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BOUNDEDNESS OF FRACTIONAL MAXIMAL OPERATOR AND THEIR HIGHER ORDER COMMUTATORS IN GENERALIZED MORREY SPACES ON CARNOT GROUPS*

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Abstract In the article we consider the fractional maximal operator M_α, $0 \le \alpha < Q$ on any Carnot group $\mathbb{G}$ (i.e., nilpotent stratified Lie group) in the generalized Morrey spaces $M_{p,\varphi}(\mathbb{G})$, where Q is the homogeneous dimension of $\mathbb{G}$. We find the conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operator M_α from one generalized Morrey space $M_{p,\varphi_1}(\mathbb{G})$ to another $M_{q,\varphi_2}(\mathbb{G})$, $1 < p \le q < \infty$, $1/p - 1/q = \alpha/Q$, and from the space $M_{1,\varphi_1}(\mathbb{G})$ to the weak space $WM_{q,\varphi_2}(\mathbb{G})$, $1 \le q < \infty$, $1 - 1/q = \alpha/Q$. Also find conditions on the φ which ensure the Adams type boundedness of the M_α from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$ for $1 < p < q < \infty$ and from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$ for $1 < q < \infty$. In the case $b \in \mathrm{BMO}(\mathbb{G})$ and $1 < p < q < \infty$, find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the kth-order commutator operator $M_{b,\alpha,k}$ from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$ with $1/p - 1/q = \alpha/Q$. Also find the sufficient conditions on the φ which ensures the boundedness of the operator $M_{b,\alpha,k}$ from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$ for $1 < p < q < \infty$. In all the cases the conditions for the boundedness of M_α are given it terms of supremal-type inequalities on (φ_1, φ_2) and φ, which do not assume any assumption on monotonicity of (φ_1, φ_2) and φ in r. As applications we consider the Schrödinger operator $-\Delta_{\mathbb{G}} + V$ on $\mathbb{G}$, where the nonnegative potential V belongs to the reverse Hölder class $B_\infty(\mathbb{G})$. The $M_{p,\varphi_1} - M_{q,\varphi_2}$ estimates for the operators $V^\gamma(-\Delta_{\mathbb{G}} + V)^{-\beta}$ and $V^\gamma \nabla_{\mathbb{G}}(-\Delta_{\mathbb{G}} + V)^{-\beta}$ are obtained.

Key words Carnot group; fractional maximal function; generalized Morrey space; Schrödinger operator; BMO space

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1 Introduction

Carnot groups appear in quantum physics and many parts of mathematics, including Fourier analysis, several complex variables, geometry and topology. Analysis on the groups is also motivated by their role as the simplest and the most important model in the general theory of vector fields satisfying Hormander's condition. The simplest examples of the Carnot groups are Euclidean space $\mathbb{R}^n$, Heisenberg group $\mathbb{H}_n$ and (Heisenberg)-type groups introduced by Kaplan [14]. Carnot groups form a natural habitat for extensions of many of the objects studied in Euclidean space and find applications in the study of strongly pseudoconvex domains in complex analysis, semiclassical analysis of quantum mechanics, control theory, probability theory of degenerate diffusion processes and others.

In the present paper we will prove the boundedness of the fractional maximal operator and their commutators on the Carnot group in generalized Morrey spaces.

For $x \in \mathbb{G}$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Haar measure of the ball $B(x,r)$. Given a function f which is integrable on any ball $B(x,r) \subset \mathbb{G}$, the fractional maximal function $M_\alpha f$, $0 \le \alpha < Q$ of f is defined by

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{Q}} \int_{B(x,r)} |f(y)|\mathrm{d}y.$$

The fractional maximal function $M_\alpha f$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ (see [7, 25]). The operator M_α play important role in real and harmonic analysis (see, for example, [6, 7, 25]).

For a positive integer k and a function b, the kth-order commutator $M_{b,\alpha,k}$ of the fractional maximal operator with b (see [19]) is defined by

$$M_{b,\alpha,k}(f)(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{Q}} \int_{B(x,r)} |b(x)-b(y)|^k |f(y)|\mathrm{d}y.$$

In this work, we prove the boundedness of the fractional maximal operator M_α, $0 \le \alpha < Q$ from one generalized Morrey space $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$, $1 < p \le q < \infty$, $1/p - 1/q = \alpha/Q$, and from $M_{1,\varphi_1}(\mathbb{G})$ to the weak space $WM_{q,\varphi_2}(\mathbb{G})$, $1 \le q < \infty$, $1-1/q = \alpha/Q$. We also prove the Adams type boundedness of the operator M_α from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$ for $1 < p < q < \infty$ and from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$ for $1 < q < \infty$. In the case $b \in \mathrm{BMO}(\mathbb{G})$ and $1 < p < q < \infty$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the kth-order commutator operator $M_{b,\alpha,k}$ from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$ with $1/p - 1/q = \alpha/Q$. We also find the sufficient conditions on the φ which ensures the boundedness of the operator $M_{b,\alpha,k}$ from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$ for $1 < p < q < \infty$. In all the cases the conditions for the boundedness are given it terms of supremal-type inequalities on (φ_1, φ_2) and φ, which do not assume any assumption on monotonicity of (φ_1, φ_2) and φ in r.

Let $L = -\Delta_{\mathbb{G}} + V$ be a Schrödinger operator on $\mathbb{G}$, where $\Delta_{\mathbb{G}}$ is the sub-Laplacian and the nonnegative potential V belongs to the reverse Hölder class $B_\infty(\mathbb{G})$. As applications we establish the boundedness of the operators $V^\gamma(-\Delta_{\mathbb{G}} + V)^{-\beta}$ and $V^\gamma \nabla_{\mathbb{G}}(-\Delta_{\mathbb{G}} + V)^{-\beta}$ on generalized Morrey spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Notations

We begin with some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [7] and [28] for analysis on stratified groups.

Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that there is a direct sum vector space decomposition

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_m \tag{2.1}$$

so that each element of V_j, $2 \leq j \leq m$, is a linear combination of $(j-1)$th order commutator of elements of V_1. Equivalently, (2.1) is a stratification provided $[V_i, V_j] = V_{i+j}$ whenever $i+j \leq m$ and $[V_i, V_j] = 0$ otherwise. Let $X = X_1, \cdots, X_n$ be a basis for V_1 and X_{ij}, $1 \leq i \leq k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i$, $i = 1, \cdots, n$ and $k_1 = n$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in \mathbb{R}^N$, $1 \leq i \leq k_j$, $1 \leq j \leq m$, $N = \sum\limits_{j=1}^{m} k_j$, such that $g = \exp\left(\sum x_{ij} X_{ij}\right)$. A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by $|g| = \left(\sum |x_{ij}|^{2m!/j}\right)^{1/(2m!)}$, and $Q = \sum\limits_{j=1}^{m} j k_j$ is said to be the homogeneous dimension of $\mathbb{G}$, since $\mathrm{d}(\delta_r x) = r^Q \mathrm{d}x$ for $r > 0$. The dilation δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp\left(\sum r^j x_{ij} X_{ij}\right) \quad \text{if} \quad g = \exp\left(\sum x_{ij} X_{ij}\right).$$

The convolution operation on $\mathbb{G}$ is defined by

$$f * h(x) = \int_{\mathbb{G}} f(xy^{-1}) h(y) \mathrm{d}y = \int_{\mathbb{G}} f(y) h(y^{-1}x) \mathrm{d}y,$$

where y^{-1} is the inverse of y and xy^{-1} denotes the group multiplication of x by y^{-1}. It is known that for any left invariant vector field X on $\mathbb{G}$, $X(f * h) = f * (Xh)$.

