

## An Innovative Exhaustion Approach for Estimating the Circumference of a Circle and a Visualization by Geogebra

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**Abstract:** The method of exhaustion is a technique that Archimedes used to find the area and circumference of the circle or any shape which is bounded by non-linear borders. More generally, the exhaustion technique can be assumed as an earlier form of integral calculus. In this paper first, we will give a modern calculation of Archimedes' own approach to find the circumference of a circle. Additionally, we will propose a new approach and related calculation of the circumference of the circle. We will visualize new approach to calculation of the circumference of a circle by using GeoGebra which is dynamic mathematics software.

**Key words:** GeoGebra · Circumference of a circle · Dynamic Mathematics · The method of exhaustion · Visualization

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### INTRODUCTION

Around 225 BC, Archimedes wrote the treatise *Measurement of a Circle* which contained the first derivation of a formula for the area of a circle and the first formal approximation of the constant we now call pi. At that time, it was already known that the ratio of the areas of two circles equals to the ratio of the squares of their diameters. This result is stated by Euclid in his *Elements* as proposition XII.2 [1].

From this relationship, Archimedes understood that ratio of the area of a circle to the square of its diameter must be the same constant for every circle. He sought not only to find this constant ratio, but also to find a formula for the area of a circle. Archimedes proposes to draw inscribing or circumscribing regular polygons to a circle. When the number of sides increases, the circumference and area will approach to the circle (Figure 1).

Of course, Archimedes used some special calculation techniques to calculate the area and circumference of an n-gon. This process is an actually an approximation of the constant which can be described as the ratio of a circle's circumference to its diameter and we call this number as pi today.

Approximating pi and calculating the circumference of a circle or area of a disc by using different methods has always been of interest. Janjic proposed a geometric definition and approximation by nested radicals [2]. Reece defined a method to estimate pi by using Monte Carlo method in n-dimensions [3]. Linn and Neal used golden ratio finding an approximation to pi [4]. Hendel [5] used to an exciting method to find the area of a disc using triangle. He cut any circle along its radius and converted the form into a triangle with a base of circumference of circle and the height of radius.

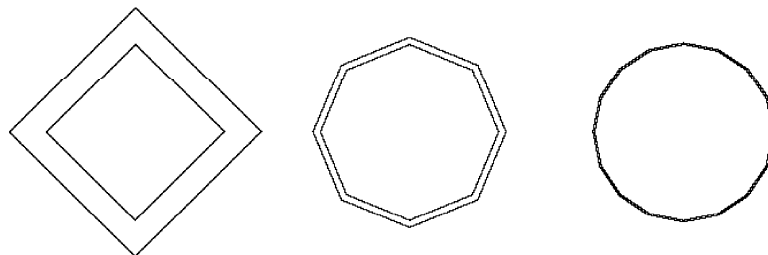


Fig. 1: Inscribing and circumscribing regular polygons to the circle

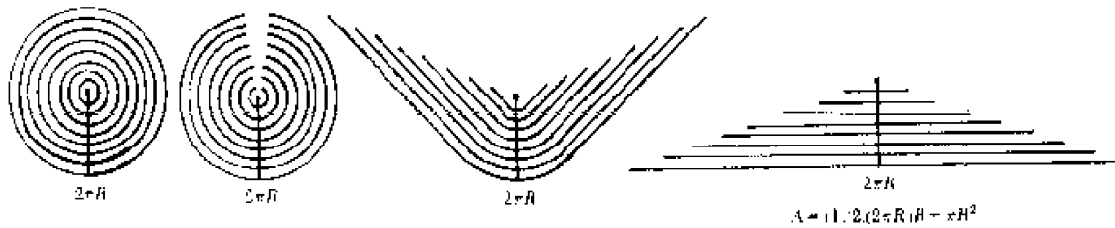


Fig. 2: Hendel's method of finding the area of disc

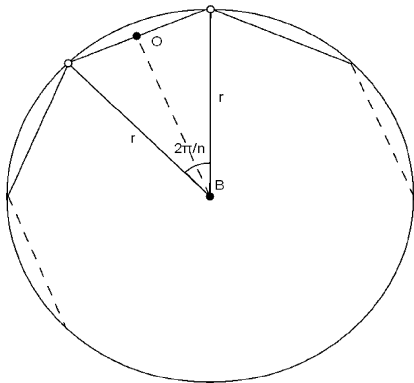


Fig. 3: Finding the length of the inscribed polygon

In this paper, first we will give an analytical calculation of the circumference of the circle obtained from the Archimedes' method. Then, we will define a different approach for finding circumference of a circle and give the analytical calculation.

**Analytical Calculation for Archimedes' Method:** By using inscribed regular polygons, following analysis can be reached [6, 7].

