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RIESZ POTENTIAL IN GENERALIZED MORREY SPACES ON THE HEISENBERG GROUP

V. S. Guliyev *

Ahi Evran University, Kirsehir 40100, Turkey

Institute of Mathematics and Mechanics, Azerbaijan NAS
9, F. Agaev St., Baku AZ 1141, Azerbaijan
vagif@guliyev.com

A. Eroglu

Nigde University, Nigde 51245, Turkey
aeroglu@nigde.edu.tr

Y. Y. Mammadov

Institute of Mathematics and Mechanics, Azerbaijan NAS
9, F. Agaev St., Baku AZ 1141, Azerbaijan
Nakhchivan Teacher-Training Institute, Nakhchivan AZ 7003, Azerbaijan
yagubmammadov@yahoo.com

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We consider the Riesz potential operator $\mathscr{I}_\alpha$, on the Heisenberg group $\mathbb{H}_n$ in generalized Morrey spaces $M_{p,\varphi}(\mathbb{H}_n)$ and find conditions for the boundedness of $\mathscr{I}_\alpha$ as an operator from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{p,\varphi_2}(\mathbb{H}_n)$, $1 < p < \infty$, and from $M_{1,\varphi_1}(\mathbb{H}_n)$ to a weak Morrey space $WM_{1,\varphi_2}(\mathbb{H}_n)$. The boundedness conditions are formulated it terms of Zygmund type integral inequalities. Based on the properties of the fundamental solution of the sub-Laplacian on $\mathbb{H}_n$, we prove two Sobolev–Stein embedding theorems for generalized Morrey and Besov–Morrey spaces. Bibliography: 40 titles.

1 Introduction

The Heisenberg group appears in quantum physics and many fields of mathematics, including Fourier analysis, functions of several complex variables, geometry, and topology.

In this paper, we establish the boundedness of the Riesz potential on the Heisenberg group in generalized Morrey spaces. We start with some basic results concerning the Heisenberg group and refer the interested reader to [1]–[3] and the references therein for more details. The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_n$ is the Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and multiplication

$$(x,t)(y,s) = \Big(x + y, t + s + 2\sum_{j=1}^{n}(x_{n+j}y_j - x_j y_{n+j})\Big).$$

* To whom the correspondence should be addressed.

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The inverse element of $g = (x, t)$ is $g^{-1} = (-x, -t)$ and the identity is denoted by $e = (0, 0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$ for $r > 0$ by the formula

$$\delta_r(x, t) = (rx, r^2 t).$$

These dilations are group automorphisms and the Jacobian determinant is r^Q, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. The homogeneous norm on $\mathbb{H}_n$ is given by the formula

$$|g| = |(x, t)| = (|x|^2 + |t|)^{1/2}.$$

This norm satisfies the triangle inequality and leads to the left-invariant distance $d(g, h) = |g^{-1}h|$. Using this norm, we define the Heisenberg ball

$$B(g, r) = \{h \in \mathbb{H}_n : |g^{-1}h| < r\}$$

with center $g = (x, t)$ and radius r and denote by ${}^{\complement}B(g, r) = \mathbb{H}_n \setminus B(g, r)$ its complement. The volume of the ball $B(g, r)$ is $C_n r^Q$, where C_n is the volume of the unit ball B_1:

$$C_n = |B(e, 1)| = \frac{2\pi^{n+1/2}\Gamma\left(\frac{n}{2}\right)}{(n+1)\Gamma(n)\Gamma\left(\frac{n+1}{2}\right)}.$$

Using the coordinates $g = (x, t)$ of points in $\mathbb{H}_n$, we can write the left-invariant vector fields $X_1, \ldots, X_{2n}, X_{2n+1}$ on $\mathbb{H}_n$ equal to $\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_{2n}}, \dfrac{\partial}{\partial t}$ at the origin as follows:

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, \ldots, n,$$

$$X_{2n+1} = \frac{\partial}{\partial t}.$$

These $2n+1$ vector fields form a basis for the Lie algebra of $\mathbb{H}_n$ with the commutation relations

$$[X_j, X_{n+j}] = -4X_{2n+1}, \quad j = 1, \ldots, n,$$

whereas the other commutators vanish. The sub-Laplacian $\Delta_{\mathbb{H}_n}$ is defined by the formula

$$\Delta_{\mathbb{H}_n} = -\sum_{j=1}^{2n} X_j^2$$

and the gradient $\nabla_{\mathbb{H}_n}$ is defined as

$$\nabla_{\mathbb{H}_n} = (X_1, \ldots, X_{2n}).$$

It is known that the sub-Laplacian operator (hypoelliptic by the Hörmander theorem [4]) plays the same fundamental role on the group $\mathbb{H}_n$ as the Laplacian on $\mathbb{R}^n$.

Let f be a given integrable function on a ball $B(g, r) \subset \mathbb{H}_n$. The *fractional maximal function* $M_\alpha f$, $0 \leqslant \alpha < Q$, of f is defined by the formula

$$M_\alpha f(g) = \sup_{r>0} |B(g, r)|^{-1+\alpha/Q} \int\limits_{B(g,r)} |f(h)| dh.$$

In the case $\alpha = 0$, the fractional maximal function $M_\alpha f$ coincides with the Hardy–Littlewood maximal function $Mf \equiv M_0 f$ (cf. [1, 3]) and is closely related to the fractional integral

$$I_\alpha f(g) = \int_{\mathbb{H}_n} |h^{-1}g|^{\alpha - Q} f(h)dh, \quad 0 < \alpha < Q.$$

The operators M_α and I_α play important role in real and harmonic analysis [1, 3, 5].

The *classical Riesz potential* $\mathscr{I}_\alpha$ is defined on $\mathbb{R}^n$ by the formula

$$\mathscr{I}_\alpha f = (-\Delta)^{-\alpha/2} f, \quad 0 < \alpha < n,$$

where Δ is the Laplacian operator. It is known, that

$$\mathscr{I}_\alpha f(x) = \gamma(\alpha)^{-1} \int_{\mathbb{R}^n} |x - y|^{\alpha - n} f(y)dy \equiv I_\alpha f(x),$$

where $\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2)/\Gamma(n/2 - \alpha/2)$. The Riesz potential on the Heisenberg group is defined in terms of the sub-Laplacian $\mathscr{L} = \Delta_{\mathbb{H}_n}$.

Definition 1.1. For $0 < \alpha < Q$ the *Riesz potential* $\mathscr{I}_\alpha$ is defined on the Schwartz space $\mathbb{S}(\mathbb{H}_n)$ by the formula

$$\mathscr{I}_\alpha f(g) = \mathscr{L}^{-\frac{\alpha}{2}} f(g) \equiv \int_0^\infty e^{-r\mathscr{L}} f(g) r^{\alpha/2 - 1} dr,$$

where

$$e^{-r\mathscr{L}} f(g) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_{\mathbb{H}_n} K_r(h, g) f(h)dh$$

is the semigroups of the operator $\mathscr{L}$.

