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On the Boundedness of the Fractional Maximal Operator, Riesz Potential and Their Commutators in Generalized Morrey Spaces

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Dedicated to the 70th birthday of Prof. S. Samko

Abstract. In the paper the authors find conditions on the pair (φ_1, φ_2) which ensure the Spanne type boundedness of the fractional maximal operator M_α and the Riesz potential operator I_α from one generalized Morrey spaces M_{p,φ_1} to another M_{q,φ_2}, $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$, and from M_{1,φ_1} to the weak space WM_{q,φ_2}, $1 < q < \infty$, $1 - 1/q = \alpha/n$. We also find conditions on φ which ensure the Adams type boundedness of the M_α and I_α from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $1 < q < \infty$. As applications of those results, the boundeness of the commutators of operators M_α and I_α on generalized Morrey spaces is also obtained. In the case $b \in BMO(\mathbb{R}^n)$ and $1 < p < q < \infty$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators $M_{b,\alpha}$ and $[b, I_\alpha]$ from M_{p,φ_1} to M_{q,φ_2} with $1/p - 1/q = \alpha/n$. We also find the sufficient conditions on φ which ensures the boundedness of the operators $M_{b,\alpha}$ and $[b, I_\alpha]$ from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$. In all cases conditions for the boundedness are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) and φ, which do not assume any assumption on monotonicity of φ_1, φ_2 and φ in r. As applications, we get some estimates for Marcinkiewicz operator and fractional powers of the some analytic semigroups on generalized Morrey spaces.

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1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with weighted Lebesgue spaces, general Morrey-type spaces also play an important role.

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For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy, \qquad 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n.$$

If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

It is well known that fractional maximal operator, Riesz potential and Calderón-Zygmund operators play an important role in harmonic analysis (see [22, 29, 30]).

In [3] (see also [17]), we prove the boundedness of the maximal operator M and the Calderón-Zygmund operators T from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1 < p < \infty$, and from M_{1,φ_1} to the weak space WM_{1,φ_2}. In the case $b \in BMO(\mathbb{R}^n)$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operators M_b and $[b,T]$ from M_{p,φ_1} to M_{p,φ_2}, $1 < p < \infty$, where $M_b f(x) = M\big((b(\cdot) - b(x))f\big)(x)$ and $[b,T]f(x) = b(x)\,Tf(x) - T(bf)(x)$.

In this work, we prove the boundedness of the operators M_α and I_α, $\alpha \in (0,n)$ from one generalized Morrey space M_{p,φ_1} to another one M_{q,φ_2}, $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$, and from M_{1,φ_1} to the weak space WM_{q,φ_2}, $1 < q < \infty$, $1 - 1/q = \alpha/n$. We also prove the Adams type boundedness of the operators M_α and I_α from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $1 < q < \infty$. In the case $b \in BMO(\mathbb{R}^n)$, $1 < p < q < \infty$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the commutator of operators $M_{b,\alpha}$ and $[b, I_\alpha]$ from M_{p,φ_1} to M_{q,φ_2}, $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$, where $M_{b,\alpha} f(x) = M_\alpha\big((b(\cdot) - b(x))f\big)(x)$ and $[b, I_\alpha]f(x) = b(x)\,I_\alpha f(x) - I_\alpha(bf)(x)$. We also find the sufficient conditions on φ which ensures the boundedness of the operators $M_{b,\alpha}$ and $[b, I_\alpha]$ from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$. Finally, as applications we apply this result to several particular operators such as Marcinkiewicz operator and fractional powers of the some analytic semigroups.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [24] to study the local behavior of solutions to second-order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [24, 26].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned}\|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)}\\ &= \sup_{t>0} t\,|\{y\in B(x,r):\,|f(y)|>t\}|^{1/p}.\end{aligned}$$

The classical result by Hardy-Littlewood-Sobolev states that if $1<p<q<\infty$, then the operator I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p}-\frac{1}{q}\right)$ and for $p=1<q<\infty$, the operator I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(1-\frac{1}{q}\right)$. S. Spanne and D.R. Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results can be summarized as follows.

Theorem 2.1 (Spanne, but published by Peetre [26]). *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $0<\lambda<n-\alpha p$. Moreover, let $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ and $\frac{\lambda}{p}=\frac{\mu}{q}$. Then for $p>1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p=1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Theorem 2.2 (Adams [1]). *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $0<\lambda<n-\alpha p$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. Then for $p>1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p=1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Recall that, for $0<\alpha<n$,

$$M_\alpha f(x) \le v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x),$$

hence Theorems 2.1 and 2.2 also imply boundedness of the fractional maximal operator M_α, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

3. Generalized Morrey spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 3.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and $1\le p<\infty$. We denote by $M_{p,\varphi}\equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f\in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\,|B(x,r)|^{-\frac{1}{p}}\,\|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi}\equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f\in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\,|B(x,r)|^{-\frac{1}{p}}\,\|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x,r)=r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In [15]–[18], [20], [23] and [25] there were obtained sufficient conditions on φ_1 and φ_2 for the boundedness of the maximal operator M and Calderón-Zygmund operator from M_{p,φ_1} to M_{p,φ_2}, $1<p<\infty$ and of the fractional maximal operator

M_α and Riesz potential operator I_α from M_{p,φ_1} to M_{q,φ_2}, $1 < p < q < \infty$ (see also [5]–[9]). In [25] the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\,\varphi(x,r) \tag{3.1}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition:

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\,\varphi(x,r)^p, \tag{3.2}$$

for the singular integral operator T, and the condition

$$\int_r^\infty t^{\alpha p}\varphi(x,t)^p \frac{dt}{t} \le C\,r^{\alpha p}\varphi(x,r)^p \tag{3.3}$$

for the Riesz potential operator I_α, where $C(>0)$ does not depend on r and $x \in \mathbb{R}^n$.

