$

(c) $\Omega \in \text{Lip}_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is there exists a constant $C > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq C|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, STEIN [33] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

The continuity of Marcinkiewicz operator μ_Ω has been extensively studied in [24, 34, 35, 37].

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|$.

By Minkowski inequality and the above conditions on Ω , we get

$$\mu_\Omega(f)(x) \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy.$$

Thus, μ_Ω satisfies the condition (1.1). It is known that μ_Ω is bounded on $L_p(w)$ for $1 < p < \infty$ and $w \in A_p$ (see [38]), then from Theorem 4.8 we get the following new results.

Corollary 5.7. *Let $1 < p < \infty$, $w \in A_p$. Suppose that (φ_1, φ_2) satisfy the condition (4.7) and Ω satisfies the conditions (a)–(c). Then μ_Ω is bounded from $M_{p, \varphi_1}(w)$ to $M_{p, \varphi_2}(w)$.*

Corollary 5.8. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$. Suppose that Ω satisfies the conditions (a)–(c). Then μ_Ω is bounded on $L_{p, \kappa}(w)$.*

5.4. Bochner-Riesz operator

Let $\delta > (n-1)/2$, $B_t^\delta(f)(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(x) = t^{-n} B^\delta(x/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [23, 20])

$$B_{\delta, *}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|.$$

Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then it is clear that $B_{\delta, *}(f)(x) = \|B_t^\delta(f)(x)\|$.

By the condition on B_r^δ (see [11]), we have

$$\begin{aligned} |B_r^\delta(x-y)| &\leq Cr^{-n}(1+|x-y|/r)^{-(\delta+(n+1)/2)} \\ &= C \left(\frac{r}{r+|x-y|} \right)^{\delta-(n-1)/2} \frac{1}{(r+|x-y|)^n} \leq C|x-y|^{-n}, \end{aligned}$$

and

$$B_{\delta, *}(f)(x) \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy.$$

Thus, $B_{\delta, *}$ satisfies the condition (1.1). It is known that $B_{\delta, *}$ is bounded on $L_p(w)$ for $1 < p < \infty$ and $w \in A_p$, and bounded from $L_1(w)$ to $WL_1(w)$ for $w \in A_1$ (see [31, 39]), then from Theorem 4.8 we get the following new results.

Corollary 5.9. *Let $1 \leq p < \infty$, $w \in A_p$, (φ_1, φ_2) satisfy the condition (4.7) and $\delta > (n-1)/2$. Then the Bochner-Riesz operator $B_{\delta, *}$ is bounded from $M_{p, \varphi_1}(w)$ to $M_{p, \varphi_2}(w)$ for $p > 1$ and from $M_{1, \varphi_1}(w)$ to $WM_{1, \varphi_2}(w)$.*

Corollary 5.10. *Let $1 \leq p < \infty$, $0 < \kappa < 1$, $w \in A_p$ and $\delta > (n-1)/2$. Then the Bochner-Riesz operator $B_{\delta, *}$ is bounded on $L_{p, \kappa}(w)$ for $p > 1$ and from $L_{1, \kappa}(w)$ to $WL_{1, \kappa}(w)$.*

Remark 5.4. Recall that, under the assumptions $\varphi(x, r)$ satisfy the conditions (3.1) and (3.2), Corollary 5.9 were proved in [23].

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Department of Mathematics,
Ahi Evran University,
Kirsehir,
TURKEY
tkaraman@ahievran.edu.tr

Department of Mathematics,
Ahi Evran University,
Kirsehir,
TURKEY
and

Institute of Mathematics and Mechanics,
ANAS, Baku,
AZERBAIJAN
vagif@guliyev.com

Department of Mathematics,
Ankara University,
Ankara,
TURKEY
serbetci@ankara.edu.tr