In [6], relations between the Riesz potential and the heat kernel on the Heisenberg group are studied. The following assertion [6, Theorem 1] yields an expression for $\mathscr{I}_\alpha$, which allows us to discuss the boundedness of the Riesz potential.

Theorem A. *Let $q_s(g)$ be the heat kernel on $\mathbb{H}_n$. If $0 \leqslant \alpha < Q$, then for $f \in \mathbb{S}(\mathbb{H}_n)$*

$$\mathscr{I}_\alpha f(g) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty s^{\alpha/2 - 1} q_s(\cdot) ds * f(g).$$

The Riesz potential $\mathscr{I}_\alpha$ satisfies the estimate [6, Theorem 2]

$$|\mathscr{I}_\alpha f(g)| \lesssim I_\alpha f(g), \tag{1.1}$$

which provides a suitable estimate for the Riesz potential on the Heisenberg group.

In this paper, we establish the boundedness of the Riesz potential $\mathscr{I}_\alpha$, $0 < \alpha < Q$, from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{q,\varphi_2}(\mathbb{H}_n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/Q$, and from $M_{1,\varphi_1}(\mathbb{H}_n)$ to the weak space $WM_{q,\varphi_2}(\mathbb{H}_n)$, $1 < q < \infty$, $1 - 1/q = \alpha/Q$. We also find conditions on φ for the Adams type boundedness of $\mathscr{I}_\alpha$ from $M_{p,\varphi^{1/p}}$ to $M_{q,\varphi^{1/q}}$ in the case $1 < p < q < \infty$ and from $M_{1,\varphi}$ to

$WM_{q,\varphi^{1/q}}$ in the case $1 < q < \infty$. In all the cases, the boundedness conditions are expressed in terms of Zygmund type integral inequalities for (φ_1, φ_2) and φ, without any assumption about the monotonicity of (φ_1, φ_2) and φ in r.

As an application of the properties of the fundamental solution of sub-Laplacian $\mathscr{L}$ on $\mathbb{H}_n$, we prove (Theorem 6.1) the following generalized Morrey version of Sobolev inequality on $\mathbb{H}_n$: for every $u \in C_0^\infty(\mathbb{H}_n)$

$$\|u\|_{M_{q,\varphi_2}} \leqslant C\|\nabla_{\mathscr{L}} u\|_{M_{p,\varphi_1}},$$

where $1 < p < q < \infty$, $1/p - 1/q = 1/Q$, and (φ_1, φ_2) satisfy the condition (5.6). In Theorem 6.3, we establish the boundedness of the operator $\mathscr{I}_\alpha$ from $BM^s_{p\theta,\varphi_1}(\mathbb{H}_n)$ to $BM^s_{q\theta,\varphi_2}(\mathbb{H}_n)$, where $1 < p < q < \infty$, $1/p - 1/q = \alpha/Q$, $1 \leqslant \theta \leqslant \infty$, $0 < s < 1$, and (φ_1, φ_2) satisfy (5.6).

For another application, we prove (Theorem 6.5) the following Sobolev–Stein embedding inequality in the generalized Besov–Morrey space on $\mathbb{H}_n$: for every $u \in C_0^\infty(\mathbb{H}^n)$

$$\|u\|_{BM^s_{q\theta,\varphi_2}} \leqslant C\|\nabla_{\mathscr{L}} u\|_{BM^s_{p\theta,\varphi_1}},$$

where, $1 < p < q < \infty$, $1/p - 1/q = 1/Q$, $1 \leqslant \theta \leqslant \infty$, $0 < s < 1$, and (φ_1, φ_2) satisfy (5.6).

We write $A \lesssim B$ if $A \leqslant CB$, where C is a positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are *equivalent.*

2 Notation

The Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$, together with the weighted spaces $L_{p,w}(\mathbb{R}^n)$, play an important role in the theory of partial differential equations. The Morrey spaces were introduced by Morrey [7] in connection with certain problems in elliptic partial differential equations and Calculus of Variations. Later, the Morrey spaces were applied to the study of the Navier-Stokes equations [8, 9], the Schrödinger equations [10]–[12], elliptic problems with discontinuous coefficients [13, 14], and the potential theory [15, 16]. More information about the Morrey spaces can be found in [17].

Definition 2.1. Suppose that $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant Q$, and $[t]_1 = \min\{1, t\}$. The *generalized Morrey space* $M_{p,\varphi}(\mathbb{H}_n)$ is the set of locally integrable functions $f(g)$, $u \in \mathbb{H}_n$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \sup_{g \in \mathbb{H}_n,\, \tau > 0} \left(\tau^{-\lambda} \int\limits_{B(g,\tau)} |f(h)|^p dh \right)^{1/p}.$$

If $\lambda = 0$, then $L_{p,0}(\mathbb{H}_n) = L_p(\mathbb{H}_n)$; if $\lambda = Q$, then $L_{p,Q}(\mathbb{H}_n) = L_\infty(\mathbb{H}_n)$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda}(\mathbb{H}_n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{H}_n$.

Definition 2.2 (cf. [18])**.** Suppose that $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant Q$. The *weak Morrey space* $WL_{p,\lambda}(\mathbb{H}_n)$ is the set of locally integrable functions $f(g)$, $u \in \mathbb{H}_n$ with the finite norm

$$\|f\|_{WL_{p,\lambda}} = \sup_{r > 0} r \sup_{g \in \mathbb{H}_n,\, \tau > 0} \left(\tau^{-\lambda} \; |\{h \in B(g,\tau) : \; |f(h)| > r\}| \right)^{1/p}.$$

We note that $WL_p(\mathbb{H}_n) = WL_{p,0}(\mathbb{H}_n)$, $L_{p,\lambda}(\mathbb{H}_n) \subset WL_{p,\lambda}(\mathbb{H}_n)$, and $\|f\|_{WL_{p,\lambda}} \leqslant \|f\|_{L_{p,\lambda}}$.

By the classical Hardy–Littlewood–Sobolev result, I_α is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ if and only if $\alpha = Q\,(1/p - 1/q)$ in the case $1 < p < q < \infty$ and I_α is bounded from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ if and only if $\alpha = Q(1 - 1/q)$ in the case $p = 1 < q < \infty$.

Spanne [19] and Adams [15] proved the boundedness of I_α on $\mathbb{R}^n$ in the Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$. This result was reproved by Chiarenza and Frasca [20]. Using the more general results of Guliyev [21] (cf., also [18, 22, 23]), it is possible to prove the following generalization of the results of [15, 19, 20] to the case of the Heisenberg group (cf., also [24]).