Since $\mathbb{G}$ is nilpotent, the exponential map is diffeomorphism from $\mathbb{G}$ onto $\mathbb{G}$ which takes Lebesgue measure on $\mathbb{G}$ to a biinvariant Haar measure $\mathrm{d}x$ on $\mathbb{G}$. The group identity of $\mathbb{G}$ will be referred to as the origin and denoted by e.

A homogenous norm on $\mathbb{G}$ is a continuous function $x \to \rho(x)$ from $\mathbb{G}$ to $[0, \infty)$, which is C^∞ on $\mathbb{G} \backslash \{0\}$ and satisfies $\rho(x^{-1}) = \rho(x)$, $\rho(\delta_t x) = t\rho(x)$ for all $x \in \mathbb{G}$, $t > 0$; $\rho(e) = 0$ (the group identity). Moreover, there exists a constant $c_0 \geq 1$ such that $\rho(xy) \leq c_0 \left(\rho(x) + \rho(y)\right)$ for all $x, \ y \in G$.

We call a curve $\gamma : [a, b] \to \mathbb{G}$ a horizontal curve connecting two points $x, y \in \mathbb{G}$ if $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma'(t) \in V_1$ for all t. Then the Carnot-Caratheodory distance between x and y is defined as

$$d_{cc}(x, y) = \inf_{\gamma} \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}} \mathrm{d}t,$$

where the infimum is taken over all horizontal curves γ connecting x and y. It is known that any two points x, y on $\mathbb{G}$ can be joined by a horizontal curve of finite length and then d_{cc} is a left invariant metric on $\mathbb{G}$. We can define the metric ball centered at x and with radius r associated with this metric by

$$B_{cc}(x, r) = \{y \in \mathbb{G} : d_{cc}(x, y) < r\}.$$

We must notice that this metric d_{cc} is equivalent to the pseudo-metric $\rho(x,y) = |x^{-1}y|$ defined by the homogeneous norm $|\cdot|$ in the following sense (see [7]):

$$C^{-1}\rho(x,y) \le d_{cc}(x,y) \le C\rho(x,y).$$

We denote the metric ball associated with ρ as $D(x,r) = \{y \in \mathbb{G} : \rho(x,y) < r\}$. An important feature of both of these distance functions is that these distances and thus the associated metric balls are left invariant, namely,

$$d_{cc}(zx,zy) = d_{cc}(x,y), \quad B_{cc}(x,r) = xB_{cc}(e,r)$$

and

$$\rho(zx,zy) = \rho(x,y), \quad D(x,r) = xD(e,r).$$

From now on, we will always use the metric d_{cc} and drop the subscript from d_{cc}. Similarly, we will use $B(x,r)$ to denote $B_{cc}(x,r)$.

With this norm, we define the $\mathbb{G}$ - ball centered at x with radius r by $D(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\}$, and we denote by $D_r = D(e,r) = \{y \in \mathbb{G} \ : \ \rho(y) < r\}$ the open ball centered at e, the identity element of $\mathbb{G}$, with radius r.

One easily recognizes that there exist $c_1 = c_1(\mathbb{G})$, and $c_2 = c_2(\mathbb{G})$ such that

$$|B(x,r)| = c_1\, r^Q, \quad |D(x,r)| = c_2\, r^Q, \quad x \in \mathbb{G}, \ r > 0.$$

The most basic partial differential operator in a Carnot group is the sub- Laplacian associated with X is the second-order partial differential operator on G given by $\mathcal{L} = \sum\limits_{i=1}^{n} X_i^2$.

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(\mathbb{G})$ play an important role, see [8]. They were introduced by Morrey in 1938 [20]. The Morrey space in a Carnot group is defined as follows: for $1 \le p \le \infty$, $0 \le \lambda \le Q$, a function $f \in L_{p,\lambda}(\mathbb{G})$ if $f \in L_p^{\rm loc}(\mathbb{H}_n)$ and

$$\|f\|_{L_{p,\lambda}} := \sup_{u \in \mathbb{G},\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty$$

(if $\lambda = 0$, then $L_{p,0}(\mathbb{G}) = L_p(\mathbb{G})$; if $\lambda = Q$, then $L_{p,Q}(\mathbb{G}) = L_\infty(\mathbb{G})$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda}(\mathbb{G}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{G}$.)

We also denote by $WL_{p,\lambda}(\mathbb{G})$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{H}_n)$ for which

$$\|f\|_{WL_{p,\lambda}} = \sup_{x \in \mathbb{G},\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} = \sup_{\tau>0} \tau \left|\{y \in B(x,r) : |f(y)| > \tau\}\right|^{1/p}.$$

Note that

$$WL_p(\mathbb{G}) = WL_{p,0}(\mathbb{G}), \ L_{p,\lambda}(\mathbb{G}) \subset WL_{p,\lambda}(\mathbb{G}) \ \text{ and } \ \|f\|_{WL_{p,\lambda}} \le \|f\|_{L_{p,\lambda}}.$$

Everywhere in the sequel the functions $\varphi(x,r)$, $\varphi_1(x,r)$ and $\varphi_2(x,r)$ used in the body of the paper, are non-negative measurable function on $\mathbb{G} \times (0,\infty)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1 Let $1 \le p < \infty$. The generalized Morrey space $M_{p,\varphi}(\mathbb{G})$ is defined as the space of all functions $f \in L_p^{\rm loc}(\mathbb{G})$ having the following finite norm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{G}, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

According to this definition, we recover the space $L_{p,\lambda}(\mathbb{G})$ under the choice $\varphi(x,r) = r^{\frac{\lambda-Q}{p}}$:

$$L_{p,\lambda}(\mathbb{G}) = M_{p,\varphi}(\mathbb{G})\Big|_{\varphi(x,r)=r^{\frac{\lambda-Q}{p}}}.$$

In [9, 10, 21] and [22] there were obtained sufficient conditions on weights φ_1 and φ_2 for the boundedness of the maximal operator M and the singular integral operators T from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{p,\varphi_2}(\mathbb{G})$. In [21, 22] the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,\tau) \le c\,\varphi(x,r), \tag{2.2}$$

whenever $r \le \tau \le 2r$, where $c(\ge 1)$ does not depend on t, r and $x \in \mathbb{G}$, jointly with the condition:

$$\int_r^\infty \varphi(x,\tau)^p \frac{{\rm d}\tau}{\tau} \le C\,\varphi(x,r)^p \tag{2.3}$$

for the maximal or singular operators and the condition

$$\int_r^\infty \tau^{\alpha p}\varphi(x,\tau)^p \frac{{\rm d}\tau}{\tau} \le C\,r^{\alpha p}\varphi(x,r)^p \tag{2.4}$$

for potential and fractional maximal operators, where $C(>0)$ does not depend on r and $x \in \mathbb{G}$.