4. Boundedness of the fractional maximal operator in generalized Morrey spaces

4.1. Spanne type result

Sufficient conditions on φ for the boundedness of M and M_α in generalized Morrey spaces $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ have been obtained in [2], [4], [5], [6], [8], [17], [18], [23], [25].

The following lemma is true.

Lemma 4.1. *Let $1 \le p < \infty$, $0 \le \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$ the inequality*

$$\|M_\alpha f\|_{L_q(B(x,r))} \lesssim \|f\|_{L_p(B(x,2r))} + r^{\frac{n}{q}} \sup_{t>2r} t^{-n+\alpha}\|f\|_{L_1(B(x,t))} \tag{4.1}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|M_\alpha f\|_{WL_q(B(x,r))} \lesssim \|f\|_{L_1(B(x,2r))} + r^{\frac{n}{q}} \sup_{t>2r} t^{-n+\alpha}\|f\|_{L_1(B(x,t))} \tag{4.2}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. For arbitrary ball $B = B(x,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$.

$$\|M_\alpha f\|_{L_q(B)} \le \|M_\alpha f_1\|_{L_q(B)} + \|M_\alpha f_2\|_{L_q(B)}.$$

By the continuity of the operator $M_\alpha : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$ we have

$$\|M_\alpha f_1\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)}.$$

Let y be an arbitrary point from B. If $B(y,t) \cap {}^\complement(2B) \ne \emptyset$, then $t > r$. Indeed, if $z \in B(y,t) \cap {}^\complement(2B)$, then $t > |y-z| \ge |x-z| - |x-y| > 2r - r = r$.

On the other hand, $B(y,t) \cap {}^\complement(2B) \subset B(x,2t)$. Indeed, $z \in B(y,t) \cap {}^\complement(2B)$, then we get $|x-z| \le |y-z| + |x-y| < t + r < 2t$.

Hence

$$\begin{aligned} M_\alpha f_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|^{1-\alpha/n}} \int_{B(y,t)\cap{}^\complement(2B)} |f(z)|dz \\ &\le 2^{n-\alpha} \sup_{t>r} \frac{1}{|B(x,2t)|^{1-\alpha/n}} \int_{B(x,2t)} |f(z)|dz \\ &= 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)} |f(z)|dz. \end{aligned}$$

Therefore, for all $y \in B$ we have

$$M_\alpha f_2(y) \le 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)} |f(z)|dz. \tag{4.3}$$

Thus

$$\|M_\alpha f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + |B|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)} |f(z)|dz \right).$$

Let $p = 1$. It is obvious that for any ball $B = B(x,r)$

$$\|M_\alpha f\|_{WL_q(B)} \le \|M_\alpha f_1\|_{WL_q(B)} + \|M_\alpha f_2\|_{WL_q(B)}.$$

By the continuity of the operator $M_\alpha : L_1(\mathbb{R}^n) \to WL_q(\mathbb{R}^n)$ we have

$$\|M_\alpha f_1\|_{WL_q(B)} \lesssim \|f\|_{L_1(2B)}.$$

Then by (4.3) we get the inequality (4.2). □

Lemma 4.2. *Let $1 \le p < \infty$, $0 \le \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$, the inequality*

$$\|M_\alpha f\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L_p(B(x,t))} \tag{4.4}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|M_\alpha f\|_{WL_q(B)} \lesssim r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L_1(B(x,t))} \tag{4.5}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Denote

$$\mathcal{M}_1 := |B|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)} |f(z)|dz \right),$$
$$\mathcal{M}_2 := \|f\|_{L_p(2B)}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim |B|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{\frac{1}{q}}} \left(\int_{B(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \right).$$

On the other hand,

$$|B|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{\frac{1}{q}}} \left(\int_{B(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \right)$$
$$\gtrsim |B|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{\frac{1}{q}}} \right) \|f\|_{L_p(2B)} \approx \mathcal{M}_2.$$

Since by Lemma 4.1

$$\|M_\alpha f\|_{L_q(B)} \le \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (4.4).

Let $p = 1$. The inequality (4.5) directly follows from (4.2). □

Theorem 4.3. *Let $1 \le p < \infty$, $0 \le \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} t^\alpha \varphi_1(x,t) \le C \varphi_2(x,r), \tag{4.6}$$

where C does not depend on x and r. Then for $p > 1$, M_α is bounded from M_{p,φ_1} to M_{q,φ_2} and for $p = 1$, M_α is bounded from M_{1,φ_1} to WM_{q,φ_2}.

Proof. By Lemma 4.2 we get

$$\begin{aligned}\|M_\alpha f\|_{M_{q,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \left(\sup_{t>r} t^{-\frac{n}{q}} \|f\|_{L_p(B(x,t))}\right) \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^{\frac{n}{p}-\frac{n}{q}} \varphi_1(x,t) \left(\varphi_1(x,r)^{-1} t^{-\frac{n}{p}} \|f\|_{L_p(B(x,t))}\right) \\ &\lesssim \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^\alpha \varphi_1(x,t) \\ &\lesssim \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n)}\end{aligned}$$

if $p \in (1,\infty)$ and

$$\begin{aligned}\|M_\alpha f\|_{WM_{q,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \left(\sup_{t>r} t^{-\frac{n}{q}} \|f\|_{L_1(B(x,t))}\right) \\ &\lesssim \|f\|_{M_{1,\varphi_1}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^\alpha \varphi_1(x,t) \\ &\lesssim \|f\|_{M_{1,\varphi_1}(\mathbb{R}^n)}\end{aligned}$$

if $p = 1$. □

In the case $\alpha = 0$ and $p = q$ from Theorem 4.3 we get the following corollary, which was proved in [3].

