Journal of
Mathematical
Inequalities
Volume 5, Number 1 (2011), 87–106

THE STEIN–WEISS TYPE INEQUALITIES FOR THE B–RIESZ POTENTIALS

A. D. GADJIEV, V. S. GULIYEV, A. SERBETCI AND E. V. GULIYEV

(*Communicated by J. Pečarić*)

Abstract. We establish two inequalities of Stein-Weiss type for the Riesz potential operator $I_{\alpha,\gamma}$ ($B-$Riesz potential operator) generated by the Laplace-Bessel differential operator Δ_B in the weighted Lebesgue spaces $L_{p,|x|^\beta,\gamma}$. We obtain necessary and sufficient conditions on the parameters for the boundedness of $I_{\alpha,\gamma}$ from the spaces $L_{p,|x|^\beta,\gamma}$ to $L_{q,|x|^{-\lambda},\gamma}$, and from the spaces $L_{1,|x|^\beta,\gamma}$ to the weak spaces $WL_{q,|x|^{-\lambda},\gamma}$. In the limiting case $p=Q/\alpha$ we prove that the modified $B-$Riesz potential operator $\widetilde{I}_{\alpha,\gamma}$ is bounded from the spaces $L_{p,|x|^\beta,\gamma}$ to the weighted $B-BMO$ spaces $BMO_{|x|^{-\lambda},\gamma}$.

As applications, we get the boundedness of $I_{\alpha,\gamma}$ from the weighted B-Besov spaces $B^s_{p\theta,|x|^\beta,\gamma}$ to the spaces $B^s_{q\theta,|x|^{-\lambda},\gamma}$. Furthermore, we prove two Sobolev embedding theorems on weighted Lebesgue $L_{p,|x|^\beta,\gamma}$ and weighted B-Besov spaces $B^s_{p\theta,|x|^\beta,\gamma}$ by using the fundamental solution of the B-elliptic equation $\Delta_B^{\alpha/2}$.

1. Introduction and main results

Let $\mathbb{R}^n_{k,+}=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_1>0,\dots,x_k>0\}$, $1\leqslant k\leqslant n$. We denote by $L_{p,\gamma}\equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$ the set of all classes of measurable functions f with finite norm

$$\|f\|_{L_{p,\gamma}}=\left(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p(x')^\gamma dx\right)^{1/p},\quad 1\leqslant p<\infty,$$

where $x'=(x_1,\dots,x_k)$, and $\gamma=(\gamma_1,\dots,\gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma|=\gamma_1+\ldots+\gamma_k$ and $(x')^\gamma=x_1^{\gamma_1}\cdot\ldots\cdot x_k^{\gamma_k}$. If $p=\infty$, we assume

$$L_{\infty,\gamma}\equiv L_\infty=\{f:\|f\|_{L_{\infty,\gamma}}=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}}|f(x)|<\infty\}.$$

For any measurable set $E\subset\mathbb{R}^n_{k,+}$, let $|E|_\gamma=\int_E(x')^\gamma dx$. The weak $L_{p,\gamma}$ space $WL_{p,\gamma}\equiv WL_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1\leqslant p<\infty$, is defined as the set of locally integrable functions f, with

Mathematics subject classification (2010): 42B20, 42B25, 42B35.
Keywords and phrases: Laplace-Bessel differential operator, B-Riesz potential, Stein-Weiss type inequalities, weighted Lebesgue space, weighted B-Besov space.
A. D. Gadjiev was partially supported by the grant of INTAS (project 06-1000017-8792). V. S. Guliyev was partially supported by BGP II (project ANSF/AZM1-3110-BA-08).

Paper JMI-05-09

finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left| \left\{ x \in \mathbb{R}^n_{k,+} \ : \ |f(x)| > r \right\} \right|_\gamma^{1/p}.$$

Let w be a weight function on $\mathbb{R}^n_{k,+}$, i.e., w is a non-negative and measurable function on $\mathbb{R}^n_{k,+}$, then for all measurable functions f on $\mathbb{R}^n_{k,+}$ the weighted Lebesgue space $L_{p,w,\gamma} \equiv L_{p,w,\gamma}(\mathbb{R}^n_{k,+})$ and the weak weighted Lebesgue space $WL_{p,w,\gamma} \equiv WL_{p,w,\gamma}(\mathbb{R}^n_{k,+})$ are defined by

$$L_{p,w,\gamma} = \{f : \|f\|_{L_{p,w,\gamma}} = \|wf\|_{L_{p,\gamma}} < \infty\}$$

and

$$WL_{p,w,\gamma} = \{f : \|f\|_{WL_{p,w,\gamma}} = \|wf\|_{WL_{p,\gamma}} < \infty\},$$

respectively.

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. The potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \ \ B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad i = 1, \dots, k$$

have been the research interests of many mathematicians such as B. Muckenhoupt and E. Stein [26], K. Stempak [28], K. Trimèche [30], I. Kipriyanov [17], A. D. Gadjiev and I. A. Aliev [8], L. Lyakhov [23], I. A. Aliev and B. Rubin [1], V. S. Guliyev [12]-[14], and others.

In this paper we study the Riesz potential associated with Δ_B (B-Riesz potential) defined by

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha-Q} f(y) (y')^\gamma dy,$$

and the modified B-Riesz potential by

$$\widetilde{I}_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} \left(T^y |x|^{\alpha-Q} - |y|^{\alpha-Q} \chi_{\complement_{B_1}}(y) \right) f(y) (y')^\gamma dy$$

in weighted Lebesgue spaces $L_{p,|x|^\beta,\gamma}$, where T^y is B-shift operators is defined below, $B(x,r) = \{y \in \mathbb{R}^n_{k,+} \ : \ |x-y| < r\}$ is the open ball centered at x with radius r in $\mathbb{R}^n_{k,+}$ and $B_r = B(0,r)$, ${}^\complement B_r = \mathbb{R}^n_{k,+} \setminus B_r$, and $0 < \alpha < Q$, $Q = n + |\gamma|$.