In [22] the following statements was proved.

Theorem 2.1 Let $1 \le p < \infty, 0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and $\varphi(x,\tau)$ satisfy conditions (2.2) and (2.4). Then for $p > 1$ the operator M_α is bounded from $M_{p,\varphi}(\mathbb{G})$ to $M_{q,\varphi}(\mathbb{G})$ and for $p = 1$ M_α is bounded from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,\varphi}(\mathbb{G})$.

The following statement, containing results obtained in [22] was proved in [9] (see also [10–12]).

Theorem 2.2 Let $1 \le p < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and (φ_1, φ_2) satisfy the condition

$$\int_\tau^\infty r^\alpha \varphi_1(x,r) \frac{{\rm d}r}{r} \le C\,\varphi_2(x,\tau), \tag{2.5}$$

where C does not depend on x and τ. Then the operator M_α is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$ for $p = 1$.

3 Boundedness of the Fractional Maximal Operator in the Spaces $M_{p,\varphi}(\mathbb{G})$

3.1 Spanne Type Result

We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [4].

Theorem 3.1 Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0,\infty)$.

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\|v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\right)\right\|_{L_\infty(0,\infty)} < \infty. \tag{3.1}$$

Sufficient conditions on φ for the boundedness of M and M_α in generalized Morrey spaces $\mathcal{M}_{p,\varphi}(\mathbb{G})$ were obtained in [2, 4, 11–13, 22].

The following lemma is true.

Lemma 3.1 Let $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then for $p > 1$ and any ball $B = B(x,r)$ in $\mathbb{G}$ the inequality

$$\|M_\alpha f\|_{L_q(B(x,r))} \lesssim \|f\|_{L_p(B(x,2r))} + r^{\frac{Q}{q}} \sup_{\tau > 2r} \tau^{-Q+\alpha} \|f\|_{L_1(B(x,\tau))} \tag{3.2}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{G})$.

Moreover for $p = 1$ the inequality

$$\|M_\alpha f\|_{WL_q(B(x,r))} \lesssim \|f\|_{L_1(B(x,2r))} + r^{\frac{Q}{q}} \sup_{\tau > 2r} \tau^{-Q+\alpha} \|f\|_{L_1(B(x,\tau))} \tag{3.3}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{G})$.

Proof Let $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$. For arbitrary ball $B = B(x,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$,

$$\|M_\alpha f\|_{L_q(B)} \le \|M_\alpha f_1\|_{L_q(B)} + \|M_\alpha f_2\|_{L_q(B)}.$$

By the continuity of the operator $M_\alpha : L_p(\mathbb{G}) \to L_q(\mathbb{G})$ (see, for example, [7]) we have

$$\|M_\alpha f_1\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)}.$$

Let y be an arbitrary point from B. If $B(y,\tau) \cap {}^{\complement}(2B) \ne \emptyset$, then $\tau > r$. Indeed, if $z \in B(y,\tau) \cap {}^{\complement}(2B)$, then $\tau > |y^{-1}z| \ge |x^{-1}z| - |x^{-1}y| > 2r - r = r$.

On the other hand, $B(y,\tau) \cap {}^{\complement}(2B) \subset B(x,2\tau)$. Indeed, $z \in B(y,\tau) \cap {}^{\complement}(2B)$, then we get $|x^{-1}z| \le |y^{-1}z| + |x^{-1}y| < \tau + r < 2\tau$.

Hence

$$
\begin{aligned}
M_\alpha f_2(y) &= \sup_{\tau>0} \frac{1}{|B(y,\tau)|^{1-\alpha/Q}} \int_{B(y,\tau)\cap \complement(2B)} |f(z)|\mathrm{d}z \\
&\le 2^{Q-\alpha} \sup_{\tau>r} \frac{1}{|B(x,2\tau)|^{1-\alpha/Q}} \int_{B(x,2\tau)} |f(z)|\mathrm{d}z \\
&= 2^{Q-\alpha} \sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |f(z)|\mathrm{d}z.
\end{aligned}
$$

Therefore, for all $y \in B$ we have

$$
M_\alpha f_2(y) \le 2^{Q-\alpha} \sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |f(z)|\mathrm{d}z. \tag{3.4}
$$

Thus

$$
\|M_\alpha f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + |B|^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |f(z)|\mathrm{d}z \Big).
$$

Let $p = 1$. It is obvious that for any ball $B = B(x,r)$,

$$
\|M_\alpha f\|_{WL_q(B)} \le \|M_\alpha f_1\|_{WL_q(B)} + \|M_\alpha f_2\|_{WL_q(B)}.
$$

By the continuity of the operator $M_\alpha : L_1(\mathbb{G}) \to WL_q(\mathbb{G})$ we have

$$
\|M_\alpha f_1\|_{WL_q(B)} \lesssim \|f\|_{L_1(2B)}.
$$

Then by (3.4) we get inequality (3.3). □

Lemma 3.2 Let $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then for $p > 1$ and any ball $B = B(x,r)$ in $\mathbb{G}$, the inequality

$$
\|M_\alpha f\|_{L_q(B(x,r))} \lesssim r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{-\frac{Q}{q}} \|f\|_{L_p(B(x,\tau))} \tag{3.5}
$$

holds for all $f \in L_p^{\rm loc}(\mathbb{G})$.

Moreover for $p = 1$ the inequality

$$
\|M_\alpha f\|_{WL_q(B(x,r))} \lesssim r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{-\frac{Q}{q}} \|f\|_{L_1(B(x,\tau))} \tag{3.6}
$$

holds for all $f \in L_1^{\rm loc}(\mathbb{G})$.

Proof Let $1 < p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Denote

$$
\begin{aligned}
&\mathcal{M}_1 := |B|^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |f(z)|\mathrm{d}z \Big), \\
&\mathcal{M}_2 := \|f\|_{L_p(2B)}.
\end{aligned}
$$

Applying Hölder's inequality, we get

$$
\mathcal{M}_1 \lesssim |B|^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{\frac{1}{q}}} \Big(\int_{B(x,\tau)} |f(z)|^p \mathrm{d}z \Big)^{\frac{1}{p}} \Big).
$$

On the other hand,

$$\begin{aligned}&|B|^{\frac{1}{q}}\Big(\sup_{\tau>2r}\frac{1}{|B(x,\tau)|^{\frac{1}{q}}}\Big(\int_{B(x,\tau)}|f(z)|^p\mathrm{d}z\Big)^{\frac{1}{p}}\Big)\\ \gtrsim\ &|B|^{\frac{1}{q}}\Big(\sup_{\tau>2r}\frac{1}{|B(x,\tau)|^{\frac{1}{q}}}\Big)\|f\|_{L_p(2B)}\approx\mathcal{M}_2.\end{aligned}$$

Since by Lemma 3.1

$$\|M_\alpha f\|_{L_q(B)}\le\mathcal{M}_1+\mathcal{M}_2,$$

we arrive at (3.5).