Theorem A. *Suppose that* $0 < \alpha < Q$, $0 \leqslant \lambda < Q - \alpha$, *and* $1 \leqslant p < (Q - \lambda)/\alpha$.

1. *If* $1 < p < (Q - \lambda)/\alpha$, *then the condition* $1/p - 1/q = \alpha/(n - \lambda)$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $L_{p,\lambda}(\mathbb{H}_n)$ *to* $L_{q,\lambda}(\mathbb{H}_n)$.

2. *If* $p = 1$, *then the condition* $1 - 1/q = \alpha/(Q - \lambda)$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $L_{1,\lambda}(\mathbb{H}_n)$ *to* $WL_{q,\lambda}(\mathbb{H}_n)$.

If $\alpha = Q/p - Q/q$, then $\lambda = 0$ and Theorem A reduces to the aforementioned Hardy–Littlewood–Sobolev result.

We recall that for $0 < \alpha < Q$

$$M_\alpha f(g) \leqslant v_n^{\alpha/Q-1} I_\alpha(|f|)(g). \tag{2.1}$$

Hence Theorem A implies the boundedness of the fractional maximal operator M_α, where $v_n = |B(e,1)|$ is the volume of the unit ball in $\mathbb{H}_n$.

3 Generalized Morrey Spaces

Throughout the paper, $\varphi(g,r)$, $\varphi_1(g,r)$, and $\varphi_2(g,r)$ are nonnegative measurable functions on $\mathbb{H}_n \times (0,\infty)$. It is convenient to define generalized Morrey spaces as follows.

Definition 3.1. Let $1 \leqslant p < \infty$. The *generalized Morrey space* $M_{p,\varphi}(\mathbb{H}_n)$ is the set of all functions $f \in L_p^{loc}(\mathbb{H}_n)$ equipped with the norm

$$\|f\|_{M_{p,\varphi}} = \sup_{u \in \mathbb{H}_n, r>0} \frac{r^{-Q/p}}{\varphi(g,r)} \|f\|_{L_p(B(g,r))}.$$

According to this definition, we recover the space $L_{p,\lambda}(\mathbb{H}_n)$ under the choice $\varphi(g,r) = r^{\frac{\lambda-Q}{p}}$:

$$L_{p,\lambda}(\mathbb{H}_n) = M_{p,\varphi}(\mathbb{H}_n)\Big|_{\varphi(g,r) = r^{\frac{\lambda-Q}{p}}}.$$

Sufficient conditions on weights φ_1 and φ_2 for the boundedness of a singular operator T from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{p,\varphi_2}(\mathbb{H}_n)$ were obtained in [21, 22, 25]. In [25], the following conditions were imposed:

$$c^{-1}\varphi(g,r) \leqslant \varphi(g,\tau) \leqslant c\,\varphi(g,r), \quad r \leqslant \tau \leqslant 2r, \tag{3.1}$$

on $\varphi(g,r)$, where c ($\geqslant 1$) is independent of t, r, and $u \in \mathbb{H}_n$,

$$\int\limits_r^\infty \varphi(g,\tau)^p \frac{d\tau}{\tau} \leqslant C\,\varphi(g,r)^p \tag{3.2}$$

on the maximal or singular operator, and

$$\int_r^\infty \tau^{\alpha p}\varphi(g,\tau)^p \frac{d\tau}{\tau} \leqslant C\, r^{\alpha p}\varphi(g,r)^p \tag{3.3}$$

on the potential and fractional maximal operators, where $C(>0)$ is independent of r and $g \in \mathbb{H}_n$.

The following assertion was proved in [25].

Theorem 3.1. *Suppose that* $1 \leqslant p < \infty, 0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, *and* $\varphi(g,\tau)$ *satisfies the conditions* (3.1) *and* (3.3). *Then the operator* I_α *is bounded from* $M_{p,\varphi}(\mathbb{H}_n)$ *to* $M_{q,\varphi}(\mathbb{H}_n)$ *for* $p > 1$ *and from* $M_{1,\varphi}(\mathbb{H}_n)$ *to* $WM_{q,\varphi}(\mathbb{H}_n)$ *for* $p = 1$.

The following assertion, containing the results of [25], was proved in [21] (cf. also [18, 22, 23, 27, 28, 29]).

Theorem 3.2. *Suppose that* $1 \leqslant p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \tau^{\alpha}\varphi_1(g,r) \frac{d\tau}{\tau} \leqslant C\, \varphi_2(g,r), \tag{3.4}$$

where C *is independent of* g *and* r. *Then the operator* I_α *is bounded from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $M_{q,\varphi_2}(\mathbb{H}_n)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(\mathbb{H}_n)$ *to* $WM_{q,\varphi_2}(\mathbb{H}_n)$ *for* $p = 1$.

4 Maximal Operator in $M_{p,\varphi}(\mathbb{H}_n)$

We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$, with the finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

and set $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue measurable functions on $(0,\infty)$, and let $\mathfrak{M}^+(0,\infty)$ be its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ that are nondecreasing on $(0,\infty)$ and set

$$\mathbb{A} = \big\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\big\}.$$

Let u be a continuous nonnegative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by the formula

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [26].

Theorem 4.1. *Let* v_1, v_2 *be nonnegative measurable functions such that*

$$0 < \|v_1\|_{L_\infty(t,\infty)} < \infty \quad \textit{for any } t > 0,$$

and let u *be a continuous nonnegative function on* $(0,\infty)$. *Then the operator* $\overline{S}_u$ *is bounded from* $L_{\infty,v_1}(0,\infty)$ *to* $L_{\infty,v_2}(0,\infty)$ *on the cone* $\mathbb{A}$ *if and only if*

$$\big\|v_2 \overline{S}_u\big(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\big)\big\|_{L_\infty(0,\infty)} < \infty. \tag{4.1}$$

Sufficient conditions on φ for the boundedness of M and M_α in generalized Morrey spaces $\mathscr{M}_{p,\varphi}(\mathbb{H}_n)$ were obtained in [21, 25, 27, 28].

Lemma 4.1. *Let $1 \leqslant p < \infty$. Then for $p > 1$ and any ball $B = B(g,r)$*

$$\|Mf\|_{L_p(B(g,r))} \lesssim \|f\|_{L_p(B(g,2r))} + r^{Q/p} \sup_{\tau > 2r} \tau^{-Q} \|f\|_{L_1(B(g,\tau))} \tag{4.2}$$

for all $f \in L_p^{\rm loc}(\mathbb{H}_n)$. Moreover, for $p = 1$

$$\|Mf\|_{WL_1(B(g,r))} \lesssim \|f\|_{L_1(B(g,2r))} + r^{Q} \sup_{\tau > 2r} \tau^{-Q} \|f\|_{L_1(B(g,\tau))} \tag{4.3}$$

for all $f \in L_1^{\rm loc}(\mathbb{H}_n)$.