V. Kokilashvili and A. Meskhi [21] proved the Stein-Weiss inequality for the fractional integral operator defined on nonhomogeneous spaces. In this paper we establish two inequalities of Stein-Weiss type (see [27]) for the B-Riesz potential $I_{\alpha,\gamma} f$. We give the Stein-Weiss type inequality in Theorem 1, and a weak version of the Stein-Weiss inequality in Theorem 2 for $I_{\alpha,\gamma} f$.

THEOREM 1. *Let* $0<\alpha<Q$, $1<p\leqslant q<\infty$, $\beta<Q/p'$, $\lambda<Q/q$, $\beta+\lambda\geqslant 0$ *(*$\beta+\lambda>0$*, if* $p=q$*),* $1/p-1/q=(\alpha-\beta-\lambda)/Q$ *and* $f\in L_{p,|x|^{\beta},\gamma}$*. Then* $I_{\alpha,\gamma}f\in L_{q,|x|^{-\lambda},\gamma}$ *and the following inequality holds*

$$\left(\int_{\mathbb{R}^n_{k,+}}|x|^{-\lambda q}\left|I_{\alpha,\gamma}f(x)\right|^q(x')^{\gamma}dx\right)^{1/q}\leqslant C\left(\int_{\mathbb{R}^n_{k,+}}|x|^{\beta p}|f(x)|^p(x')^{\gamma}dx\right)^{1/p}, \qquad (1)$$

where C *is independent of* f*.*

THEOREM 2. *Let* $0<\alpha<Q$, $1<q<\infty$, $\beta\leqslant 0$, $\lambda<Q/q$, $\beta+\lambda\geqslant 0$, $1-1/q=(\alpha-\beta-\lambda)/Q$ *and* $f\in L_{1,|x|^{\beta},\gamma}$*. Then* $I_{\alpha,\gamma}f\in WL_{q,|x|^{-\lambda},\gamma}$ *and the following inequality holds*

$$\left(\int_{\{x\in\mathbb{R}^n_{k,+}:|x|^{-\lambda}\left|I_{\alpha,\gamma}f(x)\right|>\tau\}}(x')^{\gamma}dx\right)^{1/q}\leqslant\frac{C}{\tau}\int_{\mathbb{R}^n_{k,+}}|x|^{\beta}|f(x)|(x')^{\gamma}dx, \qquad (2)$$

where C *is independent of* f*.*

In the following, by using Stein-Weiss type Theorems 1 and 2, we obtain necessary and sufficient conditions on the parameters for the boundedness of the B-Riesz potential operator $I_{\alpha,\gamma}$ from the spaces $L_{p,|x|^{\beta},\gamma}$ to $L_{q,|x|^{\lambda},\gamma}$, and from the spaces $L_{1,|x|^{\beta},\gamma}$ to the weak spaces $WL_{q,|x|^{\lambda},\gamma}$. In the limiting case $p=Q/\alpha$ we prove that the modified B-Riesz potential operator $\widetilde{I_{\alpha}}$ is bounded from the space $L_{p,|x|^{\beta},\gamma}$ to the weighted B-BMO space $BMO_{|x|^{-\lambda},\gamma}$.

THEOREM 3. *Let* $0<\alpha<Q$, $1\leqslant p\leqslant q<\infty$, $\beta<Q/p'$ *(*$\beta\leqslant 0$*, if* $p=1$*),* $\lambda<Q/q$ *(*$\lambda\leqslant 0$*, if* $q=\infty$*),* $\alpha\geqslant\beta+\lambda\geqslant 0$ *(*$\beta+\lambda>0$*, if* $p=q$*).*

1) If $1<p<Q/(\alpha-\beta-\lambda)$*, then the condition* $1/p-1/q=(\alpha-\beta-\lambda)/Q$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{p,|x|^{\beta},\gamma}$ *to* $L_{q,|x|^{-\lambda},\gamma}$*.*

2) If $p=1$*, then the condition* $1-1/q=(\alpha-\beta-\lambda)/Q$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{1,|x|^{\beta},\gamma}$ *to* $WL_{q,|x|^{-\lambda},\gamma}$*.*

3) If $1<p=Q/(\alpha-\beta-\lambda)$*, then the operator* $\widetilde{I}_{\alpha,\gamma}$ *is bounded from* $L_{p,|x|^{\beta},\gamma}$ *to* $BMO_{|x|^{-\lambda},\gamma}$*.*

Moreover, if the integral $I_{\alpha,\gamma}f$ *exists almost everywhere for* $f\in L_{p,|x|^{\beta},\gamma}$*, then* $I_{\alpha,\gamma}f\in BMO_{|x|^{-\lambda},\gamma}$ *and the following inequality holds*

$$\|I_{\alpha,\gamma}f\|_{BMO_{|x|^{-\lambda},\gamma}}\leqslant C\|f\|_{L_{p,|x|^{\beta},\gamma}},$$

where $C>0$ *is independent of* f*.*

REMARK 1. Note that in the case of $k=1$ the statements 1) and 2) in Theorem 3 were proved in [9], and the statement 3) in [10].

Here the weighted $B-BMO$ space $BMO_{w,\gamma}$ is defined as the set of locally integrable functions f with finite norm

$$\|f\|_{*,w,\gamma} = \sup_{x\in\mathbb{R}^n_{k,+},\, r>0} w(B_r)^{-1} \int_{B_r} |T^y f(x) - f_{B_r}(x)|(y')^\gamma dy < \infty,$$

and $B-BMO$ space (see [13]) $BMO_\gamma(\mathbb{R}^n_{k,+}) \equiv BMO_{1,\gamma}(\mathbb{R}^n_{k,+})$, where

$$f_{B_r}(x) = |B_r|_\gamma^{-1} \int_{B_r} T^y f(x)(y')^\gamma dy,$$

$|B_r|_\gamma = \omega(n,k,\gamma) r^Q$ and

$$\omega(n,k,\gamma) = \int_{B_1} (x')^\gamma dx = \pi^{(n-k)/2}\, 2^{-k} \prod_{i=1}^{k} \Gamma((\gamma_i+1)/2)[\Gamma(\gamma_i/2)]^{-1}.$$

Besov spaces in the setting of the Bessel differential operator on $(0,\infty)$ is studied by G. Altenburg [2], D. I. Cruz-Baez and J. Rodriguez, [5], M. Assal and H. Ben Abdallah [3], and the setting of the Laplace-Bessel differential operator on $\mathbb{R}^n_{k,+}$ studied by V. S. Guliyev, A. Serbetci and Z. V. Safarov [16].