Let $p=1$. Inequality (3.6) directly follows from (3.3). □

Theorem 3.2 Let $1\le p<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, and (φ_1,φ_2) satisfies the condition

$$\sup_{r<t<\infty}t^{\alpha-\frac{Q}{p}}\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)\,s^{\frac{Q}{p}}\le C\,\varphi_2(x,r),\tag{3.7}$$

where C does not depend on x and r. Then for $p>1$, M_α is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$ and for $p=1$, M_α is bounded from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$.

Proof By Theorem 3.1 and Lemma 3.2 we get

$$\begin{aligned}\|M_\alpha f\|_{M_{q,\varphi_2}}&\lesssim\sup_{x\in\mathbb{G},r>0}\varphi_2(x,r)^{-1}\sup_{\tau>r}\tau^{-\frac{Q}{q}}\|f\|_{L_p(B(x,\tau))}\\&\lesssim\sup_{x\in\mathbb{G},r>0}\varphi_1(x,r)^{-1}\,r^{-\frac{Q}{p}}\|f\|_{L_p(B(x,r))}\\&=\|f\|_{M_{p,\varphi_1}},\end{aligned}$$

if $p\in(1,\infty)$ and

$$\begin{aligned}\|M_\alpha f\|_{WM_{q,\varphi_2}}&\lesssim\sup_{x\in\mathbb{G},r>0}\varphi_2(x,r)^{-1}\sup_{\tau>r}\tau^{-\frac{Q}{q}}\|f\|_{L_1(B(x,\tau))}\\&\lesssim\sup_{x\in\mathbb{G},r>0}\varphi_1(x,r)^{-1}\,r^{-Q}\|f\|_{L_1(B(x,r))}\\&=\|f\|_{M_{1,\varphi_1}},\end{aligned}$$

if $p=1$. □

In the case $\alpha=0$ and $p=q$ from Theorem 3.2 we get the following corollary, which proven in [2] on $\mathbb{R}^n$.

Corollary 3.1 Let $1\le p<\infty$ and (φ_1,φ_2) satisfies the condition

$$\sup_{r<t<\infty}t^{-\frac{Q}{p}}\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)\,s^{\frac{Q}{p}}\le C\,\varphi_2(x,r),\tag{3.8}$$

where C does not depend on x and r. Then for $p>1$, M is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{p,\varphi_2}(\mathbb{G})$ and for $p=1$, M is bounded from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{1,\varphi_2}(\mathbb{G})$.

Corollary 3.2 Let $p\in[1,\infty)$ and let $\varphi:(0,\infty)\to(0,\infty)$ be an decreasing function. Assume that the mapping $r\mapsto\varphi(r)\,r^{\frac{Q}{p}}$ is almost increasing (there exists a positive constant c such that for $s<r$ we have $\varphi(s)\,s^{\frac{Q}{p}}\le c\varphi(r)\,r^{\frac{Q}{p}}$). Then there exists a constant $C>0$ such that

$$\|Mf\|_{\mathcal{M}_{p,\varphi}}\le C\|f\|_{\mathcal{M}_{p,\varphi}}\quad\text{if}\quad p>1,$$

and

$$\|Mf\|_{W\mathcal{M}_{1,\varphi}} \le C\|f\|_{\mathcal{M}_{1,\varphi}}.$$

3.2 Adams Type Result

The following is a result of Adams type for the fractional maximal operator (see [1]).

Theorem 3.3 Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and let $\varphi(x,\tau)$ satisfy the condition

$$\sup_{r<t<\infty} t^{-Q} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s)\, s^Q \le C\,\varphi(x,r) \tag{3.9}$$

and

$$\sup_{r<\tau<\infty} t^{\alpha}\varphi(x,\tau)^{\frac{1}{p}} \le Cr^{-\frac{\alpha p}{q-p}}, \tag{3.10}$$

where C does not depend on $x \in \mathbb{G}$ and $r > 0$.

Then the operator M_α is bounded from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$.

Proof Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$. Write $f = f_1 + f_2$, where $B = B(x,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

For $M_\alpha f_2(y)$ for all $y \in B$ from (3.4) we have

$$\begin{aligned} M_\alpha(f_2)(y) &\le 2^{Q-\alpha} \sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |f(z)|\mathrm{d}z \\ &\lesssim \sup_{\tau>2r} \tau^{-\frac{Q}{q}} \|f\|_{L_p(B(x,\tau))}. \end{aligned} \tag{3.11}$$

Then from conditions (3.10) and (3.11) we get

$$\begin{aligned} M_\alpha f(y) &\lesssim r^\alpha\, Mf(y) + \sup_{\tau>2r} t^{\alpha-\frac{Q}{p}} \|f\|_{L_p(B(x,\tau))} \\ &\le r^\alpha\, Mf(y) + \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{\tau>2r} \tau^\alpha \varphi(x,\tau)^{\frac{1}{p}} \\ &\lesssim r^\alpha\, Mf(y) + r^{-\frac{\alpha p}{q-p}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned}$$

Hence choose $r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p}}}}{Mf(y)}\right)^{\frac{q-p}{\alpha q}}$ for every $y \in B$, we have

$$|M_\alpha f(y)| \lesssim (Mf(y))^{\frac{p}{q}}\, \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ provided by Corollary 3.1 in virtue of condition (3.9).

$$\begin{aligned} \|M_\alpha f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{G},\,\tau>0} \varphi(x,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|M_\alpha f\|_{L_q(B(x,\tau))} \\ &\lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{x\in\mathbb{G},\,\tau>0} \varphi(x,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|Mf\|^{\frac{p}{q}}_{L_p(B(x,\tau))} \\ &= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \left(\sup_{x\in\mathbb{G},\,\tau>0} \varphi(x,\tau)^{-\frac{1}{p}} \tau^{-\frac{Q}{p}} \|Mf\|_{L_p(B(x,\tau))} \right)^{\frac{p}{q}} \\ &= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \|Mf\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \\ &\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned}\|M_\alpha f\|_{WM_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{G},\,\tau>0} \varphi(x,\tau)^{-\frac{1}{q}} t^{-\frac{Q}{q}} \|M_\alpha f\|_{WL_q(B(x,\tau))} \\ &\lesssim \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \sup_{x\in\mathbb{G},\,\tau>0} \varphi(x,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|Mf\|_{WL_1(B(x,\tau))}^{\frac{1}{q}} \\ &= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \left(\sup_{x\in\mathbb{G},\,\tau>0} \varphi(x,\tau)^{-1} \tau^{-Q} \|Mf\|_{WL_1(B(x,\tau))} \right)^{\frac{1}{q}} \\ &= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \|Mf\|_{WM_{1,\varphi}}^{\frac{1}{q}} \\ &\lesssim \|f\|_{M_{1,\varphi}},\end{aligned}$$

if $1 < q < \infty$. □

In the case $\varphi(x,r) = r^{\lambda-Q}$, $0 < \lambda < Q$ from Theorem 3.3 we get the following Adams type result [1] for the fractional maximal operator.