Proof. Let $1 < p < \infty$. For an arbitrary ball $B = B(g,r)$ we set $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Then

$$\|Mf\|_{L_p(B)} \leqslant \|Mf_1\|_{L_p(B)} + \|Mf_2\|_{L_p(B)}.$$

By the continuity of the operator $M : L_p(\mathbb{H}_n) \to L_p(\mathbb{H}_n)$ (cf., for example, [1]), we have

$$\|Mf_1\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)}.$$

Let h be an arbitrary point in B. If $B(h,\tau) \cap {}^{\complement}(2B) \neq \varnothing$, then $\tau > r$. Indeed, if $w \in B(h,\tau) \cap {}^{\complement}(2B)$, then $\tau > |h^{-1}w| \geqslant |g^{-1}w| - |g^{-1}h| > 2r - r = r$. On the other hand, $B(h,\tau) \cap {}^{\complement}(2B) \subset B(g, 2\tau)$. Indeed, for $w \in B(h,\tau) \cap {}^{\complement}(2B)$ we have $|g^{-1}w| \leqslant |h^{-1}w| + |g^{-1}h| < \tau + r < 2\tau$. Hence

$$\begin{aligned} Mf_2(h) &= \sup_{\tau > 0} \frac{1}{|B(h,\tau)|} \int\limits_{B(h,\tau) \cap {}^{\complement}(2B)} |f(w)| dw \leqslant 2^Q \sup_{\tau > r} \frac{1}{|B(g,2\tau)|} \int\limits_{B(g,2\tau)} |f(w)| dw \\ &= 2^Q \sup_{\tau > 2r} \frac{1}{|B(g,\tau)|} \int\limits_{B(g,\tau)} |f(w)| dw. \end{aligned}$$

Therefore, for all $h \in B$

$$Mf_2(h) \leqslant 2^Q \sup_{\tau > 2r} \frac{1}{|B(g,\tau)|} \int\limits_{B(g,\tau)} |f(w)| dw. \tag{4.4}$$

Thus,

$$\|Mf\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + |B|^{1/p} \Bigg(\sup_{\tau > 2r} \frac{1}{|B(g,\tau)|} \int\limits_{B(g,\tau)} |f(w)| dw \Bigg).$$

Let $p = 1$. It is obvious that for any ball $B = B(g,r)$

$$\|Mf\|_{WL_1(B)} \leqslant \|Mf_1\|_{WL_1(B)} + \|Mf_2\|_{WL_1(B)}.$$

By the continuity of the operator $M : L_1(\mathbb{H}_n) \to WL_1(\mathbb{H}_n)$,

$$\|Mf_1\|_{WL_1(B)} \lesssim \|f\|_{L_1(2B)}.$$

By (4.4), we get the inequality (4.3). ☐

Lemma 4.2. *Let $1 \leqslant p < \infty$. Then for $p > 1$ and any ball $B = B(g,r)$ in $\mathbb{H}_n$*

$$\|Mf\|_{L_p(B(g,r))} \lesssim r^{Q/p} \sup_{\tau > 2r} \tau^{-Q/p} \|f\|_{L_p(B(g,\tau))} \tag{4.5}$$

for all $f \in L_p^{\rm loc}(\mathbb{H}_n)$. Moreover, for $p = 1$

$$\|Mf\|_{WL_1(B(g,r))} \lesssim r^{Q} \sup_{\tau > 2r} \tau^{-Q} \|f\|_{L_1(B(g,\tau))} \tag{4.6}$$

for all $f \in L_1^{\rm loc}(\mathbb{H}_n)$.

Proof. Let $1 < p < \infty$. Denote

$$\mathscr{M}_1 := |B|^{1/p} \Bigg(\sup_{\tau > 2r} \frac{1}{|B(g,\tau)|} \int\limits_{B(g,\tau)} |f(w)| dw \Bigg),$$

$$\mathscr{M}_2 := \|f\|_{L_p(2B)}.$$

Applying the Hölder inequality, we get

$$\mathscr{M}_1 \lesssim |B|^{1/p} \Bigg(\sup_{\tau > 2r} \frac{1}{|B(g,\tau)|^{\frac{1}{p}}} \Bigg(\int\limits_{B(g,\tau)} |f(w)|^p dw \Bigg)^{\frac{1}{p}} \Bigg).$$

On the other hand,

$$|B|^{\frac{1}{p}} \Bigg(\sup_{\tau > 2r} \frac{1}{|B(g,\tau)|^{\frac{1}{p}}} \Bigg(\int\limits_{B(g,\tau)} |f(w)|^p dw \Bigg)^{\frac{1}{p}} \Bigg) \gtrsim |B|^{\frac{1}{p}} \Bigg(\sup_{\tau > 2r} \frac{1}{|B(g,\tau)|^{\frac{1}{p}}} \Bigg) \|f\|_{L_p(2B)} \approx \mathscr{M}_2.$$

Since $\|Mf\|_{L_p(B)} \leqslant \mathscr{M}_1 + \mathscr{M}_2$ in view of Lemma 4.1, we obtain (4.5).

Let $p = 1$. The inequality (4.6) directly follows from (4.3). $\square$

Theorem 4.2. *Suppose that $1 \leqslant p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r < t < \infty} t^{-Q/p} \operatorname*{ess\,inf}_{t < s < \infty} \varphi_1(g,s)\, s^{Q/p} \leqslant C\, \varphi_2(g,r), \tag{4.7}$$

where C is independent of g and r. If $p > 1$, then M is bounded as an operator from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{p,\varphi_2}(\mathbb{H}_n)$. If $p = 1$, then M is bounded as an operator from $M_{1,\varphi_1}(\mathbb{H}_n)$ to $WM_{1,\varphi_2}(\mathbb{H}_n)$.