Corollary 4.1. *Let $1 \le p < \infty$ and let (φ_1, φ_2) satisfy the condition* $\Longleftarrow$ ok?

$$\sup_{r<t<\infty} \varphi_1(x,t) \le C\, \varphi_2(x,r), \tag{4.7}$$

where C does not depend on x and r. Then for $p > 1$, M is bounded from M_{p,φ_1} to M_{p,φ_2} and for $p = 1$, M is bounded from M_{1,φ_1} to WM_{1,φ_2}.

4.2. Adams type result

The following is a result of Adams type for the fractional maximal operator.

Theorem 4.4. *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$ and let $\varphi(x,t)$ satisfy the condition*

$$\sup_{r<t<\infty} \varphi(x,t) \le C\, \varphi(x,r), \tag{4.8}$$

and

$$\sup_{r<t<\infty} t^\alpha \varphi(x,t)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{4.9}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator M_α is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}}}$. Write $f = f_1 + f_2$, where $B = B(x,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

For $M_\alpha f_2(x)$ and for all $y \in B$ from (4.3) we have

$$\begin{aligned}M_\alpha(f_2)(y) &\le 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)} |f(z)|dz \\ &\lesssim \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L_p(B(x,t))}. \end{aligned} \tag{4.10}$$

Then from conditions (4.9) and (4.10) we get

$$\begin{aligned}M_\alpha f(x) &\lesssim r^\alpha\, Mf(x) + \sup_{t>2r} t^{\alpha-\frac{n}{p}} \|f\|_{L_p(B(x,t))} \\ &\le r^\alpha\, Mf(x) + \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{t>2r} t^\alpha \varphi(x,t)^{\frac{1}{p}} \\ &\lesssim r^\alpha\, Mf(x) + r^{-\frac{\alpha p}{q-p}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}.\end{aligned}$$

Consequently choosing $r = \Big(\frac{\|f\|_{M_{p,\varphi^{1/p}}}}{Mf(x)}\Big)^{\frac{q-p}{\alpha q}}$ for every $x \in \mathbb{R}^n$, we have

$$|M_\alpha f(x)| \lesssim (Mf(x))^{\frac{p}{q}} \, \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{\frac{1}{p}}}$ provided by Corollary 4.1 in virtue of condition (4.8).

$$\begin{aligned}
\|M_\alpha f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|M_\alpha f\|_{L_q(B(x,t))} \\
&\lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|^{\frac{p}{q}}_{L_p(B(x,t))} \\
&= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{n}{p}} \|Mf\|_{L_p(B(x,t))} \right)^{\frac{p}{q}} \\
&= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \|Mf\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \\
&\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}},
\end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned}
\|M_\alpha f\|_{WM_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|M_\alpha f\|_{WL_q(B(x,t))} \\
&\lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|^{\frac{1}{q}}_{WL_1(B(x,t))} \\
&= \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-1} t^{-n} \|Mf\|_{WL_1(B(x,t))} \right)^{\frac{1}{q}} \\
&= \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}} \|Mf\|^{\frac{1}{q}}_{WM_{1,\varphi}} \\
&\lesssim \|f\|_{M_{1,\varphi}},
\end{aligned}$$

if $1 < q < \infty$. □

In the case $\varphi(x,t) = t^{\lambda - n}$, $0 < \lambda < n$ from Theorem 4.4 we get the following Adams type result for the fractional maximal operator.

Corollary 4.2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$, the operator M_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$, M_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

5. Riesz potential operator in the spaces $M_{p,\varphi}$

5.1. Spanne type result

In [25] the following statements was proved by Riesz potential operator I_α.

Theorem 5.5. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\varphi(x,r)$ satisfy the conditions* (3.1) *and* (3.3)*. Then the operator I_α is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.*

The following statements, containing results obtained in [23], [25] were proved in [15, 17] (see also [5]–[9], [16, 18]).

Theorem 5.6. *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and let (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty r^\alpha \varphi_1(x,r) \frac{dr}{r} \le C\, \varphi_2(x,t), \tag{5.1}$$

where C does not depend on x and t. Then the operators M_α and I_α are bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2}.

5.2. Adams type result

The following is a result of Adams type.

Theorem 5.7. *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $q > p$ and let $\varphi(x,t)$ satisfy the conditions* (4.8) *and*

$$\int_r^\infty t^\alpha \,\varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \le C r^{-\frac{\alpha p}{q-p}}, \tag{5.2}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator I_α is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $q > p$ and $f \in M_{p,\varphi^{\frac{1}{p}}}$. Write $f = f_1 + f_2$, where $B = B(x,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

For $I_\alpha f_2(x)$ we have

$$\begin{aligned} |I_\alpha f_2(x)| &\le \int_{\complement_{B(x,2r)}} |x-y|^{\alpha-n} |f(y)| dy \\ &\lesssim \int_{\complement_{B(x,2r)}} |f(y)| dy \int_{|x-y|}^\infty t^{\alpha-n-1} dt \\ &\lesssim \int_{2r}^\infty \left(\int_{2r<|x-y|<t} |f(y)| dy \right) t^{\alpha-n-1} dt \\ &\lesssim \int_r^\infty t^{\alpha-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))} dt. \end{aligned} \tag{5.3}$$