In Theorem 4 we prove the boundedness of $I_{\alpha,\gamma}$ in the weighted Besov spaces associated with the Laplace-Bessel differential operator on $\mathbb{R}^n_{k,+}$ (weighted B-Besov spaces)

$$B^s_{p\theta,w,\gamma} = \left\{ f : \|f\|_{B^s_{p\theta,w,\gamma}} = \|f\|_{L_{p,w,\gamma}} + \left(\int\limits_{\mathbb{R}^n_{k,+}} \frac{\|T^x f(\cdot) - f(\cdot)\|^\theta_{L_{p,w,\gamma}}}{|x|^{Q+s\theta}} (x')^\gamma dx \right)^{\frac{1}{\theta}} < \infty \right\} \tag{3}$$

for a power weight w, $1 \leqslant p,\theta \leqslant \infty$ and $0 < s < 1$.

THEOREM 4. *Let $0<\alpha<Q$, $1<p\leqslant q<\infty$, $\beta<Q/p'$, $\lambda<Q/q$, $\alpha \geqslant \beta+\lambda \geqslant 0$ ($\beta+\lambda>0$, if $p=q$).*

If $1<p<Q/(\alpha-\beta-\lambda)$, $1/p-1/q=(\alpha-\beta-\lambda)/Q$, $1\leqslant\theta\leqslant\infty$ and $0<s<1$, then the operator $I_{\alpha,\gamma}$ is bounded from $B^s_{p\theta,|x|^\beta,\gamma}$ to $B^s_{q\theta,|x|^{-\lambda},\gamma}$. More precisely, there is a constant $C>0$ such that

$$\|I_{\alpha,\gamma} f\|_{B^s_{q\theta,|x|^{-\lambda},\gamma}} \leqslant C \|f\|_{B^s_{p\theta,|x|^\beta,\gamma}}$$

holds for all $f \in B^s_{p\theta,|x|^\beta,\gamma}$.

It is known that (see [18], [19]) there exists a positive constant C_0 such that $G(x) = C_0|x|^{2-Q}$ is the fundamental solution of the Laplace-Bessel differential operator Δ_B.

THEOREM 5. [19] *Let α is an even positive integer such that $0<\alpha<Q$. If the function f is finite, even with respect to the variables $x_1,\dots,x_k$ having α continuous*

derivatives by the variables $x_1,\ldots,x_k$ *and* $\alpha/2$ *continuous derivatives by* $x_{k+1},\ldots,x_n$, *then the potential* $I_{\alpha,\gamma}f$ *is a solution of the* B*-elliptic equation*

$$\Delta_B^{\alpha/2}u(x) = f(x).$$

In the following we prove two Sobolev embedding theorems on weighted Lebesgue $L_{p,|x|^\beta,\gamma}$ and weighted B-Besov spaces $B^s_{p\theta,|x|^\beta,\gamma}$ by using the fundamental solution of the B-elliptic equation $\Delta_B^{\alpha/2}$. We expect that these results will be useful to investigate the regularity properties of B-elliptic differential equations.

From Theorems 3 and 5 we have

THEOREM 6. *Let* f *be defined as in Theorem* 5 *and* α *be an even positive integer,* $0<\alpha<Q$, $1\leqslant p\leqslant q<\infty$, $\beta<Q/p'$ *(*$\beta\leqslant 0$, *if* $p=1$*),* $\lambda<Q/q$ *(*$\lambda\leqslant 0$, *if* $q=\infty$*),* $\alpha\geqslant\beta+\lambda\geqslant 0$ *(*$\beta+\lambda>0$, *if* $p=q$*).*

1) If $f\in L_{p,|x|^\beta,\gamma}$, $1<p<Q/(\alpha-\beta-\lambda)$, $1/p-1/q=(\alpha-\beta-\lambda)/Q$, *then the following estimation holds:*

$$\|u\|_{L_{q,|x|^{-\lambda},\gamma}} \leqslant C\|\Delta_B^{\alpha/2}u\|_{L_{p,|x|^\beta,\gamma}},$$

where $C>0$ *is independent of* u.

2) If $f\in L_{1,|x|^\beta,\gamma}$, $1-1/q=(\alpha-\beta-\lambda)/Q$, *then the following estimation holds:*

$$\|u\|_{WL_{q,|x|^{-\lambda},\gamma}} \leqslant C\|\Delta_B^{\alpha/2}u\|_{L_{1,|x|^\beta,\gamma}},$$

where $C>0$ *is independent of* u.

From Theorems 4 and 5 we have

THEOREM 7. *Let* α *be an even positive integer,* $0<\alpha<Q$, $1<p\leqslant q<\infty$, $\beta<Q/p'$, $\lambda<Q/q$, $\alpha\geqslant\beta+\lambda\geqslant 0$ *(*$\beta+\lambda>0$, *if* $p=q$*).*

If $f\in B^s_{p\theta,|x|^\beta,\gamma}$, $1<p<Q/(\alpha-\beta-\lambda)$, $1/p-1/q=(\alpha-\beta-\lambda)/Q$, $1\leqslant\theta\leqslant\infty$ *and* $0<s<1$, *then the following estimation holds:*

$$\|u\|_{B^s_{q\theta,|x|^{-\lambda},\gamma}} \leqslant C\|\Delta_B^{\alpha/2}u\|_{B^s_{p\theta,|x|^\beta,\gamma}},$$

where $C>0$ *is independent of* u.