Proof. By Theorem 4.1 and Lemma 4.2,

$$\begin{aligned} \|Mf\|_{M_{p,\varphi_2}} &\lesssim \sup_{g \in \mathbb{H}_n, r > 0} \varphi_2(g,r)^{-1} \sup_{\tau > r} \tau^{-Q/p} \|f\|_{L_p(B(g,\tau))} \\ &\lesssim \sup_{g \in \mathbb{H}_n, r > 0} \varphi_1(g,r)^{-1}\, r^{-Q/p} \|f\|_{L_p(B(g,r))} = \|f\|_{M_{p,\varphi_1}} \end{aligned}$$

if $p \in (1,\infty)$ and

$$\begin{aligned} \|Mf\|_{WM_{1,\varphi_2}} &\lesssim \sup_{g \in \mathbb{H}_n, r > 0} \varphi_2(g,r)^{-1} \sup_{\tau > r} \tau^{-Q} \|f\|_{L_1(B(g,\tau))} \\ &\lesssim \sup_{g \in \mathbb{H}_n, r > 0} \varphi_1(g,r)^{-1}\, r^{-Q} \|f\|_{L_1(B(g,r))} = \|f\|_{M_{1,\varphi_1}} \end{aligned}$$

if $p = 1$. $\square$

Corollary 4.1. *Let $p \in [1,\infty)$ and let $\varphi : (0,\infty) \to (0,\infty)$ be a decreasing function. Assume that the mapping $r \mapsto \varphi(r)\, r^{Q/p}$ is almost increasing (there exists a constant c such that $\varphi(s)\, s^{Q/p} \leqslant c\varphi(r)\, r^{Q/p}$ for $s < r$). Then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\mathscr{M}_{p,\varphi}} \leqslant C\|f\|_{\mathscr{M}_{p,\varphi}}, \quad p > 1,$$

$$\|Mf\|_{W\mathscr{M}_{1,\varphi}} \leqslant C\|f\|_{\mathscr{M}_{1,\varphi}}.$$

5 Riesz Potential Operator in $M_{p,\varphi}(\mathbb{H}_n)$

5.1 Spanne–Guliyev type result

In this section, we use the following statement on the boundedness of the Hardy operator

$$(H\phi)(t) := \frac{1}{t}\int_0^t \phi(r)dr, \ 0 < t < \infty.$$

Theorem 5.1 (cf. [30])**.** *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)H\phi(t) \leqslant c \operatorname*{ess\,sup}_{t>0} v(t)\phi(t)$$

holds for all nonnegative and nonincreasing ϕ on $(0,\infty)$ if and only if

$$A := \sup_{t>0} \frac{w(t)}{t} \int_0^t \frac{dr}{\operatorname*{ess\,sup}\limits_{0<s<r} v(s)} < \infty \quad \text{and} \quad c \approx A.$$

Lemma 5.1. *Suppose that $1 \leqslant p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, and $f \in L_p^{loc}(\mathbb{H}_n)$. If $p > 1$, then*

$$\|I_\alpha f\|_{L_q(B(g,r))} \lesssim r^{Q/q} \int_r^\infty \tau^{-Q/q-1}\|f\|_{L_p(B(g,\tau))}d\tau$$

for any ball $B(g,r)$ and all $f \in L_p^{\rm loc}(\mathbb{H}_n)$. If $p = 1$, then

$$\|I_\alpha f\|_{WL_q(B(g,r))} \lesssim \tau^{Q/q} \int_\tau^\infty r^{-Q/q-1}\|f\|_{L_1(B(g,r))}dr \tag{5.1}$$

for any ball $B(g,r)$ and all $f \in L_1^{\rm loc}(\mathbb{H}_n)$.

Proof. Suppose that $1 < p < \infty$, $0 < \alpha < Q/p$ and $1/q = 1/p - \alpha/Q$. For an arbitrary ball $B = B(g,r)$ we set $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, $f_2 = f\chi_{\complement_{(2B)}}$ and $2B = B(g,2r)$. Then

$$\|I_\alpha f\|_{L_q(B)} \leqslant \|I_\alpha f_1\|_{L_q(B)} + \|I_\alpha f_2\|_{L_q(B)}.$$

By the boundedness of the operator I_α from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$,

$$\|I_\alpha f_1\|_{L_q(B)} \leqslant \|I_\alpha f_1\|_{L_q(\mathbb{H}_n)} \lesssim \|f_1\|_{L_p(\mathbb{H}_n)} = \|f\|_{L_p(2B)}.$$

It is clear that for $g \in B$ and $h \in {}^{\complement}(2B)$

$$\frac{1}{2}|h^{-1}w| \leqslant |g^{-1}h| \leqslant \frac{3}{2}|h^{-1}w|.$$

Therefore,

$$|I_\alpha f_2(g)| \leqslant 2^Q \int\limits_{{}^{\complement}(2B)} \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh.$$

By the Fubini theorem,

$$\int\limits_{{}^{\complement}(2B)} \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh \approx \int\limits_{{}^{\complement}(2B)} |f(h)| \int\limits_{|g^{-1}h|}^{\infty} \frac{d\tau}{\tau^{Q+1-\alpha}} dh$$

$$\approx \int\limits_{2r}^{\infty} \int\limits_{2r \leqslant |g^{-1}h| \leqslant \tau} |f(h)| dh \frac{d\tau}{\tau^{Q+1-\alpha}} \lesssim \int\limits_{2r}^{\infty} \int\limits_{B(g,\tau)} |f(h)| dh \frac{d\tau}{\tau^{Q+1-\alpha}}.$$

Applying the Hölder inequality, we get

$$\int\limits_{{}^{\complement}(2B)} \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh \lesssim \int\limits_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}}. \tag{5.2}$$

Moreover, for all $p \in [1, \infty)$

$$\|I_\alpha f_2\|_{L_q(B)} \lesssim r^{Q/q} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}}. \tag{5.3}$$

Thus,

$$\|I_\alpha f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^{Q/q} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}}.$$

On the other hand,

$$\|f\|_{L_p(2B)} \approx r^{Q/q} \|f\|_{L_p(2B)} \int\limits_{2r}^{\infty} \frac{d\tau}{\tau^{Q/q+1}} \leqslant r^{Q/q} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}}. \tag{5.4}$$

Thus,

$$\|I_\alpha f\|_{L_q(B)} \lesssim r^{Q/q} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}}.$$

Let $p = 1$. From the weak $(1, q)$ boundedness of I_α and (5.4) it follows that

$$\|I_\alpha f_1\|_{WL_q(B)} \leqslant \|I_\alpha f_1\|_{WL_q(\mathbb{H}_n)} \lesssim \|f_1\|_{L_1(\mathbb{H}_n)}$$

$$= \|f\|_{L_1(2B)} \lesssim r^{Q/q} \int\limits_{2r}^{\infty} \|f\|_{L_1(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}}. \tag{5.5}$$

Then from (5.3) and (5.5) we get the inequality (5.1). □

Theorem 5.2. *Suppose that $1 \leqslant p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(g,s) s^{Q/p}}{\tau^{Q/q+1}} d\tau \leqslant C\, \varphi_2(g,r), \tag{5.6}$$

where C is independent of g and r. Then the operators $\mathscr{I}_\alpha$ and I_α are bounded from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{q,\varphi_2}(\mathbb{H}_n)$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{H}_n)$ to $WM_{q,\varphi_2}(\mathbb{H}_n)$ for $p = 1$. Moreover,

$$\|\mathscr{I}_\alpha f\|_{M_{q,\varphi_2}} \lesssim \|I_\alpha f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \quad p > 1,$$

$$\|\mathscr{I}_\alpha f\|_{WM_{q,\varphi_2}} \lesssim \|I_\alpha f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}, \quad p = 1.$$