Then from condition (5.2) and inequality (5.3) we get

$$\begin{aligned} |I_\alpha f(x)| &\lesssim r^\alpha \, Mf(x) + \int_r^\infty t^{\alpha-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))} dt \\ &\le r^\alpha \, Mf(x) + \|f\|_{M_{p,\varphi}} \int_r^\infty t^\alpha \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \\ &\lesssim r^\alpha \, Mf(x) + r^{-\frac{\alpha p}{q-p}} \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned} \tag{5.4}$$

Hence choosing $r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p}}}}{Mf(x)} \right)^{\frac{q-p}{\alpha q}}$ for every $x \in \mathbb{R}^n$, we have

$$|I_\alpha f(x)| \lesssim (Mf(x))^{\frac{p}{q}} \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Consequently the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{\frac{1}{p}}}$ provided by Corollary 4.1 in virtue of condition (4.8).

$$\begin{aligned} \|I_\alpha f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|I_\alpha f\|_{L_q(B(x,t))} \\ &\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{L_p(B(x,t))}^{\frac{p}{q}} \\ &= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{n}{p}} \|Mf\|_{L_p(B(x,t))} \right)^{\frac{p}{q}} \\ &= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \, \|Mf\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}} \\ &\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned}\|I_\alpha f\|_{WM_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|I_\alpha f\|_{WL_q(B(x,t))} \\ &\lesssim \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{WL_1(B(x,t))}^{\frac{1}{q}} \\ &= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-1} t^{-n} \|Mf\|_{WL_1(B(x,t))} \right)^{\frac{1}{q}} \\ &= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \|Mf\|_{WM_{1,\varphi}}^{\frac{1}{q}} \\ &\lesssim \|f\|_{M_{1,\varphi}},\end{aligned}$$

if $1 < q < \infty$. □

In the case $\varphi(x,t) = t^{\lambda-n}$, $0 < \lambda < n$ from Theorem 5.7 we get Adams Theorem 2.2.

6. Commutators of fractional maximal operators in the spaces $M_{p,\varphi}$

6.1. Spanne type result

The theory of commutator was originally studied by Coifman, Rochberg and Weiss in [12]. Since then, many authors have been interested in studying this theory. When $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, Chanillo [10] proved that the commutator operator $[b, I_\alpha]f$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ whenever $b \in BMO(\mathbb{R}^n)$.

First we introduce the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 6.2. Suppose that $f \in L_1^{\rm loc}(\mathbb{R}^n)$, and let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{ f \in L_1^{\rm loc}(\mathbb{R}^n) \ : \ \|f\|_* < \infty \}.$$

If one regards two functions whose difference is a constant as one, then space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 6.1.

(1) The John-Nirenberg inequality: there are constants C_1, $C_2 > 0$, such that for all $f \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \ : \ |f(x) - f_B| > \beta\}| \le C_1 |B| e^{-C_2\beta/\|f\|_*}, \ \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{6.1}$$

for $1 < p < \infty$.

(3) Let $f \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left| f_{B(x,r)} - f_{B(x,t)} \right| \le C \|f\|_* \ln \frac{t}{r} \text{ for } 0 < 2r < t, \tag{6.2}$$

where C is independent of f, x, r and t.

For the sublinear commutator of the fractional maximal operator

$$M_{b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)||f(y)|dy$$

in [13] the following statement was proved, containing the result in [23, 25].

Theorem 6.8. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, $\varphi(x,r)$ satisfies the conditions* (3.1) *and* (3.3)*. Then the operator $M_{b,\alpha}$ is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.*

Lemma 6.3. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$.*
Then the inequality

$$\|M_{b,\alpha} f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* \, r^{\frac{n}{q}} \sup_{t>2r} \Big(1 + \ln \frac{t}{r}\Big) t^{-\frac{n}{q}} \, \|f\|_{L_p(B(x_0,t))}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Write $f = f_1 + f_2$, where $B = B(x_0,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence,

$$\|M_{b,\alpha} f\|_{L_q(B)} \le \|M_{b,\alpha} f_1\|_{L_q(B)} + \|M_{b,\alpha} f_2\|_{L_q(B)}.$$

From the boundedness of $M_{b,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that:

$$\begin{aligned}\|M_{b,\alpha} f_1\|_{L_q(B)} &\le \|M_{b,\alpha} f_1\|_{L_q(\mathbb{R}^n)} \\ &\lesssim \|b\|_* \, \|f_1\|_{L_p(\mathbb{R}^n)} = \|b\|_* \, \|f\|_{L_p(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}M_{b,\alpha} f_2(x) &\lesssim \sup_{t>0} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)} |b(y) - b(x)| \, f_2(y)|dy \\ &= \sup_{t>0} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)\cap \complement_{(2B)}} |b(y) - b(x)| \, |f(y)|dy.\end{aligned}$$

Let x be an arbitrary point from B. If $B(x,t) \cap \{\complement(2B)\} \ne \emptyset$, then $t > r$. Indeed, if $y \in B(x,t) \cap \{\complement(2B)\}$, then $t > |x-y| \ge |x_0 - y| - |x_0 - x| > 2r - r = r$.

On the other hand, $B(x,t) \cap \{\complement(2B)\} \subset B(x_0, 2t)$. Indeed, $y \in B(x,t) \cap \{\complement(2B)\}$, then we get $|x_0 - y| \le |x-y| + |x_0 - x| < t + r < 2t$.