We need to find the length of one side of the polygon shown in the Figure 2 as MN. Since MNB is an isosceles triangle the perpendicular line segment OB will divide the angle B into two equal parts. Then, you can calculate the MO, which is half of the MN, as below.

$$\frac{|MO|}{r} = \sin \frac{\pi}{n} \Rightarrow |MO| = r \sin \frac{\pi}{n}$$

So, one side of the n-gon will be  $2r \sin \frac{\pi}{n}$ . Therefore, the circumference of the n-gon can be determined as  $2nr \sin \frac{\pi}{n}$ . This result is a circumference function of the side number of inscribed regular polygon. When we translate Archimedes suggestion into modern calculus, the limit of the function  $2nr \sin \frac{\pi}{n}$ , as n goes to infinity, must be the circumference of the circle.

$$\begin{aligned} \lim_{n \rightarrow \infty} 2nr \sin \frac{\pi}{n} &= 2r \lim_{n \rightarrow \infty} n \sin \frac{\pi}{n} \\ &= 2r \lim_{n \rightarrow \infty} \frac{\sin \pi \frac{1}{n}}{\frac{1}{n}} \\ &= 2r\pi \left( \text{by using the theorem } \lim_{n \rightarrow \infty} a_n = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{\sin ka_n}{a_n} = k \right) \end{aligned}$$

And this result is exactly the circumference of any circle.

**Different Method of Exhaustion:** We have a circle centered at B. First, divide the complete circular arc into n equal parts gradually.

Then, expand all arcs as shown in the Figure 5. We will obtain a shape similar to consecutive waves composed of circular arcs. When you complete the radius line DB as DE and draw the line FE, where F is the end point of the waves, we will obtain a perpendicular triangle, whose hypotenuse is not exactly linear.

When the partition of arcs goes to infinity the hypotenuse will be linear and the DEF will be exactly a perpendicular triangle. You can observe this process by using the attached GeoGebra dynamic applet, whose interface is shown in Figure 6.

After being satisfied that the hypotenuse of the triangle DEF will be the circumference of the circle, let's check this result by analytical calculation.

By thinking the bottom right triangle of the Figure 3;

$$|DE| = n|DA| \text{ and } |EF| = n|AC|$$

$$|AB| = r \cos \left( \frac{2\pi}{n} \right), |DA| = r - |AB| = r - r \cos \left( \frac{2\pi}{n} \right)$$

$$\text{So, it can be written as } |AC| = r \sin \left( \frac{2\pi}{n} \right)$$

Since  $|DE| = n|DA|$  ve  $|EF| = n|AC|$ , it can written that

$$|DE| = n \left[ r - r \cos \left( \frac{2\pi}{n} \right) \right] \text{ ve } |EF| = n \left[ r \sin \left( \frac{2\pi}{n} \right) \right]$$