Proof. By Lemma 5.1 and Theorem 5.1, for $p > 1$

$$\begin{aligned}
\|I_\alpha f\|_{M_{q,\varphi_2}} &\lesssim \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_2(g,r)^{-1} \int_r^\infty \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}} \\
&\approx \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_2(g,r)^{-1} \int_0^{r^{-Q/q}} \|f\|_{L_p(B(g,\tau^{-q/Q}))} d\tau \\
&= \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_2(g,r^{-q/Q})^{-1} \int_0^r \|f\|_{L_p(B(g,\tau^{-q/Q}))} d\tau \\
&\lesssim \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_1(g,r^{-q/Q})^{-1}\, r^{q/p}\, \|f\|_{L_p(B(g,r^{-q/Q}))} = \|f\|_{M_{p,\varphi_1}}
\end{aligned}$$

and for $p = 1$

$$\begin{aligned}
\|I_\alpha f\|_{WM_{q,\varphi_2}} &\lesssim \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_2(g,r)^{-1} \int_r^\infty \|f\|_{L_1(B(g,\tau))} \frac{d\tau}{\tau^{Q/q+1}} \\
&\approx \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_2(g,r)^{-1} \int_0^{r^{-Q/q}} \|f\|_{L_1(B(g,\tau^{-Q/q}))} d\tau \\
&= \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_2(g,r^{-q/Q})^{-1} \int_0^r \|f\|_{L_1(B(g,\tau^{-q/Q}))} d\tau \\
&\lesssim \sup_{g\in\mathbb{H}_n,\, r>0} \varphi_1(g,r^{-q/Q})^{-1}\, r^{q}\, \|f\|_{L_1(B(g,r^{-q/Q}))} = \|f\|_{M_{1,\varphi_1}}.
\end{aligned}$$

□

Remark 5.1. It is obvious that (3.4) implies (5.6). Indeed, if the condition (3.4) holds, then

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(s) s^{n/p}}{\tau^{n/p+1-\alpha}} d\tau \leqslant \int_r^\infty \tau^\alpha \varphi_1(\tau) \frac{d\tau}{\tau}, \qquad r \in (0,\infty),$$

so the condition (5.6) holds.

In general, (5.6) does not imply (3.4). For example, the functions

$$\varphi_1(r) = r^{\beta-n/p} \left|\sin\left(\max\{1, \pi/(2r)\}\right)\right|, \quad \varphi_2(r) = \begin{cases} 1, & r \in (0,1), \\ r^{\beta-n/q}, & r \in (1,\infty), \quad 0 < \beta < n/(2q) \end{cases}$$

satisfy the condition (5.6). But, in the case $r \in (0,2)$, we have $\operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(s) s^{n/p} = 0$ and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(s) s^{n/p}}{\tau^{n/p+1-\alpha}} d\tau \approx \begin{cases} 1, & r \in (0,1), \\ r^{\beta-n/q}, & r \in (1,\infty), \end{cases} \lesssim \varphi_2(r), \quad r \in (0,\infty),$$

which means that these functions do not satisfy the condition (3.4). Another example is presented by the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{Q/p-\beta}}, \quad \varphi_2(r) = r^{-Q/q}(1+r^\beta), \quad 0 < \beta < Q/q$$

which satisfy (5.6), but do not satisfy (3.4).

5.2 Adams–Guliyev type result

The following assertion is a result of Adams–Guliyev type.

Theorem 5.3. *Suppose that $1 \leqslant p < \infty$, $0 < \alpha < Q/p$, and $q > p$. Let $\varphi(g,r)$ satisfy the conditions*

$$\sup_{r<t<\infty} t^{-Q} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(g,s)\, s^Q \leqslant C\, \varphi(g,r), \tag{5.7}$$

$$\int_r^\infty \tau^\alpha\, \varphi(g,\tau)^{1/p} \frac{d\tau}{\tau} \leqslant C r^{-\frac{\alpha p}{q-p}}, \tag{5.8}$$

where C is independent of $g \in \mathbb{H}_n$ and $r > 0$. Then the operators $\mathscr{I}_\alpha$ and I_α are bounded from $M_{p,\varphi^{1/p}}(\mathbb{H}_n)$ to $M_{q,\varphi^{1/q}}(\mathbb{H}_n)$ for $p > 1$ and from $M_{1,\varphi}(\mathbb{H}_n)$ to $WM_{q,\varphi^{1/q}}(\mathbb{H}_n)$. Moreover,

$$\|\mathscr{I}_\alpha f\|_{M_{q,\varphi^{1/q}}} \lesssim \|I_\alpha f\|_{M_{q,\varphi^{1/q}}} \lesssim \|f\|_{M_{p,\varphi^{1/p}}}, \quad p > 1,$$

$$\|\mathscr{I}_\alpha f\|_{WM_{q,\varphi^{1/q}}} \lesssim \|I_\alpha f\|_{WM_{q,\varphi^{1/q}}} \lesssim \|f\|_{M_{1,\varphi}}, \quad p = 1.$$

Proof. Suppose that $1 < p < \infty$, $0 < \alpha < Q/p$, $q > p$, and $f \in M_{p,\varphi^{1/p}}(\mathbb{H}_n)$. Suppose that $f = f_1 + f_2$, $B = B(g,r)$, $f_1 = f\chi_{2B}$, and $f_2 = f\chi_{\complement(2B)}$. For $I_\alpha f_2(g)$ we have

$$|I_\alpha f_2(g)| \leqslant \int_{\complement B(g,2r)} |g^{-1}h|^{\alpha-Q} |f(h)| dh \lesssim \int_{\complement B(g,2r)} |f(h)| dh \int_{|g^{-1}h|}^\infty \tau^{\alpha-Q-1} d\tau$$

$$\lesssim \int_{2r}^\infty \left(\int_{2r<|g^{-1}h|<\tau} |f(h)| dh \right) \tau^{\alpha-Q-1} d\tau \lesssim \int_r^\infty \tau^{\alpha-Q/p-1} \|f\|_{L_p(B(g,\tau))} d\tau. \tag{5.9}$$

Then from (5.8) and (5.9) we get

$$|I_\alpha f(g)| \lesssim r^\alpha\, Mf(g) + \int\limits_r^\infty t^{\alpha-Q/p-1}\|f\|_{L_p(B(g,\tau))}dt \leqslant r^\alpha\, Mf(g) + \|f\|_{M_{p,\varphi}} \int\limits_r^\infty \tau^\alpha \varphi(g,\tau)^{1/p}\frac{d\tau}{\tau}$$

$$\lesssim r^\alpha\, Mf(g) + r^{-\frac{\alpha p}{q-p}}\, \|f\|_{M_{p,\varphi^{1/p}}}.$$