Hence

$$\begin{aligned}M_{b,\alpha}(f_2)(x) &= \sup_{t>0} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)\cap \complement_{(2B)}} |b(y) - b(x)| \, |f(y)|dy \\ &\le 2^{n-\alpha} \sup_{t>r} \frac{1}{|B(x_0,2t)|^{1-\alpha/n}} \int_{B(x_0,2t)} |b(y) - b(x)| \, |f(y)|dy \\ &= 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x_0,t)|^{1-\alpha/n}} \int_{B(x_0,t)} |b(y) - b(x)| \, |f(y)|dy.\end{aligned}$$

Therefore, for all $x \in B$ we have

$$M_{b,\alpha}(f_2)(x) \le 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x_0,t)|^{1-\alpha/n}} \int_{B(x_0,t)} |b(y) - b(x)| \, |f(y)|dy. \qquad (6.3)$$

Then

$$\begin{aligned}\|M_{b,\alpha}f_2\|_{L_q(B)} &\lesssim \left(\int_B \left(\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\alpha/n}}\int_{B(x_0,t)}|b(y)-b(x)|\,|f(y)|dy\right)^q dx\right)^{\frac{1}{q}} \\ &\le \left(\int_B \left(\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\alpha/n}}\int_{B(x_0,t)}|b(y)-b_B|\,|f(y)|dy\right)^q dx\right)^{\frac{1}{q}} \\ &+ \left(\int_B \left(\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\alpha/n}}\int_{B(x_0,t)}|b(x)-b_B|\,|f(y)|dy\right)^q dx\right)^{\frac{1}{q}} \\ &= J_1+J_2.\end{aligned}$$

Let us estimate J_1.

$$\begin{aligned}J_1 &= r^{\frac{n}{q}}\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\alpha/n}}\int_{B(x_0,t)}|b(y)-b_B|\,|f(y)|dy \\ &\approx r^{\frac{n}{q}}\sup_{t>2r}t^{\alpha-n}\int_{B(x_0,t)}|b(y)-b_B|\,|f(y)|dy.\end{aligned}$$

Applying Hölder's inequality and by (6.1), (6.2), we get

$$\begin{aligned}J_1 &\lesssim r^{\frac{n}{q}}\sup_{t>2r}t^{\alpha-n}\int_{B(x_0,t)}|b(y)-b_{B(x_0,t)}|\,|f(y)|dy \\ &+ r^{\frac{n}{q}}\sup_{t>2r}t^{\alpha-n}|b_{B(x_0,r)}-b_{B(x_0,t)}|\int_{B(x_0,t)}|f(y)|dy \\ &\lesssim r^{\frac{n}{q}}\sup_{t>2r}t^{\alpha-\frac{n}{p}}\left(\frac{1}{|B(x_0,t)|}\int_{B(x_0,t)}|b(y)-b_{B(x_0,t)}|^{p'}dy\right)^{\frac{1}{p'}}\|f\|_{L_p(B(x_0,t))} \\ &+ r^{\frac{n}{q}}\sup_{t>2r}t^{\alpha-\frac{n}{p}}|b_{B(x_0,r)}-b_{B(x_0,t)}|\|f\|_{L_p(B(x_0,t))} \\ &\lesssim \|b\|_*\,r^{\frac{n}{q}}\sup_{t>2r}t^{\frac{n}{q}}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}.\end{aligned}$$

In order to estimate J_2 note that

$$\begin{aligned}J_2 &= \left(\int_B|b(x)-b_B|^q dx\right)^{\frac{1}{q}}\sup_{t>2r}t^{\alpha-n}\int_{B(x_0,t)}|f(y)|dy \\ &\lesssim \|b\|_*\,r^{\frac{n}{q}}\sup_{t>2r}t^{\frac{n}{q}}\|f\|_{L_p(B(x_0,t))}.\end{aligned}$$

Summing up J_1 and J_2, for all $p\in(1,\infty)$ we get

$$\|M_{b,\alpha}f_2\|_{L_q(B)} \lesssim \|b\|_*\,r^{\frac{n}{q}}\sup_{t>2r}t^{\frac{n}{q}}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}. \tag{6.4}$$

Finally,

$$\begin{aligned}\|M_{b,\alpha}f\|_{L_q(B)} &\lesssim \|b\|_*\,\|f\|_{L_p(2B)} + \|b\|_*\,r^{\frac{n}{q}}\sup_{t>2r}t^{\frac{n}{q}}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))} \\ &\lesssim \|b\|_*\,r^{\frac{n}{q}}\sup_{t>2r}t^{\frac{n}{q}}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}.\end{aligned}$$ □

The following theorem is true.

Theorem 6.9. *Let $1<p<\infty$, $0\le\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $b\in BMO(\mathbb{R}^n)$ and let (φ_1,φ_2) satisfy the condition*

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)t^{\alpha}\,\varphi_1(x,t)\le C\,\varphi_2(x,r), \tag{6.5}$$

where C does not depend on x and r.

Then the operator $M_{b,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2}. Moreover

$$\|M_{b,\alpha}f\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

Proof. The statement of Theorem 6.9 follows by Lemma 6.3 in the same manner as in the proof of Theorem 4.3. □

In the case $\alpha = 0$ and $p = q$ from Theorem 6.9 we get the following corollary.

Corollary 6.3. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and let (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \varphi_1(x,t) \le C \, \varphi_2(x,r), \tag{6.6}$$

where C does not depend on x and r.

Then the operator M_b is bounded from M_{p,φ_1} to M_{p,φ_2}. Moreover

$$\|M_b f\|_{M_{p,\varphi_2}} \lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

6.2. Adams type result

The following is a result of Adams type.

Theorem 6.10. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $b \in BMO(\mathbb{R}^n)$ and let $\varphi(x,t)$ satisfy the conditions*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big)^p \varphi(x,t) \le C \, \varphi(x,r) \tag{6.7}$$

and

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{\alpha} \varphi(x,t)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{6.8}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then $M_{b,\alpha}$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}}}$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$.