2. Preliminaries

Denote the generalized shift operator (B–shift operator) by T^y, acting according to the law

$$T^y f(x) = C_{\gamma,k}\int_0^\pi\ldots\int_0^\pi f\left((x',y')_\beta, x''-y''\right)\,d\nu(\beta),$$

where $(x',y')_\beta = \left((x_1,y_1)_{\beta_1},\ldots,(x_k,y_k)_{\beta_k}\right)$, $(x_i,y_i)_{\beta_i} = (x_i^2 - 2x_iy_i\cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \leqslant i \leqslant k$,, $d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i-1}\beta_i \; d\beta_1 \ldots d\beta_k$, $1 \leqslant k \leqslant n$ and

$$C_{\gamma,k} = \pi^{-k/2} \prod_{i=1}^{k} \Gamma((\gamma_i+1)/2)\,[\Gamma(\gamma_i/2)]^{-1} = 2^k\pi^{-k}\omega(2k,k,\gamma).$$

We remark that the generalized shift operator T^y is closely connected with the Laplace-Bessel differential operator Δ_B (see [17, 22, 23] for details). Furthermore, T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)[T^y g(x)](y')^\gamma dy.$$

LEMMA 1. [9] *Let* $0 < \alpha < Q$. *Then for* $2|x| \leqslant |y|$, $x,y \in \mathbb{R}^n_{k,+}$, *the following inequality holds*

$$\left|T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\right| \leqslant 2^{Q-\alpha+1}|y|^{\alpha-Q-1}|x|. \tag{4}$$

We will need the following Hardy-type transforms defined on $\mathbb{R}^n_{k,+}$:

$$H_\gamma f(x) = \int_{B_{|x|}} f(y)(y')^\gamma dy,$$

and

$$H'_\gamma f(x) = \int_{\complement_{B_{|x|}}} f(y)(y')^\gamma dy.$$

The following two theorems related to the boundedness of these transforms were proved in [6] (see also [7], Section 1.1).

THEOREM A. *Let* $1 < q < \infty$. *Suppose that* v *and* w *are a.e. positive functions on* $\mathbb{R}^n_{k,+}$. *Then*

(a) *The operator* H_γ *is bounded from* $L_{1,w,\gamma}$ *to* $WL_{q,v,\gamma}$ *if and only if*

$$A_1 \equiv \sup_{t>0} \left(\int_{\complement_{B_t}} v^q(x)(x')^\gamma dx\right)^{1/q} \sup_{B_t} w^{-1}(x) < \infty;$$

(b) *The operator* H'_γ *is bounded from* $L_{1,w,\gamma}$ *to* $WL_{q,v,\gamma}$ *if and only if*

$$A_2 \equiv \sup_{t>0} \left(\int_{B_t} v^q(x)(x')^\gamma dx\right)^{1/q} \sup_{\complement_{B_t}} w^{-1}(x) < \infty.$$

Moreover, there exist positive constants a_j, $j = 1,\ldots,4$, *depending only on* q *such that* $a_1A_1 \leqslant \|H\| \leqslant a_2A_1$ *and* $a_3A_2 \leqslant \|H'\| \leqslant a_4A_2$.

THEOREM B. *Let $1 < p \leqslant q < \infty$. Suppose that v and w are a.e. positive functions on $\mathbb{R}^n_{k,+}$. Then*

(a) The operator H_γ is bounded from $L_{p,w,\gamma}$ to $L_{q,v,\gamma}$ if and only if

$$A_3 \equiv \sup_{t>0} \left(\int_{\complement B_t} v^q(x)(x')^\gamma dx \right)^{1/q} \left(\int_{B_t} w^{-p'}(x)(x')^\gamma dx \right)^{1/p'} < \infty,$$

$p' = p/(p-1)$;

(b) The operator H'_γ is bounded from $L_{p,w,\gamma}$ to $L_{q,v,\gamma}$ if and only if

$$A_4 \equiv \sup_{t>0} \left(\int_{B_t} v^q(x)(x')^\gamma dx \right)^{1/q} \left(\int_{\complement B_t} w^{-p'}(x)(x')^\gamma dx \right)^{1/p'} < \infty.$$

Moreover, there exist positive constants b_j, $j = 1, \ldots, 4$, depending only on p and q such that $b_1 A_3 \leqslant \|H\| \leqslant b_2 A_3$ and $b_3 A_4 \leqslant \|H'\| \leqslant b_4 A_4$.

We will need the case that we substitute $L_{p,\upsilon,\gamma}$ with the homogeneous space (X, ρ, μ) in Theorems A and B in which $X = \mathbb{R}^n_{k,+}$, $\rho(x,y) = |x-y|$ and $d\mu(x) = (x')^\gamma dx$.

DEFINITION 1. The weight function w belongs to the class $A_{p,\gamma}$ for $1 < p, q < \infty$, if

$$\sup_{x,r} \left(|B(x,r)|_\gamma^{-1} \int\limits_{B(x,r)} w(y)(y')^\gamma dy \right) \left(|B(x,r)|_\gamma^{-1} \int\limits_{B(x,r)} w^{-\frac{1}{p-1}}(y)(y')^\gamma dy \right)^{p-1} < \infty$$

and w belongs to $A_{1,\gamma}$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n_{k,+}$ and $r > 0$

$$|B(x,r)|_\gamma^{-1} \int_{B(x,r)} w(y)(y')^\gamma dy \leqslant C \operatorname*{ess\,inf}_{y \in B(x,r)} w(y).$$

The properties of the class $A_{p,\gamma}$ are analogous to those of the Muckenhoupt classes. In particular, if $w \in A_{p,\gamma}$, then $w \in A_{p-\varepsilon,\gamma}$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p_1,\gamma}$ for any $p_1 > p$.

Note that, $|x|^\alpha \in A_{p,\gamma}$, $1 < p < \infty$, if and only if $-\frac{Q}{p} < \alpha < \frac{Q}{p'}$; and $|x|^\alpha \in A_{1,\gamma}$, if and only if $-Q < \alpha \leqslant 0$.