Corollary 3.3 Let $0 < \alpha < Q$, $1 \le p < \frac{Q}{\alpha}$, $0 < \lambda < Q - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$. Then for $p > 1$, the operator M_α is bounded from $L_{p,\lambda}(\mathbb{G})$ to $L_{q,\lambda}(\mathbb{G})$ and for $p = 1$, M_α is bounded from $L_{1,\lambda}(\mathbb{G})$ to $WL_{q,\lambda}(\mathbb{G})$.

4 Commutators of Fractional Maximal Operators in the Spaces $M_{p,\varphi}(\mathbb{G})$

4.1 Spanne Type Result

First we introduce the definition of the space of BMO($\mathbb{G}$) (see, for example, [7, 19, 23, 26]).

Definition 4.1 Suppose that $f \in L_1^{\rm loc}(\mathbb{G})$, and let

$$\|f\|_* = \sup_{x\in\mathbb{G},r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \mathrm{d}y < \infty,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \mathrm{d}y.$$

Define

$$\mathrm{BMO}(\mathbb{G}) = \{ f \in L_1^{\rm loc}(\mathbb{G}) \ : \ \|f\|_* < \infty \}.$$

If one regards two functions whose difference is a constant as one, then space BMO($\mathbb{G}$) is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 4.1 [7, 26] (1) The John-Nirenberg inequality : there are constants C_1, $C_2 > 0$, such that for all $f \in \mathrm{BMO}(\mathbb{G})$ and $\beta > 0$

$$|\{x \in B \ : \ |f(x) - f_B| > \beta\}| \le C_1 |B| e^{-C_2\beta/\|f\|_*}, \quad \forall B \subset \mathbb{G}.$$

(2) The John-Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{G},r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p \mathrm{d}y \right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(3) Let $f \in \mathrm{BMO}(\mathbb{G})$. Then there is a constant $C > 0$ such that

$$\left|f_{B(x,r)} - f_{B(x,\tau)}\right| \le C\|f\|_* \log\frac{\tau}{r} \quad \text{for} \quad 0 < 2r < \tau, \tag{4.2}$$

where C is independent of f, x, r and τ.

For the kth-order commutator of the fractional maximal operator $M_{b,\alpha,k}$.

Lemma 4.1 Let $1 < p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $b \in \mathrm{BMO}(\mathbb{G})$.

Then the inequality

$$\|M_{b,\alpha,k}f\|_{L_q(B(x,r))} \lesssim \|b\|_*^k\, r^{\frac{Q}{q}} \sup_{\tau>2r} \log^k\left(e + \frac{\tau}{r}\right)\tau^{-\frac{Q}{q}}\, \|f\|_{L_p(B(x,\tau))}$$

holds for any ball $B(x,r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{G})$.

Proof Let $1 < p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Write $f = f_1 + f_2$, where $B = B(x,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{{}^{\complement}(2B)}$. Hence

$$\|M_{b,\alpha,k}f\|_{L_q(B)} \le \|M_{b,\alpha,k}f_1\|_{L_q(B)} + \|M_{b,\alpha,k}f_2\|_{L_q(B)}.$$

From the boundedness of $M_{b,\alpha,k}$ from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ (see, for example, [3, 7, 26]) it follows that:

$$\|M_{b,\alpha,k}f_1\|_{L_q(B)} \le \|M_{b,\alpha,k}f_1\|_{L_q(\mathbb{G})} \lesssim \|b\|_*^k\|f_1\|_{L_p(\mathbb{G})} = \|b\|_*^k\|f\|_{L_p(2B)}.$$

For $z \in B$ we have

$$\begin{aligned} M_{b,\alpha,k}f_2(z) &\lesssim \sup_{\tau>0}\frac{1}{|B(z,\tau)|^{1-\alpha/Q}}\int_{B(z,\tau)}|b(y)-b(z)|^k\,|f_2(y)|\mathrm{d}y \\ &= \sup_{\tau>0}\frac{1}{|B(z,\tau)|^{1-\alpha/Q}}\int_{B(z,\tau)\cap{}^{\complement}(2B)}|b(y)-b(z)|^k\,|f(y)|\mathrm{d}y. \end{aligned}$$

Let z be an arbitrary point from B. If $B(z,\tau)\cap{}^{\complement}(2B) \ne \emptyset$, then $\tau > r$. Indeed, if $y \in B(z,\tau)\cap{}^{\complement}(2B)$, then $\tau > |y^{-1}z| \ge |x^{-1}z| - |x^{-1}y| > 2r - r = r$.

On the other hand, $B(z,\tau)\cap{}^{\complement}(2B) \subset B(x,2\tau)$. Indeed, $y \in B(z,\tau)\cap{}^{\complement}(2B)$, then we get $|x^{-1}y| \le |y^{-1}z| + |x^{-1}z| < \tau + r < 2\tau$.

Hence

$$\begin{aligned} M_{b,\alpha,k}(f_2)(z) &= \sup_{\tau>0}\frac{1}{|B(z,\tau)|^{1-\alpha/Q}}\int_{B(z,\tau)\cap{}^{\complement}(2B)}|b(y)-b(z)|^k\,|f(y)|\mathrm{d}y \\ &\le 2^{Q-\alpha}\sup_{\tau>r}\frac{1}{|B(x,2\tau)|^{1-\alpha/Q}}\int_{B(x,2\tau)}|b(y)-b(z)|^k\,|f(h)|\mathrm{d}h \\ &= 2^{Q-\alpha}\sup_{\tau>2r}\frac{1}{|B(x_0,\tau)|^{1-\alpha/Q}}\int_{B(x_0,\tau)}|b(y)-b(z)|^k\,|f(y)|\mathrm{d}y. \end{aligned}$$

Therefore, for all $z \in B$ we have

$$M_{b,\alpha,k}(f_2)(z) \le 2^{Q-\alpha}\sup_{\tau>2r}\frac{1}{|B(x,\tau)|^{1-\alpha/Q}}\int_{B(x,\tau)}|b(y)-b(z)|^k\,|f(y)|\mathrm{d}y. \tag{4.3}$$

Then

$$\|M_{b,\alpha,k}f_2\|_{L_q(B)} \lesssim \left(\int_B\left(\sup_{\tau>2r}\frac{1}{|B(x,\tau)|^{1-\alpha/Q}}\int_{B(x,\tau)}|b(y)-b(z)|^k\,|f(y)|\mathrm{d}y\right)^q\mathrm{d}g\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\lesssim \Big(\int_B \Big(\sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |b(y)-b_B|^k\,|f(y)| \mathrm{d}y \Big)^q \mathrm{d}g \Big)^{\frac{1}{q}} \\
&\quad + \Big(\int_B \Big(\sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |b(z)-b_B|^k\,|f(y)| \mathrm{d}y \Big)^q \mathrm{d}g \Big)^{\frac{1}{q}} \\
&= J_1 + J_2.
\end{aligned}
$$