For $x \in B$ we have

$$\begin{aligned} |M_{b,\alpha} f_2(x)| &\lesssim \sup_{t>0} t^{\alpha-n} \int_{B(x,t)} |b(y) - b(x)| \, |f_2(y)| dy \\ &\approx \sup_{t>2r} t^{\alpha-n} \int_{B(x,t)} |b(y) - b(x)| \, |f_2(y)| dy. \end{aligned}$$

Applying Hölder's inequality and by (6.1), (6.2), we get

$$\begin{aligned} |M_{b,\alpha} f_2(x)| &\lesssim \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| \, |f(y)| dy \\ &\quad + \sup_{t>2r} t^{\alpha-n} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| dy \\ &\lesssim \sup_{t>2r} t^{\alpha-\frac{n}{p}} \left(\frac{1}{|B(x_0,t)|} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}|^{p'} dy \right)^{\frac{1}{p'}} \|f\|_{L_p(B(x_0,t))} \\ &\quad + \sup_{t>2r} t^{\alpha-\frac{n}{p}} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L_p(B(x_0,t))} \\ &\lesssim \|b\|_* \sup_{t>2r} t^{\alpha-\frac{n}{p}} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))}. \end{aligned}$$

Consequently, for all $p \in (1,\infty)$ and $x \in B$ we get

$$|M_{b,\alpha} f_2(x)| \lesssim \|b\|_* \sup_{t>2r} t^{\alpha-\frac{n}{p}} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))}. \tag{6.9}$$

Then from conditions (6.8) and (6.9) we get

$$\begin{aligned} M_{b,\alpha}f(x) &\lesssim \|b\|_* \, r^\alpha \, M_b f(x) + \|b\|_* \sup_{t>2r} t^{\alpha-\frac{n}{p}} \Big(1+\ln\frac{t}{r}\Big) \, \|f\|_{L_p(B(x_0,t))} \\ &\le \|b\|_* \, r^\alpha \, M_b f(x) + \|b\|_* \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{t>r} \Big(1+\ln\frac{t}{r}\Big) \, t^\alpha \, \varphi(x,t)^{\frac{1}{p}} \\ &\lesssim \|b\|_* \, r^\alpha \, M_b f(x) + \|b\|_* \, r^{-\frac{\alpha p}{q-p}} \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned} \tag{6.10}$$

Hence choose $r = \Big(\frac{\|f\|_{M_{p,\varphi^{\frac{1}{p}}}}}{M_b f(x)}\Big)^{\frac{q-p}{\alpha q}}$ for every $x \in B$, we have

$$|M_{b,\alpha}f(x)| \lesssim \|b\|_* \, (M_b f(x))^{\frac{p}{q}} \, \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}}.$$

Hence the statement of the theorem follows in view of the boundedness of the commutator of maximal operator M_b in $M_{p,\varphi^{\frac{1}{p}}}$ provided by Corollary 6.3 in virtue of condition (6.7).

$$\begin{aligned} \|M_{b,\alpha}f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-\frac{1}{q}} r^{-\frac{n}{q}} \|M_{b,\alpha}f\|_{L_q(B(x,r))} \\ &\lesssim \|b\|_* \, \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-\frac{1}{q}} r^{-\frac{n}{q}} \|M_b f\|^{\frac{p}{q}}_{L_p(B(x,r))} \\ &= \|b\|_* \, \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \Big(\sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-\frac{1}{p}} r^{-\frac{n}{p}} \|M_b f\|_{L_p(B(x,r))}\Big)^{\frac{p}{q}} \\ &= \|b\|_* \, \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \, \|M_b f\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}} \\ &\lesssim \|b\|_* \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned}$$

□

In the case $\varphi(x,t) = t^{\lambda-n}$, $0 < \lambda < n$ from Theorem 6.10 we get the following Adams type result for the commutator of fractional maximal operator.

Corollary 6.4. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ and $b \in BMO(\mathbb{R}^n)$. Then, the operator $M_{b,\alpha}$ is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$.*

7. Commutators of Riesz potential operators in the spaces $M_{p,\varphi}$

7.1. Spanne type result

In [13] the following statement was proved for the commutators of Riesz potential operators, containing the result in [23, 25].

Theorem 7.11. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, $\varphi(x,r)$ satisfies the conditions (3.1) and (3.3). Then the operator $[b, I_\alpha]$ is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.*

Lemma 7.4. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$.*

Then the inequality

$$\|[b, I_\alpha]f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Write $f = f_1 + f_2$, where $B = B(x_0,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence,

$$\|[b, I_\alpha]f\|_{L_q(B)} \le \|[b, I_\alpha]f_1\|_{L_q(B)} + \|[b, I_\alpha]f_2\|_{L_q(B)}.$$

From the boundedness of $[b, I_\alpha]$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that:

$$\begin{aligned}\|[b, I_\alpha]f_1\|_{L_q(B)} &\leq \|[b, I_\alpha]f_1\|_{L_q(\mathbb{R}^n)} \\ &\lesssim \|b\|_* \, \|f_1\|_{L_p(\mathbb{R}^n)} = \|b\|_* \, \|f\|_{L_p(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}|[b, I_\alpha]f_2(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} |f(y)| dy \\ &\approx \int_{\complement_{(2B)}} \frac{|b(y) - b(x)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy.\end{aligned}$$

Then

$$\begin{aligned}\|[b, I_\alpha]f_2\|_{L_q(B)} &\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} \frac{|b(y) - b(x)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy \right)^q dx \right)^{\frac{1}{q}} \\ &\leq \left(\int_B \left(\int_{\complement_{(2B)}} \frac{|b(y) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)| dy \right)^q dx \right)^{\frac{1}{q}} \\ &\quad + \left(\int_B \left(\int_{\complement_{(2B)}} \frac{|b(x) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)| dy \right)^q dx \right)^{\frac{1}{q}} \\ &= J_1 + J_2.\end{aligned}$$