For the B-maximal function (see [12, 13])

$$M_\gamma f(x) = \sup_{r>0} |B_r|_\gamma^{-1} \int_{B_r} T^y |f(x)| (y')^\gamma dy$$

the following analogue of Muckenhoupt theorem (see [25]) is valid.

THEOREM C. *1. If $f \in L_{1,w,\gamma}$ and $w \in A_{1,\gamma}$, then $M_\gamma f \in WL_{1,w,\gamma}$ and*

$$\|M_\gamma f\|_{WL_{1,w,\gamma}} \leqslant C_{1,w,\gamma} \|f\|_{L_{1,w,\gamma}}, \tag{5}$$

where $C_{1,w,\gamma}$ depends only on γ, k and n.

2. If $f \in L_{p,w,\gamma}$ and $w \in A_{p,\gamma}$, $1 < p < \infty$, then $M_\gamma f \in L_{p,w,\gamma}$ and

$$\|M_\gamma f\|_{L_{p,w,\gamma}} \leqslant C_{p,w,\gamma}\|f\|_{L_{p,w,\gamma}}, \tag{6}$$

where $C_{p,w,\gamma}$ depends only on w, p, γ, k and n.

Proof. Following [13] and [29], we define a maximal function on a space of homogeneous type (see [4]). By this we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure μ satisfying the doubling condition

$$\mu(E(x,2r)) \leqslant c\mu(E(x,r)), \tag{7}$$

where c does not depend on x and $r>0$. Here $E(x,r) = \{y \in X : \rho(x,y) < r\}$. Denote

$$M_\mu f(x) = \sup_{r>0} \mu(E(x,r))^{-1} \int_{E(x,r)} |f(y)| d\mu(y).$$

Let (X,ρ,μ) be a homogeneous type space. It is known that the maximal function M_μ is weighted weak $(1,1)$ type, $w \in A_{1,\gamma}$, that is

$$\int_{\{x\in X\,:\,M_\mu f(x)>\tau\}} w(x)\, d\mu(x) \leqslant \left(\frac{C_{1,w,\gamma}}{\tau}\int_X |f(x)|w(x)\, d\mu(x)\right), \tag{8}$$

and is weighted (p,p) type, $1 < p \leqslant \infty$ and $w \in A_{p,\gamma}$ (see [20], [24]), that is

$$\int_X \left|M_\mu f(x)\right|^p w(x)^p\, d\mu(x) \leqslant C_{p,w,\gamma}\int_X |f(x)|^p w(x)^p\, d\mu(x). \tag{9}$$

In [13] and [29] it is proved that the following inequality

$$M_\gamma f(x) \leqslant C M_\mu f(x)$$

holds, where constant $C>0$ does not depend on f and x.

In (8) and (9) if we take $X = \mathbb{R}^n_{k,+}$, $\rho(x,y) = |x-y|$ and $d\mu(x) = (x')^\gamma dx$, then we have

$$\|M_\gamma f\|_{p,w,\gamma} \leqslant C\|M_\mu f\|_{p,w,\gamma} \leqslant C_{p,w,\gamma}\|f\|_{p,w,\gamma}, \quad 1 < p \leqslant \infty,$$

and for $p=1$

$$\int_{\{x\in\mathbb{R}^n_{k,+}:\,M_\gamma f(x)>\tau\}} w(x)\, (x')^\gamma dx \leqslant \int_{\{x\in X\,:\,M_\mu f(x)>\frac{\tau}{C}\}} w(x)\, d\mu(x)$$
$$\leqslant \frac{C_{1,w,\gamma}}{\tau}\int_{\mathbb{R}^n_{k,+}} |f(x)|w(x)\, d\mu(x). \quad \square$$

REMARK 2. Note that in the case $k=1$ Theorem C was proved in [11].

We will need the following Hardy-Littlewood-Sobolev theorem for $I_{\alpha,\gamma}$.

THEOREM D. *Let* $0<\alpha<Q$ *and* $1\leqslant p<Q/\alpha$. *Then*

1) If $1<p<Q/\alpha$, *then the condition* $1/p-1/q=\alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{p,\gamma}$ *to* $L_{q,\gamma}$.

2) If $p=1$, *then the condition* $1-1/q=\alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{1,\gamma}$ *to* $WL_{q,\gamma}$.

3) If $1<p=Q/\alpha$, *then the operator* $\widetilde{I}_{\alpha,\gamma}$ *is bounded from* $L_{p,\gamma}$ *to* BMO_γ. *Moreover, if the integral* $I_{\alpha,\gamma}f$ *exists almost everywhere for* $f\in L_{p,\gamma}$, *then* $I_{\alpha,\gamma}f\in BMO_\gamma$ *and the following inequality is valid*

$$\|I_{\alpha,\gamma}f\|_{BMO_\gamma}\leqslant C\|f\|_{L_{p,\gamma}},$$

where $C>0$ *is independent of* f.

REMARK 3. Note that statements 1) and 2) in Theorem D was proved in [8] in the case $k=1$ and [12, 13] in the case $k=n$ and [14, 23] in the case $1\leqslant k\leqslant n$, and statement 3) in [13] in the case $k=1$.

3. Proof of the theorems

Proof of Theorem 1. We write

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n_{k,+}}|x|^{-\lambda q}\left|I_{\alpha,\gamma}f(x)\right|^q(x')^\gamma dx\right)^{1/q}\leqslant I_1+I_2+I_3\\
&\equiv\left(\int_{\mathbb{R}^n_{k,+}}|x|^{-\lambda q}\left(\int_{B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy\right)^q(x')^\gamma dx\right)^{1/q}\\
&+\left(\int_{\mathbb{R}^n_{k,+}}|x|^{-\lambda q}\left(\int_{B_{2|x|}\setminus B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy\right)^q(x')^\gamma dx\right)^{1/q}\\
&+\left(\int_{\mathbb{R}^n_{k,+}}|x|^{-\lambda q}\left(\int_{\complement B_{2|x|}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy\right)^q(x')^\gamma dx\right)^{1/q}.
\end{aligned}$$