Let us estimate J_1.

$$
\begin{aligned}
J_1 &= r^{\frac{Q}{q}} \sup_{\tau>2r} \frac{1}{|B(x,\tau)|^{1-\alpha/Q}} \int_{B(x,\tau)} |b(y)-b_B|^k\,|f(y)| \mathrm{d}y \\
&\approx r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\alpha-Q} \int_{B(x,\tau)} |b(y)-b_B|^k\,|f(y)| \mathrm{d}y.
\end{aligned}
$$

Applying Hölder's inequality and by (4.1), (4.2), we get

$$
\begin{aligned}
J_1 &\lesssim r^{\frac{Q}{q}} \sup_{\tau>2r} t^{\alpha-Q} \int_{B(x,\tau)} |b(y)-b_{B(x,\tau)}|^k\,|f(y)| \mathrm{d}y \\
&\quad + r^{\frac{Q}{q}} \sup_{\tau>2r} t^{\alpha-Q} |b_{B(x,r)}-b_{B(x,\tau)}|^k \int_{B(x,\tau)} |f(y)| \mathrm{d}y \\
&\lesssim r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\alpha-\frac{Q}{p}} \Big(\frac{1}{|B(x,\tau)|} \int_{B(x,\tau)} |b(y)-b_{B(x,\tau)}|^{kp'} \mathrm{d}y \Big)^{\frac{1}{p'}} \|f\|_{L_p(B(x,\tau))} \\
&\quad + r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\alpha-\frac{Q}{p}} |b_{B(x,r)}-b_{B(x,\tau)}|^k \|f\|_{L_p(B(x,\tau))} \\
&\lesssim \|b\|_*^k\, r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\frac{Q}{q}} \Big(1+\log\frac{\tau}{r}\Big)^k \|f\|_{L_p(B(x,\tau))}.
\end{aligned}
$$

In order to estimate J_2 note that

$$
\begin{aligned}
J_2 &= \Big(\int_B |b(z)-b_B|^{kq} \mathrm{d}z \Big)^{\frac{1}{q}} \sup_{\tau>2r} \tau^{\alpha-Q} \int_{B(x,\tau)} |f(y)| \mathrm{d}y \\
&\lesssim \|b\|_*^k\, r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\frac{Q}{q}} \|f\|_{L_p(B(x,\tau))}.
\end{aligned}
$$

Summing up J_1 and J_2, for all $p\in(1,\infty)$ we get

$$
\|M_{b,\alpha,k} f_2\|_{L_q(B)} \lesssim \|b\|_*^k\, r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\frac{Q}{q}} \Big(1+\log\frac{\tau}{r}\Big)^k \|f\|_{L_p(B(x,\tau))}. \tag{4.4}
$$

Finally,

$$
\begin{aligned}
\|M_{b,\alpha,k} f\|_{L_q(B)} &\lesssim \|b\|_*^k \|f\|_{L_p(2B)} + \|b\|_*^k\, r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\frac{Q}{q}} \Big(1+\log\frac{\tau}{r}\Big)^k \|f\|_{L_p(B(x,\tau))} \\
&\lesssim \|b\|_*^k\, r^{\frac{Q}{q}} \sup_{\tau>2r} \tau^{\frac{Q}{q}} \log^k \Big(e+\frac{\tau}{r}\Big) \|f\|_{L_p(B(x,\tau))}.
\end{aligned}
$$

□

The following theorem is true.

Theorem 4.1 Let $1<p<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $b\in \mathrm{BMO}(\mathbb{G})$ and (φ_1,φ_2) satisfies the condition

$$
\sup_{r<t<\infty} t^{\alpha-\frac{Q}{p}} \log^k \Big(e+\frac{t}{r}\Big) \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{Q}{p}} \le C\, \varphi_2(x,r), \tag{4.5}
$$

where C does not depend on x and r.

Then the operator $M_{b,\alpha,k}$ is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$. Moreover

$$\|M_{b,\alpha,k}f\|_{M_{q,\varphi_2}} \lesssim \|b\|_*^k \, \|f\|_{M_{p,\varphi_1}}.$$

Proof By Theorem 3.1 and Lemma 4.1 we get

$$\begin{aligned}\|M_{b,\alpha,k}f\|_{M_{q,\varphi_2}} &\lesssim \|b\|_*^k \sup_{x\in\mathbb{G},r>0} \varphi_2(x,r)^{-1} \sup_{\tau>r} \log^k\left(e+\frac{\tau}{r}\right)\tau^{-\frac{Q}{q}}\,\|f\|_{L_p(B(x,\tau))} \\ &\lesssim \|b\|_*^k \sup_{x\in\mathbb{G},r>0} \varphi_1(x,r)^{-1}\, r^{-\frac{Q}{p}}\,\|f\|_{L_p(B(x,r))} = \|f\|_{M_{p,\varphi_1}}.\end{aligned}$$

□

In the case $\alpha=0$ and $p=q$ from Theorem 4.1 we get the following corollary.

Corollary 4.1 Let $1<p<\infty$, $b\in \mathrm{BMO}(\mathbb{G})$ and (φ_1,φ_2) satisfies the condition

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}}\, \log^k\left(e+\frac{t}{r}\right) \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{Q}{p}} \le C\,\varphi_2(x,r), \tag{4.6}$$

where C does not depend on x and r.

Then the operator $M_{b,k}$ is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{p,\varphi_2}(\mathbb{G})$. Moreover

$$\|M_{b,k}f\|_{M_{p,\varphi_2}} \lesssim \|b\|_*^k \, \|f\|_{M_{p,\varphi_1}}.$$

4.2 Adams Type Result

The following is a result of Adams type.

Theorem 4.2 Let $1<p<q<\infty$, $0<\alpha<\frac{Q}{p}$, $b\in \mathrm{BMO}(\mathbb{G})$ and let $\varphi(x,r)$ satisfy the conditions

$$\sup_{r<t<\infty} t^{-Q}\, \log^{kp}\left(e+\frac{t}{r}\right) \operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s)\, s^{Q} \le C\,\varphi(x,r) \tag{4.7}$$

and

$$\sup_{r<\tau<\infty} \log^k\left(e+\frac{t}{r}\right)\tau^{\alpha}\varphi(x,\tau)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{4.8}$$

where C does not depend on $x\in\mathbb{G}$ and $r>0$.

Then $M_{b,\alpha,k}$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{G})$.

Proof Let $1<p<q<\infty$, $0<\alpha<\frac{Q}{p}$ and $f\in M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$. For arbitrary $x\in\mathbb{G}$, set $B=B(x,r)$ for the ball centered at x and of radius r. Write $f=f_1+f_2$ with $f_1=f\chi_{2B}$ and $f_2=f\chi_{\complement_{(2B)}}$.