Let us estimate J_1.

$$\begin{aligned}J_1 &= r^{\frac{n}{q}} \int_{\complement_{(2B)}} \frac{|b(y) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)| dy \\ &\approx r^{\frac{n}{q}} \int_{\complement_{(2B)}} |b(y) - b_B| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy \\ &\approx r^{\frac{n}{q}} \int_{2r}^{\infty} \int_{2r \leq |x_0 - y| \leq t} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality and by (6.1), (6.2), we get

$$\begin{aligned}J_1 &\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| dy \frac{dt}{t^{n+1-\alpha}} \\ &\quad + r^{\frac{n}{q}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \frac{dt}{t^{n+1-\alpha}} \int_{B(x_0,t)} |f(y)| dy \\ &\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \left(\int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}|^{p'} dy \right)^{\frac{1}{p'}} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\quad + r^{\frac{n}{q}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1-\alpha}} \\ &\lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.\end{aligned}$$

In order to estimate J_2 note that

$$J_2 = \left(\int_B |b(x) - b_B|^q dx \right)^{\frac{1}{q}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By (6.1), we get

$$J_2 \lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^{n-\alpha}}dy &\approx \int_{\complement_{(2B)}} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}}dy \\ &\approx \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} |f(y)|dy \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|dy \frac{dt}{t^{n+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^{n-\alpha}}dy \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{7.1}$$

Thus, by (7.1)

$$J_2 \lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Summing up J_1 and J_2, for all $p \in (1,\infty)$ we get

$$\|[b,I_\alpha]f_2\|_{L_q(B)} \lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{7.2}$$

Finally,

$$\begin{aligned}\|[b,I_\alpha]f\|_{L_q(B)} &\lesssim \|b\|_* \, \|f\|_{L_p(2B)} + \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.\end{aligned}$$

□

The following theorem is true.

Theorem 7.12. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ and let (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big) t^{\alpha} \, \varphi_1(x,t) \frac{dt}{t} \le C \, \varphi_2(x,r), \tag{7.3}$$

where C does not depend on x and r.

Then the operator $[b, I_\alpha]$ is bounded from M_{p,φ_1} to M_{q,φ_2}. Moreover

$$\|[b,I_\alpha]f\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

Proof. The statement of Theorem 7.12 follows from Lemma 7.4.

$$\begin{aligned}\|[b,I_\alpha]f\|_{M_{q,\varphi_2}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} r^{-\frac{n}{q}} \|[b,I_\alpha]f\|_{L_q(B(x,r))} \\ &\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}} \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) t^{\alpha} \varphi_2(x,t) \frac{dt}{t} \\ &\lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}}.\end{aligned}$$

□

7.2. Adams type result

The following is a result of Adams type.

Theorem 7.13. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $b \in BMO(\mathbb{R}^n)$ and let $\varphi(x,t)$ satisfy the conditions* (6.7) *and*

$$\int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big) \, t^{\alpha} \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \le C r^{-\frac{\alpha p}{q-p}}, \tag{7.4}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then $[b, I_\alpha]$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}}}$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

For $x \in B$ we have

$$|[b, I_\alpha]f_2(x)| \lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} |f_2(y)| dy$$
$$\approx \int_{\complement_{(2B)}} \frac{|b(y) - b(x)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy.$$

Analogously to Section 7.1, for all $p \in (1, \infty)$ and $x \in B$ we get ⟸ 7.1 ok?

$$|[b, I_\alpha]f_2(x)| \lesssim \|b\|_* \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{\alpha - \frac{n}{p} - 1} \|f\|_{L_p(B(x_0,t))} dt. \tag{7.5}$$

Then from conditions (7.4) and (7.5) we get

$$\begin{aligned} |[b, I_\alpha]f(x)| &\lesssim \|b\|_* \, r^\alpha \, M_b f(x) + \|b\|_* \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) t^{\alpha - \frac{n}{p} - 1} \|f\|_{L_p(B(x,t))} dt \\ &\le \|b\|_* \, r^\alpha \, M_b f(x) + \|b\|_* \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \varphi(x, t) \frac{dt}{t} \\ &\lesssim \|b\|_* \, r^\alpha \, M_b f(x) + \|b\|_* \, r^{-\frac{\alpha p}{q-p}} \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned} \tag{7.6}$$

Hence choose $r = \Big(\frac{\|f\|_{M_{p,\varphi^{\frac{1}{p}}}}}{M_b f(x)}\Big)^{\frac{q-p}{\alpha q}}$ for every $x \in B$, we have

$$|[b, I_\alpha]f(x)| \lesssim \|b\|_* \, (M_b f(x))^{\frac{p}{q}} \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Hence the statement of the theorem follows in view of the boundedness of the commutator of maximal operator M_b in $M_{p,\varphi^{\frac{1}{p}}}$ provided by Corollary 6.3 in virtue of condition (6.7) ⟸ 6.3 ok?

$$\begin{aligned} \|[b, I_\alpha]f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{x \in \mathbb{R}^n, \, r>0} \varphi(x, r)^{-\frac{1}{q}} r^{-\frac{n}{q}} \|[b, I_\alpha]f\|_{L_q(B(x,r))} \\ &\lesssim \|b\|_* \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x \in \mathbb{R}^n, \, r>0} \varphi(x, r)^{-\frac{1}{q}} r^{-\frac{n}{q}} \|M_b f\|_{L_p(B(x,r))}^{\frac{p}{q}} \\ &= \|b\|_* \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \Big(\sup_{x \in \mathbb{R}^n, \, r>0} \varphi(x, r)^{-\frac{1}{p}} r^{-\frac{n}{p}} \|M_b f\|_{L_p(B(x,r))}\Big)^{\frac{p}{q}} \\ &= \|b\|_* \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \, \|M_b f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}} \\ &\lesssim \|b\|_* \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned}$$

□

8. Some applications

In this section, we shall apply Theorems 4.4, 7.12 and 7.13 to several particular operators such as the Marcinkiewicz operator and fractional powers of the some analytic semigroups. ⟸ 4.4 ok?