It is easy to check that if $|y|<|x|/2$, then $|x|\leqslant|y|+|x-y|<|x|/2+|x-y|$. Hence $|x|/2<|x-y|$ and $T^y|x|^{\alpha-Q}\leqslant(|x|/2)^{\alpha-Q}$. Indeed,

$$\begin{aligned}
T^y|x|^{\alpha-Q}&=C_{\gamma,k}\int_0^\pi\cdots\int_0^\pi\left|((x',y')_\beta,x''-y'')\right|^{\alpha-Q}d\nu(\beta)\\
&\geqslant C_{\gamma,k}\int_0^\pi\cdots\int_0^\pi\left|(x'-y',x''-y'')\right|^{\alpha-Q}d\nu(\beta)\\
&=|x-y|^{\alpha-Q}\geqslant(|x|/2)^{\alpha-Q}.
\end{aligned}\tag{10}$$

Then we get

$$I_1\leqslant 2^{Q-\alpha}\left(\int_{\mathbb{R}^n_{k,+}}|x|^{(\alpha-Q-\lambda)q}\left(H_\gamma f(x)\right)^q(x')^\gamma dx\right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q < (Q-\alpha)q - Q$ ($i.e.,\ \alpha < Q/q' + \lambda$) we obtain

$$\left(\int_{\complement B_t} |x|^{(-\lambda+\alpha-Q)q}(x')^\gamma dx\right)^{1/q} = C_1 t^{\alpha-\lambda-Q/q'},$$

where $C_1 = \left(\frac{\omega(n,k,\gamma)}{q/q'+(\lambda-\alpha)q/Q}\right)^{1/q}$. Similarly, by virtue of the condition $\beta p < Q(p-1)$ ($i.e.,\ \beta < Q/p'$) it follows that

$$\left(\int_{B_t} |x|^{-\beta p'}(x')^\gamma dx\right)^{1/p'} = C_2 t^{Q/p'-\beta},$$

where $C_2 = \left(\frac{\omega(n,k,\gamma)}{1-\beta p'/Q}\right)^{1/p'}$.

Summarizing these estimates we find that

$$\sup_{t>0}\left(\int_{\complement B_t} |x|^{(-\lambda+\alpha-Q)q}(x')^\gamma dx\right)^{1/q}\left(\int_{B_t} |x|^{-\beta p'}(x')^\gamma dx\right)^{1/p'}$$
$$= C_1C_2 \sup_{t>0} t^{\alpha-\beta-\lambda+Q/q-Q/p} < \infty$$
$$\Longleftrightarrow \alpha-\beta-\lambda = Q/p - Q/q.$$

Now the first part of Theorem B gives us the inequality

$$I_1 \leqslant b_2 C_1 C_2 2^{Q-\alpha}\left(\int_{\mathbb{R}^n_{k,+}} |x|^\beta |f(x)|^p (x')^\gamma dx\right)^{1/p}.$$

If $|y| > 2|x|$, then $|y| \leqslant |x| + |x-y| < |y|/2 + |x-y|$. Hence $|y|/2 < |x-y|$ and the inequality $T^y|x|^{\alpha-Q} \leqslant (|y|/2)^{\alpha-Q}$ can be shown immediately by similar method that of the inequality (10). Consequently, we get

$$I_3 \leqslant 2^{Q-\alpha}\left(\int_{\mathbb{R}^n_{k,+}} |x|^{-\lambda q}\left(H'_\gamma\left(|f(y)||y|^{\alpha-Q}\right)(x)\right)^q (x')^\gamma dx\right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q > -Q$ ($i.e.,\ \lambda < Q/q$) we have

$$\left(\int_{B_t} |x|^{-\lambda q}(x')^\gamma dx\right)^{1/q} = C_3 t^{Q/q-\lambda},$$

where $C_3 = \left(\frac{\omega(n,k,\gamma)}{1-\lambda q/Q}\right)^{1/q}$. By the condition $\beta p > \alpha p - Q$ ($i.e.,\ \alpha < Q/p + \beta$) it follows that

$$\left(\int_{B_t} |x|^{-(\beta+Q-\alpha)p'}(x')^\gamma dx\right)^{1/p'} = C_4 t^{Q/p'-(Q+\beta-\alpha)},$$

where $C_4 = \left(\frac{\omega(n,k,\gamma)}{(1+(\beta-\alpha)/Q)p'-1}\right)^{1/p'}$.

Thus we find

$$\sup_{t>0}\left(\int_{B_t}|x|^{-\lambda q}(x')^\gamma dx\right)^{1/q}\left(\int_{\complement_{B_t}}|x|^{-(\beta+Q-\alpha)p'}(x')^\gamma dx\right)^{1/p'}$$
$$= C_3C_4\sup_{t>0}t^{\alpha-\beta-\lambda+Q/q-Q/p}<\infty$$
$$\Longleftrightarrow \alpha-\beta-\lambda=Q/p-Q/q.$$

Now the second part of Theorem B gives us the inequality

$$I_3\leqslant b_4C_3C_4 2^{Q-\alpha}\left(\int_{\mathbb{R}^n_{k,+}}|x|^\beta|f(x)|^p(x')^\gamma dx\right)^{1/p}.$$

To estimate I_2 we consider the cases $\alpha<Q/p$ and $\alpha>Q/p$, separately. If $\alpha<Q/p$, then the condition

$$\alpha=\beta+\lambda+Q/p-Q/q\geqslant Q/p-Q/q$$

implies $q\leqslant p^*$, where $p^*=Qp/(Q-\alpha p)$. Assume that $q<p^*$. In the sequel we use the notation