For $z\in B$ we have

$$\begin{aligned}M_{b,\alpha,k}f_2(z) &\lesssim \sup_{\tau>0} \tau^{\alpha-Q}\int_{B(z,\tau)} |b(y)-b(z)|^k\, f_2(y)|\mathrm{d}y \\ &\approx \sup_{\tau>2r} \tau^{\alpha-Q}\int_{B(z,\tau)} |b(y)-b(z)|^k\, f_2(y)|\mathrm{d}y.\end{aligned}$$

Analogously Section 4.1, for all $p\in(1,\infty)$ and $z\in B$ we get

$$M_{b,\alpha,k}f_2(z) \lesssim \sup_{\tau>2r} \tau^{\alpha-\frac{Q}{p}}\left(1+\log\frac{\tau}{r}\right)^k \|f\|_{L_p(B(x,\tau))}. \tag{4.9}$$

Then from conditions (4.8) and (4.9) we get

$$\begin{aligned} M_{b,\alpha,k}f(z) &\lesssim r^{\alpha}\, M_{b,k}f(z) + \|b\|_*^k \sup_{\tau>2r} \tau^{\alpha-\frac{Q}{p}} \Big(1+\log\frac{\tau}{r}\Big)^k \|f\|_{L_p(B(x,\tau))} \\ &\le r^{\alpha}\, M_{b,k}f(z) + \|b\|_*^k \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{\tau>r} \Big(1+\log\frac{\tau}{r}\Big)^k \tau^{\alpha}\, \varphi(x,\tau)^{\frac{1}{p}} \\ &\lesssim r^{\alpha}\, M_{b,k}f(z) + \|b\|_*^k\, r^{-\frac{\alpha p}{q-p}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned} \tag{4.10}$$

Hence choose $r = \Big(\frac{\|b\|_*^k \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}}{M_{b,k}f(z)}\Big)^{\frac{q-p}{\alpha q}}$ for every $z\in B$, we have

$$M_{b,\alpha,k}f(z) \lesssim \|b\|_*^{k\left(1-\frac{p}{q}\right)}\, (M_{b,k}f(z))^{\frac{p}{q}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Hence the statement of the theorem follows in view of the boundedness of the commutator of maximal operator $M_{b,k}$ in $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{G})$ provided by Corollary 4.1 in virtue of condition (4.7).

$$\begin{aligned} \|M_{b,\alpha,k}f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{G},\, r>0} \varphi(x,r)^{-\frac{1}{q}} r^{-\frac{Q}{q}} \|M_{b,\alpha,k}f\|_{L_q(B(x,r))} \\ &\lesssim \|b\|_*^{k\left(1-\frac{p}{q}\right)} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{G},\, r>0} \varphi(x,r)^{-\frac{1}{q}} r^{-\frac{Q}{q}} \|M_{b,k}f\|_{L_p(B(x,r))}^{\frac{p}{q}} \\ &= \|b\|_*^{k\left(1-\frac{p}{q}\right)} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \Big(\sup_{x\in\mathbb{G},\, r>0} \varphi(x,r)^{-\frac{1}{p}} r^{-\frac{Q}{p}} \|M_{b,k}f\|_{L_p(B(x,r))}\Big)^{\frac{p}{q}} \\ &= \|b\|_*^{k\left(1-\frac{p}{q}\right)} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|M_{b,k}f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}} \\ &\lesssim \|b\|_*^k \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned}$$

□

In the case $\varphi(x,r)=r^{\lambda-Q}$, $0<\lambda<Q$ from Theorem 4.2 we get the following Adams type result for the commutator of fractional maximal operator.

Corollary 4.2 Let $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $0<\lambda<Q-\alpha p$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q-\lambda}$ and $b\in \mathrm{BMO}(\mathbb{G})$. Then, the operator $M_{b,\alpha,k}$ is bounded from $L_{p,\lambda}(\mathbb{G})$ to $L_{q,\lambda}(\mathbb{G})$.

5 The Generalized Morrey Estimates for the Operators $V^{\gamma}(-\Delta_{\mathbb{G}}+V)^{-\beta}$ and $V^{\gamma}\nabla_{\mathbb{G}}(-\Delta_{\mathbb{G}}+V)^{-\beta}$

In this section we consider the Schrödinger operator $-\Delta_{\mathbb{G}}+V$ on $\mathbb{G}$, where the nonnegative potential V belongs to the reverse Hölder class $B_{\infty}(\mathbb{G})$. The generalized Morrey $M_{p,\varphi}(\mathbb{G})$ estimates for the operators $V^{\gamma}(-\Delta_{\mathbb{G}}+V)^{-\beta}$ and $V^{\gamma}\nabla_{\mathbb{G}}(-\Delta_{\mathbb{G}}+V)^{-\beta}$ are obtained.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [5, 24, 29]). Shen [24] studied the Schrödinger operator $-\Delta+V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q\ge n/2$ and he proved the L_p boundedness of the operators $(-\Delta+V)^{i\gamma}$, $\nabla^2(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$ and $\nabla(-\Delta+V)^{-1}$. Kurata and Sugano generalized Shens results to uniformly elliptic operators in [15].

Sugano [27] also extended some results of Shen to the operator $V^{\gamma}(-\Delta+V)^{-\beta}$, $0\le\gamma\le\beta\le 1$ and $V^{\gamma}\nabla(-\Delta+V)^{-\beta}$, $0\le\gamma\le\frac{1}{2}\le\beta\le 1$ and $\beta-\gamma\ge\frac{1}{2}$. Later, Li [16], Liu [17] and Lu [18] investigated the Schrödinger operators in a more general setting.

The main purpose of this section is investigate the generalized Morrey M_{p,φ_1}-M_{q,φ_2} boundedness of the operators

$$T_1=V^{\gamma}(-\Delta_{\mathbb{G}}+V)^{-\beta},\quad 0\le\gamma\le\beta\le 1,$$
$$T_2=V^{\gamma}\nabla_{\mathbb{G}}(-\Delta_{\mathbb{G}}+V)^{-\beta},\quad 0\le\gamma\le\frac{1}{2}\le\beta\le 1,\ \beta-\gamma\ge\frac{1}{2}.$$

Note that the operators $V(-\Delta_{\mathbb{G}}+V)^{-1}$ and $V^{\frac{1}{2}}\nabla_{\mathbb{G}}(-\Delta_{\mathbb{G}}+V)^{-1}$ in [16] are the special case of T_1 and T_2, respectively.

It is worth pointing out that we need to establish pointwise estimates for T_1, T_2 and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{G}$ in [16, 17]. And we prove the generalized Morrey estimates by using $M_{p,\varphi_1}-M_{q,\varphi_2}$ boundedness of the fractional maximal operators.

Let $V\ge 0$. We say $V\in B_{\infty}(\mathbb{G})$, if there exists a constant $C>0$ such that

$$\|V\|_{L_{\infty}(B)}\le\frac{C}{|B|}\int_B V(x)\mathrm{d}x$$

holds for every ball B in $\mathbb{G}$ (see [16]).