Choosing

$$r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p}}}}{Mf(g)}\right)^{\frac{q-p}{\alpha q}}$$

for every $g \in \mathbb{H}_n$, we have

$$|I_\alpha f(g)| \lesssim (Mf(g))^{p/q}\, \|f\|^{1-p/q}_{M_{p,\varphi^{1/p}}}.$$

Hence the required assertion follows from the boundedness of the maximal operator M in $M_{p,\varphi^{1/p}}(\mathbb{H}_n)$ provided by Theorem 4.2 in view of (5.7):

$$\begin{aligned}
\|I_\alpha f\|_{M_{q,\varphi^{1/q}}} &= \sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-1/q}\tau^{-Q/q}\|I_\alpha f\|_{L_q(B(g,\tau))} \\
&\lesssim \|f\|^{1-p/q}_{M_{p,\varphi^{1/p}}} \sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-1/q}\tau^{-Q/q}\|Mf\|^{p/q}_{L_p(B(g,\tau))} \\
&= \|f\|^{1-p/q}_{M_{p,\varphi^{1/p}}} \Big(\sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-1/p}\tau^{-Q/p}\|Mf\|_{L_p(B(g,\tau))}\Big)^{p/q} \\
&= \|f\|^{1-p/q}_{M_{p,\varphi^{1/p}}} \|Mf\|^{p/q}_{M_{p,\varphi^{1/p}}} \lesssim \|f\|_{M_{p,\varphi^{1/p}}} \quad \text{if } 1<p<q<\infty, \\
\|I_\alpha f\|_{WM_{q,\varphi^{1/q}}} &= \sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-1/q}\tau^{-Q/q}\|I_\alpha f\|_{WL_q(B(g,\tau))} \\
&\lesssim \|f\|^{1-1/q}_{M_{1,\varphi}} \sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-1/q}\tau^{-Q/q}\|Mf\|^{1/q}_{WL_1(B(g,\tau))} \\
&= \|f\|^{1-1/q}_{M_{1,\varphi}} \Big(\sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-1}\tau^{-Q}\|Mf\|_{WL_1(B(g,\tau))}\Big)^{1/q} \\
&= \|f\|^{1-1/q}_{M_{1,\varphi}} \|Mf\|^{1/q}_{WM_{1,\varphi}} \lesssim \|f\|_{M_{1,\varphi}} \quad \text{if } 1<q<\infty.
\end{aligned}$$

□

In the case $\varphi(g,r) = r^{\lambda-Q}$, $0<\lambda<Q$, from Theorem 5.3 we get Theorem A.

6 Applications

It is known (cf. [31, p. 247]) that if $|\cdot|$ is a homogeneous norm on $\mathbb{H}_n$, then there exists a positive constant C_0 such that $\Gamma(g) = C_0|x|^{2-Q}$ is the fundamental solution of $\mathscr{L}$.

From Theorem 5.2 it is easy to obtain an inequality extending the classical Sobolev embedding theorem to the Heisenberg groups.

Theorem 6.1 (Sobolev–Stein embedding on a generalized Morrey space)**.** *Suppose that* $1<p<\infty$*,* $1/q = 1/p - 1/Q$*, and* (φ_1,φ_2) *satisfy* (5.6)*. Then for every* $u \in C_0^\infty(\mathbb{H}_n)$

$$\|u\|_{M_{q,\varphi_2}} \lesssim \|\nabla_{\mathscr{L}} u\|_{M_{p,\varphi_1}}.$$

Proof. Let $u \in C_0^\infty(\mathbb{H}_n)$. Using the integral representation formula for the fundamental solution (cf. [31, p. 237]), we have

$$u(g) = \int\limits_{\mathbb{H}_n} \Gamma(g^{-1}y)\mathscr{L}u(y)dy. \tag{6.1}$$

Keeping in mind that $\mathscr{L} = \sum\limits_{i=1}^{2n} X_i^2$ and $X_i^* = -X_i$ and integrating by parts on the right-hand side of (6.1), we get

$$u(g) = \int\limits_{\mathbb{H}_n} (\nabla_{\mathscr{L}}\Gamma)(g^{-1}y)\nabla_{\mathscr{L}}u(y)dy. \tag{6.2}$$

On the other hand, out of the origin,

$$\nabla_{\mathscr{L}}\Gamma(g) = C_0\, \nabla_{\mathscr{L}}\left(|x|^{2-Q}\right) = (2-Q)C_0\, |x|^{1-Q}\, \nabla_{\mathscr{L}}|x|.$$

Therefore, since $\nabla_{\mathscr{L}}|\cdot|$ is smooth in $\mathbb{H}_n \setminus \{0\}$ and δ_λ-homogeneous of degree zero, we have

$$\nabla_{\mathscr{L}}\Gamma(g) \leqslant C|x|^{1-Q}$$

with a suitable constant $C > 0$ depending only on $\mathscr{L}$. Using this inequality in (6.2), we get

$$|u(g)| \leqslant C \int\limits_{\mathbb{H}_n} |\nabla_{\mathscr{L}}u(y)||x|^{1-Q}dy = CI_1(|\nabla_{\mathscr{L}}u|)(g). \tag{6.3}$$

Then, by Theorem 5.2,

$$\|u\|_{M_{q,\varphi_2}} \leqslant C\|I_1(|\nabla_{\mathscr{L}}u|)\|_{M_{q,\varphi_2}} \leqslant C\|\nabla_{\mathscr{L}}u\|_{M_{p,\varphi_1}}. \qquad \square$$

The following assertion is proved in the same way as Theorem 5.3.

Theorem 6.2 (Sobolev–Stein embedding on a generalized Morrey space)**.** *Suppose that* $1 < p < \infty$, $1/q = 1/p - 1/Q$, *and* φ *satisfies the conditions* (5.7) *and* (5.8). *Then for every* $u \in C_0^\infty(\mathbb{H}_n)$

$$\|u\|_{M_{q,\varphi^{1/q}}} \lesssim \|\nabla_{\mathscr{L}}u\|_{M_{p,\varphi^{1/p}}}.$$

The following theorem establishes the boundedness of $\mathscr{I}_\alpha$ in the generalized Besov–Morrey spaces on $\mathbb{H}_n$

$$BM^s_{p\theta,\varphi}(\mathbb{H}_n) = \Big\{f : \|f\|_{BM^s_{p\theta,\varphi}} = \|f\|_{M_{p,\varphi}} + \Big(\int\limits_{\mathbb{H}_n} \frac{\|f(g\cdot) - f(\cdot)\|^\theta_{M_{p,\varphi}}}{|g|^{Q+s\theta}}dg\Big)^{1/\theta} < \infty\Big\} \tag{6.4}$$

where $1 \leqslant p, \theta \leqslant \infty$ and $0 < s < 1$.