$$D_k\equiv\{x\in\mathbb{R}^n_{k,+}:2^k\leqslant|x|<2^{k+1}\},$$

and

$$\widetilde{D_k}\equiv\{x\in\mathbb{R}^n_{k,+}:2^{k-2}\leqslant|x|<2^{k+2}\}.$$

By Hölder's inequality with respect to the exponent p^*/q and Theorem D we get

$$I_2=\left(\int_{\mathbb{R}^n_{k,+}}|x|^{-\lambda q}\left(\int_{B_{2|x|}\setminus B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy\right)^q(x')^\gamma dx\right)^{1/q}$$
$$=\left(\sum_{k\in\mathbb{Z}}\int_{D_k}|x|^{-\lambda q}\left(\int_{B_{2|x|}\setminus B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy\right)^q(x')^\gamma dx\right)^{1/q}$$
$$\leqslant\left(\sum_{k\in\mathbb{Z}}\left(\int_{D_k}\left(\int_{B_{2|x|}\setminus B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy\right)^{p^*}(x')^\gamma dx\right)^{q/p^*}\right.$$
$$\left.\times\left(\int_{D_k}|x|^{\frac{-\lambda qp^*}{p^*-q}}(x')^\gamma dx\right)^{\frac{p^*-q}{p^*}}\right)^{1/q}$$
$$\leqslant C_5\left(\sum_{k\in\mathbb{Z}}2^{k[-\lambda q+\frac{p^*-q}{p^*}Q]}\left(\int_{D_k}\left|I_{\alpha,\gamma}\left(f\chi_{\widetilde{D_k}}\right)(x)\right|^{p^*}(x')^\gamma dx\right)^{q/p^*}\right)^{1/q}$$
$$\leqslant C_6\left(\sum_{k\in\mathbb{Z}}2^{k[-\lambda q+\frac{p^*-q}{p^*}Q]}\left(\int_{\widetilde{D_k}}|f(x)|^p(x')^\gamma dx\right)^{q/p}\right)^{1/q}$$
$$\leqslant C_7\left(\int_{\mathbb{R}^n_{k,+}}|x|^\beta|f(x)|^p(x')^\gamma dx\right)^{1/p}.$$

If $q = p^*$, then $\beta + \lambda = 0$. By using directly Theorem D we get

$$I_2 \leqslant C_8 \left(\sum_{k\in\mathbb{Z}} 2^{k\beta p^*} \int_{D_k} \left| I_{\alpha,\gamma}\left(f\chi_{\widetilde{D_k}}\right)(x)\right|^{p^*} (x')^\gamma dx \right)^{1/p^*}$$
$$\leqslant C_9 \left(\sum_{k\in\mathbb{Z}} 2^{k\beta p^*} \left(\int_{\widetilde{D_k}} |f(x)|^p (x')^\gamma dx \right)^{p^*/p} \right)^{1/p^*}$$
$$\leqslant C_{10} \left(\int_{\mathbb{R}^n_{k,+}} |x|^{\beta p} |f(x)|^p (x')^\gamma dx \right)^{1/p}.$$

For $\alpha > Q/p$ by Hölder's inequality with respect to the exponent p we get the following inequality

$$I_2 \leqslant \left(\int_{\mathbb{R}^n_{k,+}} |x|^{-\lambda q} \left(\int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)|^p (y')^\gamma dy \right)^{q/p} \right.$$
$$\left. \times \left(\int_{B_{2|x|}\setminus B_{|x|/2}} \left(T^y |x|^{\alpha-Q}\right)^{p'} (y')^\gamma dy \right)^{q/p'} (x')^\gamma dx \right)^{1/q}.$$

On the other hand by using (2) and the inequality $\alpha > Q/p$, we obtain

$$\int_{B_{2|x|}\setminus B_{|x|/2}} \left(T^y |x|^{\alpha-Q}\right)^{p'} (y')^\gamma dy \leqslant \int_{B_{2|x|}\setminus B_{|x|/2}} |x-y|^{(\alpha-Q)p'} (y')^\gamma dy$$
$$\leqslant \int_0^\infty \left| B_{2|x|} \cap B(x, \tau^{\frac{1}{(\alpha-Q)p'}}) \right|_\gamma d\tau$$
$$\leqslant \int_0^{|x|^{(\alpha-Q)p'}} \left|B_{2|x|}\right|_\gamma d\tau + \int_{|x|^{(\alpha-Q)p'}}^\infty \left| B(x, \tau^{\frac{1}{(\alpha-Q)p'}}) \right|_\gamma d\tau$$
$$\leqslant C_{11} |x|^{(\alpha-Q)p'+Q} + C_{12} \int_{|x|^{(\alpha-Q)p'}}^\infty \tau^{\frac{Q}{(\alpha-Q)p'}} d\tau$$
$$= C_{13} |x|^{(\alpha-Q)p'+Q},$$

where the positive constant C_{13} does not depend on x. The latter estimate yields

$$I_2 \leqslant C_{14} \left(\sum_{k\in\mathbb{Z}} \int_{D_k} |x|^{-\lambda q + [(\alpha-Q)p'+Q]q/p'} \left(\int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)|^p (y')^\gamma dy \right)^{q/p} (x')^\gamma dx \right)^{1/q}$$
$$\leqslant C_{14} \left(\sum_{k\in\mathbb{Z}} \int_{D_k} \left(\int_{\widetilde{D_k}} |f(y)|^p (y')^\gamma dy \right)^{q/p} |x|^{-\lambda q + [(\alpha-Q)p'+Q]q/p'} (x')^\gamma dx \right)^{1/q}$$
$$\leqslant C_{14} \left(\sum_{k\in\mathbb{Z}} 2^{k(-\lambda+\alpha-Q+Q/p'+Q/q)q} \left(\int_{\widetilde{D_k}} |f(y)|^p (y')^\gamma dy \right)^{q/p} \right)^{1/q}$$

$$\leqslant C_{14}\left(\sum_{k\in\mathbb{Z}}2^{k\beta q}\left(\int_{\widetilde{D_k}}|f(x)|^p(x')^\gamma dx\right)^{q/p}\right)^{1/q}$$

$$\leqslant C_{15}\left(\int_{\mathbb{R}^n_{k,+}}|x|^{\beta p}|f(x)|^p(x')^\gamma dx\right)^{q/p}.$$