By the functional calculus, we may write, for all $0<\beta<1$,

$$(-\Delta_{\mathbb{G}}+V)^{-\beta}=\frac{1}{\pi}\int_0^{\infty}\lambda^{-\beta}\ (-\Delta_{\mathbb{G}}+V+\lambda)^{-1}\,\mathrm{d}\lambda.$$

Let $f\in C_0^{\infty}(\mathbb{G})$. From

$$(-\Delta_{\mathbb{G}}+V+\lambda)^{-1}f(x)=\int_{\mathbb{G}}\Gamma(x,y,\lambda)f(y)\mathrm{d}y,$$

it follows that

$$\mathcal{T}_1f(x)=\int_{\mathbb{G}}K_1(x,y)V(y)^{\gamma}f(y)\mathrm{d}y,$$

where

$$K_1(x,y)=\begin{cases}\dfrac{1}{\pi}\displaystyle\int_0^{\infty}\lambda^{-\beta}\,\Gamma(x,y,\lambda)\mathrm{d}\lambda & \text{for}\ \ 0<\beta<1,\\ \Gamma(x,y,0) & \text{for}\ \ \beta=1.\end{cases}$$

For the potential $V\in B_{\infty}(\mathbb{G})$ the following two pointwise estimates for T_1 and T_2 was proven in [29].

Theorem B Suppose $V\in B_{\infty}(\mathbb{G})$ and $0\le\gamma\le\beta\le 1$. Then for any $f\in C_0^{\infty}(\mathbb{G})$,

$$|T_1f(x)|\lesssim M_{\alpha}f(x),\quad \left|[b,\mathcal{T}_1]^kf(x)\right|\lesssim M_{b,\alpha,k}f(x),$$

where $\alpha=2(\beta-\gamma)$ and $[b,\mathcal{T}_1]^kf(x)=\mathcal{T}_1((b(\cdot)-b(x))^kf(\cdot))(x)$.

Theorem C Suppose $V\in B_{\infty}(\mathbb{G})$, $0\le\gamma\le\frac{1}{2}\le\beta\le 1$ and $\beta-\gamma\ge\frac{1}{2}$.

Then for any $f\in C_0^{\infty}(\mathbb{G})$,

$$|T_2f(x)|\lesssim M_{\alpha}f(x),\quad \left|[b,\mathcal{T}_2]^kf(x)\right|\lesssim M_{b,\alpha,k}f(x),$$

where $\alpha = 2(\beta-\gamma)-1$ and $[b,\mathcal{T}_2]^k f(x) = \mathcal{T}_2((b(\cdot)-b(x))^k f(\cdot))(x)$.

The above theorems will yield the generalized Morrey estimates for T_1 and T_2.

Corollary 5.1 Assume that $V \in B_\infty(\mathbb{G})$, and $0 \le \gamma \le \beta \le 1$. Let $1 \le p \le q < \infty$, $2(\beta-\gamma) = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and condition (3.7) be satisfied for $\alpha = 2(\beta-\gamma)$.

Then for any $f \in C_0^\infty(\mathbb{G})$,

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}} \qquad \text{for} \ \ p>1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}} \qquad \text{for} \ \ p=1.$$

Corollary 5.2 Assume that $V \in B_\infty(\mathbb{G})$, $b \in \mathrm{BMO}(\mathbb{G})$ and $0 \le \gamma \le \beta \le 1$. Let $1 < p \le q < \infty$, $2(\beta-\gamma) = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and condition (4.5) be satisfied for $\alpha = 2(\beta-\gamma)$. Then for any $f \in C_0^\infty(\mathbb{G})$

$$\|[b,\mathcal{T}_1]^k f\|_{M_{q,\varphi_2}} \lesssim \|b\|_*^k \|f\|_{M_{p,\varphi_1}}.$$

Corollary 5.3 Assume that $V \in B_\infty(\mathbb{G})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta-\gamma \ge \frac{1}{2}$. Let $1 \le p \le q < \infty$, $2(\beta-\gamma)-1 = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and condition (3.7) be satisfied for $\alpha = 2(\beta-\gamma)-1$.

Then for any $f \in C_0^\infty(\mathbb{G})$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}} \qquad \text{for} \ \ p>1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}} \qquad \text{for} \ \ p=1.$$

Corollary 5.4 Assume that $V \in B_\infty(\mathbb{G})$, $b \in \mathrm{BMO}(\mathbb{G})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta-\gamma \ge \frac{1}{2}$. Let $1 < p \le q < \infty$, $2(\beta-\gamma)-1 = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and condition (4.5) be satisfied for $\alpha = 2(\beta-\gamma)-1$. Then for any $f \in C_0^\infty(\mathbb{G})$

$$\|[b,\mathcal{T}_2]^k f\|_{M_{q,\varphi_2}} \lesssim \|b\|_*^k \|f\|_{M_{p,\varphi_1}}.$$

Corollary 5.5 Assume that $V \in B_\infty(\mathbb{G})$ and $0 \le \gamma \le \beta \le 1$. Let $1 \le p < q < \infty$, $2(\beta-\gamma) = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and conditions (3.9), (3.10) be satisfied for $\alpha = 2(\beta-\gamma)$. Then for any $f \in C_0^\infty(\mathbb{G})$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \qquad \text{for} \ \ p>1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{1,\varphi}} \qquad \text{for} \ \ p=1.$$

Corollary 5.6 Assume that $V \in B_\infty(\mathbb{G})$, $b \in \mathrm{BMO}(\mathbb{G})$ and $0 \le \gamma \le \beta \le 1$. Let $1 < p < q < \infty$, $2(\beta-\gamma) = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and conditions (4.7), (4.8) be satisfied for $\alpha = 2(\beta-\gamma)$. Then for any $f \in C_0^\infty(\mathbb{G})$

$$\|[b,\mathcal{T}_1]^k f\|_{M_{q,\varphi^{\frac{1}{q}}}} \lesssim \|b\|_*^k \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}.$$

Corollary 5.7 Assume that $V \in B_\infty(\mathbb{G})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta-\gamma \ge \frac{1}{2}$. Let $1 \le p < q < \infty$, $2(\beta-\gamma)-1 = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and conditions (3.9), (3.10) be satisfied for $\alpha = 2(\beta-\gamma)-1$. Then for any $f \in C_0^\infty(\mathbb{G})$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \qquad \text{for} \ \ p>1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{1,\varphi}} \qquad \text{for} \quad p=1.$$

Corollary 5.8 Assume that $V \in B_\infty(\mathbb{G})$, $b \in \mathrm{BMO}(\mathbb{G})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Let $1 < p < q < \infty$, $2(\beta-\gamma)-1 = Q\big(\frac{1}{p}-\frac{1}{q}\big)$ and conditions (4.7), (4.8) be satisfied for $\alpha = 2(\beta-\gamma)-1$. Then for any $f \in C_0^\infty(\mathbb{G})$

$$\|[b,\mathcal{T}_2]^k f\|_{M_{q,\varphi^{\frac{1}{q}}}} \lesssim \|b\|_*^k \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}.$$

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