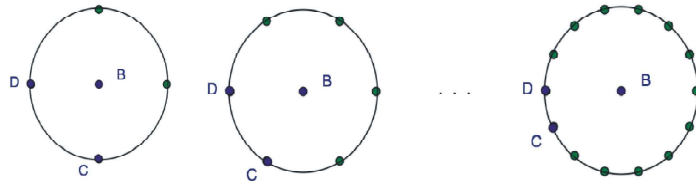


Fig. 4: Dividing the circular arc

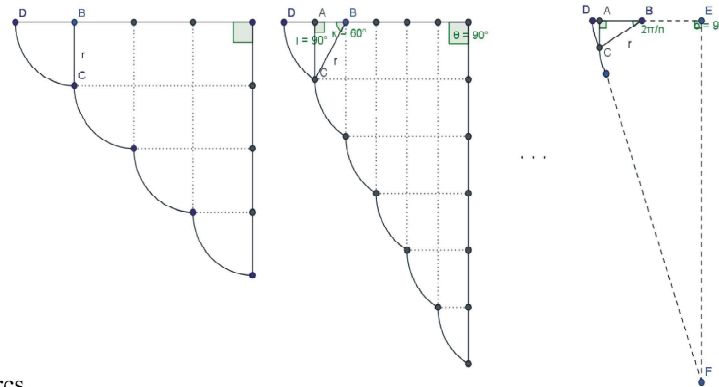


Fig. 5: Expanding the arcs

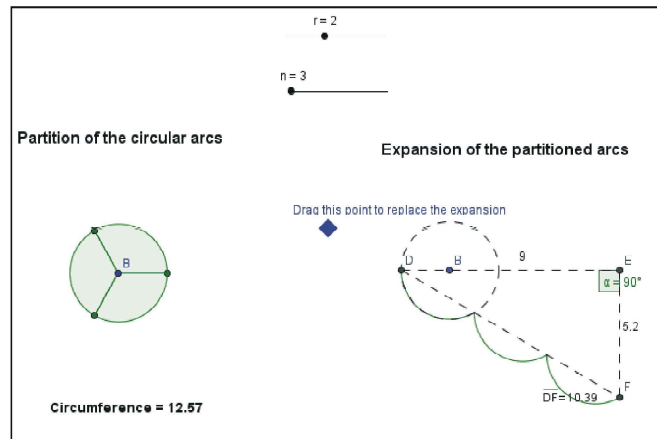


Fig. 6: Interface of GeoGebra dynamic applet

By applying Pythagoras rule for the triangle  $\triangle_{DEH}$  it can be written that  $|DF|^2 = |DE|^2 + |EF|^2$

$$\text{So, } |DF|^2 = n^2 \left[ r - r \cos\left(\frac{2\pi}{n}\right) \right]^2 + n^2 \left[ r \sin\left(\frac{2\pi}{n}\right) \right]^2$$

$$|DF| = nr \sqrt{1 - 2 \cos\left(\frac{2\pi}{n}\right) + \cos^2\left(\frac{2\pi}{n}\right) + \sin^2\left(\frac{2\pi}{n}\right)} \text{ implies}$$

$$|DF| = nr \sqrt{1 - 2 \cos\left(\frac{2\pi}{n}\right) + 1}$$

$$\text{Finally, } |DF| = nr \sqrt{2 - 2 \cos\left(\frac{2\pi}{n}\right)}$$

**RESULTS**

We must calculate the limit of last function of  $n$  as  $n$  goes to infinity;

$$\begin{aligned} \lim_{n \rightarrow \infty} nr \sqrt{2 - 2\cos\left(\frac{2\pi}{n}\right)} &= \lim_{n \rightarrow \infty} nr \sqrt{2\left(1 - \cos\left(\frac{2\pi}{n}\right)\right)} \\ &= \lim_{n \rightarrow \infty} nr \sqrt{2\left(1 - \left(1 - 2\sin^2\left(\frac{\pi}{n}\right)\right)\right)} \quad \left(\text{by using the rule } \cos(2x) = 1 - 2\sin^2 x\right) \\ &= \lim_{n \rightarrow \infty} nr \sqrt{4\sin^2\left(\frac{\pi}{n}\right)} \\ &= \lim_{n \rightarrow \infty} nr 2\sin\left(\frac{\pi}{n}\right) \\ &= 2r \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} \quad \left(\text{by using the theorem } \lim_{n \rightarrow \infty} a_n = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{\sin ka_n}{a_n} = k\right) \\ &= 2\pi r \end{aligned}$$

This last result is again the circumference formula of any circle.

We used the logic of making the partition infinitely small that we learnt from Archimedes' work. Applying same logic in a new approach produced same result. These kinds of approaches may make us to feel the historical development of mathematics.

**REFERENCES**

1. Euclid, 1956. The Thirteen Books of The Elements Vol. 3, Dover Publications.
2. Janjic, M., 2008. Geometrical Definition of  $\pi$  and its approximation by Nested Radicals, The Teaching of Mathematics, 11(1): 25-34.
3. Reece, B., 2006. Estimating  $\pi$  through Monte Carlo Methods in n-dimensions, [http:// orion.math.iastate.edu/dept/thesisarchive/MSM/ReeceMSMSS06.pdf](http://orion.math.iastate.edu/dept/thesisarchive/MSM/ReeceMSMSS06.pdf) access date: 11.05.2011.
4. Linn, S.L. and D.K. Neal, 2011. Approximating Pi with the Golden Ratio, [http:// people.wku.edu/ david.neal/ 117/Acad/GoldenPi.pdf](http://people.wku.edu/david.neal/117/Acad/GoldenPi.pdf) access date: 11.05.2011.
5. Hendel, R.J., 1990. Proof Without Words: Area of a Disk is  $\pi r^2$ , Mathematics Magazine, 63(3): 188.
6. Kabaca, T., 2006. (in Turkish) Limit Kavramının Öğretiminde Bilgisayar Cebiri Sistemlerinin Etkisi, Unpublished Doctoral Thesis, Gazi University, Turkey.
7. Aktumen, M. and A. Kacar, 2007. Maplets for the Area of the Unit Circle. Journal of Online Mathematics and its Applications, 7, access date: 10 May 2010 [http:// mathdl.maa.org/ mathDL/ 4/ ?pa=content&sa=viewDocument&nodeId=1549](http://mathdl.maa.org/mathDL/4/?pa=content&sa=viewDocument&nodeId=1549).