Besov spaces $B^s_{p\theta}(G)$ in the setting Lie groups G were studied by many authors (cf., for example [5] and [32]–[35]), unlike Besov–Morrey spaces (however, cf., for example [36]–[38]).

Theorem 6.3. *Suppose that* $1 < p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, *and* (φ_1, φ_2) *satisfy the condition* (5.6). *If* $1 \leqslant \theta \leqslant \infty$ *and* $0 < s < 1$, *then the operator* $\mathscr{I}_\alpha$ *is bounded from* $BM^s_{p\theta,\varphi_1}(\mathbb{H}_n)$ *to* $BM^s_{q\theta,\varphi_2}(\mathbb{H}_n)$. *More precisely, there is a constant* $C > 0$ *such that for all* $f \in BM^s_{p\theta,\varphi_1}(\mathbb{H}_n)$

$$\|\mathscr{I}_\alpha f\|_{BM^s_{q\theta,\varphi_2}} \leqslant C\|f\|_{BM^s_{p\theta,\varphi_1}}.$$

Proof. By the definition of the generalized Besov–Morrey spaces on $\mathbb{H}_n$, it suffices to show

$$\|\tau_h \mathscr{I}_\alpha f - \mathscr{I}_\alpha f\|_{M_{q,\varphi}} \leqslant C\|\tau_y f - f\|_{M_{p,\varphi}},$$

where $\tau_h f(g) = f(hg)$. It is easy to see that $\tau_h f$ commutes with $\mathscr{I}_\alpha$, i.e., $\tau_h \mathscr{I}_\alpha f = \mathscr{I}_\alpha(\tau_h f)$. Hence

$$|\tau_h \mathscr{I}_\alpha f - \mathscr{I}_\alpha f| = |\mathscr{I}_\alpha(\tau_h f) - \mathscr{I}_\alpha f| \leqslant \mathscr{I}_\alpha(|\tau_h f - f|).$$

Taking the $M_{p,\varphi}$-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of $\mathscr{I}_\alpha$ from $M_{p,\varphi}(\mathbb{H}_n)$ to $M_{q,\varphi}(\mathbb{H}_n)$. □

Theorem 6.4. *Suppose that* $1 < p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, *and* φ *satisfies the conditions* (5.7) *and* (5.8). *If* $1 \leqslant \theta \leqslant \infty$ *and* $0 < s < 1$, *then the operator* $\mathscr{I}_\alpha$ *is bounded from* $BM^s_{p\theta,\varphi^{1/p}}(\mathbb{H}_n)$ *to* $BM^s_{q\theta,\varphi^{1/q}}(\mathbb{H}_n)$. *More precisely, there is a constant* $C > 0$ *such that for all* $f \in BM^s_{p\theta,\varphi^{1/p}}(\mathbb{H}_n)$

$$\|\mathscr{I}_\alpha f\|_{BM^s_{q\theta,\varphi^{1/q}}} \leqslant C\|f\|_{BM^s_{p\theta,\varphi^{1/p}}}.$$

From Theorems 6.3 and 6.4 we obtain the following Sobolev–Stein embedding inequality on a generalized Besov–Morrey space.

Theorem 6.5 (Sobolev–Stein embedding on a generalized Besov–Morrey space)**.** *Suppose that* $1 < p < \infty$, $1/q = 1/p - 1/Q$, (φ_1, φ_2) *satisfy the condition* (5.6), $1 \leqslant \theta \leqslant \infty$, *and* $0 < s < 1$. *Then for every* $u \in C_0^\infty(\mathbb{H}^n)$

$$\|u\|_{BM^s_{q\theta,\varphi_2}} \lesssim \|\nabla_{\mathscr{L}} u\|_{BM^s_{p\theta,\varphi_1}}.$$

Theorem 6.6 (Sobolev–Stein embedding on a generalized Besov–Morrey space)**.** *Suppose that* $1 < p < \infty$, $1/q = 1/p - 1/Q$, φ *satisfies the conditions* (5.7) *and* (5.8), $1 \leqslant \theta \leqslant \infty$, *and* $0 < s < 1$. *Then for every* $u \in C_0^\infty(\mathbb{H}^n)$

$$\|u\|_{BM^s_{q\theta,\varphi^{1/q}}} \lesssim \|\nabla_{\mathscr{L}} u\|_{BM^s_{p\theta,\varphi^{1/p}}}.$$

The Dirichlet problem for the Kohn Laplacian on the Heisenberg group was considered in [39, 40]. Note that our results lead to the following a priori estimate for the sub-Laplacian equation $\mathscr{L} f = g$.

Theorem 6.7. *Suppose that* $1 < p < q < \infty$, $0 < s < 1$, $1 \leqslant \theta \leqslant \infty$, $g \in BM^s_{p\theta,\lambda}(\mathbb{H}_n)$, *and* $\mathscr{L} f = g$. *The following assertions hold.*

1. *If* $1/q = 1/p - 2/Q$ *and* (φ_1, φ_2) *satisfy the condition* (5.6), *then*

$$\|f\|_{BM^s_{q\theta,\varphi_2}} \lesssim \|g\|_{BM^s_{p\theta,\varphi_1}}.$$

2. *If* $1/q = 1/p - 1/Q$ *and* (φ_1, φ_2) *satisfy the condition* (5.6), *then*

$$\|X_i f\|_{BM^s_{q\theta,\varphi_2}} \lesssim \|g\|_{BM^s_{p\theta,\varphi_1}}, \quad i = 1, 2, \ldots, 2n.$$

Theorem 6.8. *Suppose that* $1 < p < q < \infty$, $0 < s < 1$, $1 \leqslant \theta \leqslant \infty$, $g \in BM^s_{p\theta,\lambda}(\mathbb{H}_n)$, *and* $\mathscr{L} f = g$. *Then the following assertions hold.*

1. *If* $1/q = 1/p - 2/Q$ *and* φ *satisfies the conditions* (5.7) *and* (5.8)*, then*

$$\|f\|_{BM^s_{q\theta,\varphi^{1/q}}} \lesssim \|g\|_{BM^s_{p\theta,\varphi^{1/p}}}.$$

2. *If* $1/q = 1/p - 1/Q$ *and* φ *satisfies the conditions* (5.7) *and* (5.8)*, then*

$$\|X_i f\|_{BM^s_{q\theta,\varphi^{1/q}}} \lesssim \|g\|_{BM^s_{p\theta,\varphi^{1/p}}}, \quad i = 1, 2, \dots, 2n.$$

The proof of Theorems 6.5 and 6.7 is similar to that of Theorem 6.1.

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