8.1. Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \text{ for any } \mu > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \le M|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [28] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [22, 29, 30].

The Marcinkiewicz operator is defined by (see [31])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

The sublinear commutator of the operator $\mu_{\Omega,\alpha}$ is defined by

$$[b, \mu_{\Omega,\alpha}](f)(x) = \left(\int_0^\infty |F^b_{\Omega,t,\alpha}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F^b_{\Omega,t,\alpha}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} [b(x) - b(y)]f(y)dy.$$

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that $\mu_{\Omega,\alpha}(f)(x) = \|F_{\Omega,\alpha,t}(x)\|$.

By Minkowski inequality and the conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

It is known that for $b \in BMO(\mathbb{R}^n)$ the operators $\mu_{\Omega,\alpha}$ and $[b, \mu_{\Omega,\alpha}]$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ (see [31]), then from Theorems 4.4 and 7.12 we get $\Longleftarrow$ 4.4 ok?

Corollary 8.5. *Let* $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* (φ_1, φ_2) *satisfy condition* (5.1) *and* Ω *satisfies conditions* (a)–(c). *Then* $\mu_{\Omega,\alpha}$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{q,φ_2}.

Corollary 8.6. *Let* $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, φ *satisfy conditions* (4.7), (5.2) *and* Ω *be satisfies the conditions* (a)–(c). *Then* $\mu_{\Omega,\alpha}$ *is bounded from* $M_{p,\varphi^{\frac{1}{p}}}$ *to* $M_{q,\varphi^{\frac{1}{q}}}$ *for* $p > 1$ *and from* $M_{1,\varphi}$ *to* $M_{q,\varphi^{\frac{1}{q}}}$ *for* $p = 1$.

Corollary 8.7. *Let* $1 < p < \infty$, $0 \le \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, (φ_1, φ_2) *satisfy condition* (7.3), $b \in BMO(\mathbb{R}^n)$ *and* Ω *be satisfies conditions* (a)–(c). *Then* $[b, \mu_{\Omega,\alpha}]$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2}.

Corollary 8.8. *Let* $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, φ *satisfy conditions* (6.7), (7.4), $b \in BMO(\mathbb{R}^n)$ *and* Ω *be satisfies the conditions* (a)–(c). *Then* $[b, \mu_{\Omega,\alpha}]$ *is bounded from* $M_{p,\varphi^{\frac{1}{p}}}$ *to* $M_{q,\varphi^{\frac{1}{q}}}$.

8.2. Fractional powers of the some analytic semigroups

The theorems of the previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \le \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{8.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where c_1, $c_2 > 0$ are independent of x, y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. See, for example, Chapter 5 in [29].

Property (8.1) is satisfied for large classes of differential operators (see, for example [7]). In [7] also other examples of operators which are estimates from above by Riesz potentials are given. In these cases Theorems 5.6, 5.7, 7.12 and 7.13 are also applicable for proving boundedness of those operators and commutators from M_{p,φ_1} to M_{q,φ_2} and from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$.

Corollary 8.9. *Let condition* (8.1) *be satisfied. Moreover, let* $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, (φ_1, φ_2) *satisfy condition* (5.1). *Then* $L^{-\alpha/2}$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{q,φ_2} *for* $p = 1$.

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (8.1), it follows that

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)$$

(see [14]). Hence by the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{M_{q,\varphi_2}} \lesssim \|I_\alpha(|f|)\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}. \qquad \square$$

Corollary 8.10. *Let condition* (8.1) *be satisfied. Moreover, let* $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, φ *satisfy conditions* (4.7) *and* (5.2). *Then* $L^{-\alpha/2}$ *is bounded from* $M_{p,\varphi^{\frac{1}{p}}}$ *to* $M_{q,\varphi^{\frac{1}{q}}}$ *for* $p > 1$ *and from* M_{1,φ_1} *to* $WM_{q,\varphi^{\frac{1}{q}}}$ *for* $p = 1$.

Let b be a locally integrable function on $\mathbb{R}^n$, the commutator of b and $L^{-\alpha/2}$ is defined as follows

$$[b, L^{-\alpha/2}] f(x) = b(x) L^{-\alpha/2} f(x) - L^{-\alpha/2}(bf)(x).$$

In [14] extended the result of [10] from $(-\Delta)$ to the more general operator L defined above. More precisely, they showed that when $b \in BMO(\mathbb{R}^n)$, then the commutator operator $[b, L^{-\alpha/2}]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then from Theorems 7.12 and 7.13 we get

Corollary 8.11. *Let condition* (8.1) *be satisfied. Moreover, let* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, *and* (φ_1, φ_2) *satisfy condition* (7.3). *Then* $[b, L^{-\alpha/2}]$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2}.

Corollary 8.12. *Let condition* (8.1) *be satisfied. Moreover, let* $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $b \in BMO(\mathbb{R}^n)$*, and* φ *satisfy conditions* (6.7) *and* (7.4)*. Then* $[b, L^{-\alpha/2}]$ *is bounded from* $M_{p,\varphi^{\frac{1}{p}}}$ *to* $M_{q,\varphi^{\frac{1}{q}}}$.

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