Thus Theorem 1 is proved. □

Proof of Theorem 2. We write

$$\left(\int_{\{x\in\mathbb{R}^n_{k,+}:|x|^{-\lambda}\left|I_{\alpha,\gamma}f(x)\right|>\tau\}}(x')^\gamma dx\right)^{1/q}\leqslant J_1+J_2+J_3$$

$$\equiv\left(\int_{\{x\in\mathbb{R}^n_{k,+}:|x|^{-\lambda}\int_{B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy>\tau/3\}}(x')^\gamma dx\right)^{1/q}$$

$$+\left(\int_{\{x\in\mathbb{R}^n_{k,+}:|x|^{-\lambda}\int_{B_{2|x|}\setminus B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy>\tau/3\}}(x')^\gamma dx\right)^{1/q}$$

$$+\left(\int_{\{x\in\mathbb{R}^n_{k,+}:|x|^{-\lambda}\int_{\complement_{B_{2|x|}}}|f(y)|\,T^y|x|^{\alpha-Q}(y')^\gamma dy>\tau/3\}}(x')^\gamma dx\right)^{1/q}.$$

Then it is clear that

$$J_1\leqslant\left(\int_{\{x\in\mathbb{R}^n_{k,+}:2^{Q-\alpha}|x|^{\alpha-Q-\lambda}H_\gamma f(x)>\tau/3\}}(x')^\gamma dx\right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q<(Q-\alpha)q-Q$ ($i.e.$, $\alpha<Q-Q/q+\lambda$) we have

$$\int_{\complement_{B_t}}|x|^{(-\lambda+\alpha-Q)q}(x')^\gamma dx=C_1^q t^{(-\lambda+\alpha-Q)q+Q}.$$

By the condition $\beta\leqslant 0$ it follows that $\sup_{B_t}|x|^{-\beta}=t^{-\beta}$.

Summarizing these estimates we find that

$$\sup_{t>0}\left(\int_{\complement_{B_t}}|x|^{(-\lambda+\alpha-Q)q}(x')^\gamma dx\right)^{1/q}\sup_{B_t}|x|^{-\beta}=C_1\sup_{t>0}t^{Q/q-\lambda+\alpha-Q-\beta}<\infty$$

$$\Leftrightarrow\alpha-\beta-\lambda=Q-Q/q.$$

Now in the case $p=1$ the first part of Theorem A gives us the inequality

$$J_1\leqslant\frac{C_{16}}{\tau}\int_{\mathbb{R}^n_{k,+}}|x|^\beta|f(x)|^p(x')^\gamma dx,$$

where the positive constant C_{16} does not depend on f.

Further, we have

$$J_3 \leqslant \left(\int_{\{x\in\mathbb{R}^n_{k,+}:\, 2^{Q-\alpha}|x|^{-\lambda} H'_\gamma\left(|f(y)||y|^{\alpha-Q}\right)(x)>\tau/3\}} (x')^\gamma dx \right)^{1/q}.$$

Taking into account the inequality $-\lambda q > -Q$ $(i.e.,\ \lambda < Q/q)$ we get

$$\int_{B_t} |x|^{-\lambda q} (x')^\gamma dx = C_{17}^q t^{-\lambda q + Q},$$

where the positive constant C_{17} depends only on α and λ. Analogously, by virtue of the condition $\beta \geqslant \alpha - Q$ it follows that

$$\sup_{\complement B_t} |x|^{-\beta+\alpha-Q} = t^{-\beta+\alpha-Q}.$$

Then we obtain

$$\sup_{t>0} \left(\int_{B_t} |x|^{-\lambda q}(x')^\gamma dx \right)^{1/q} \sup_{\complement B_t} |x|^{-\beta+\alpha-Q} = C_{17} \sup_{t>0} t^{Q/q-\lambda+\alpha-Q-\beta} < \infty$$
$$\Leftrightarrow \alpha - \beta - \lambda = Q - Q/q.$$

Now in the case $p = 1$, from the second part of Theorem A we get the inequality

$$J_3 \leqslant \frac{C_{18}}{\tau} \int_{\mathbb{R}^n_{k,+}} |x|^\beta |f(x)| (x')^\gamma dx,$$

where the positive constant C_{18} does not depend on f.

We now estimate J_2. From $\beta + \lambda \geqslant 0$ and Theorem D, we get

$$\begin{aligned}
J_2 &= \left(\sum_{k\in\mathbb{Z}} \int_{\{x\in D_k:\, |x|^{-\lambda} \int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)|\, T^y|x|^{\alpha-Q}(y')^\gamma dy > \tau/3\}} (x')^\gamma dx \right)^{1/q} \\
&\leqslant \left(\sum_{k\in\mathbb{Z}} \int_{\{x\in D_k:\, \int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)||y|^\beta\, T^y|x|^{\alpha-\beta-\lambda-Q}(y')^\gamma dy > c\tau\}} (x')^\gamma dx \right)^{1/q} \\
&\leqslant \left(\sum_{k\in\mathbb{Z}} \int_{\{x\in D_k:\, \left|I_{\alpha-\beta-\lambda,\gamma}\left(f(\cdot)|\cdot|^\beta \chi_{\widetilde{D_k}}\right)(x)\right| > c\tau\}} (x')^\gamma dx \right)^{1/q} \\
&\leqslant \left(\sum_{k\in\mathbb{Z}} \left(\frac{C_{19}}{\tau} \int_{\widetilde{D_k}} |f(x)||x|^\beta\, (x')^\gamma dx \right)^q \right)^{1/q} \\
&\leqslant \left(\frac{C_{20}}{\tau} \int_{\mathbb{R}^n_{k,+}} |x|^\beta |f(x)| (x')^\gamma dx \right)^{1/q}.
\end{aligned}$$

Thus the proof of the theorem is completed. □

Proof of Theorem 3. Sufficiency part of Theorem 3 follows from Theorems 1 and 2.

Necessity. 1) Suppose that the operator $I_{\alpha,\gamma}$ is bounded from $L_{p,|x|^{\beta},\gamma}$ to $L_{q,|x|^{-\lambda},\gamma}$ and $1<p<Q/(\alpha-\beta-\lambda)$.

Define $f_t(x)=:f(tx)$ for $t>0$. Then it can be easily shown that

$$\|f_t\|_{L_{p,|x|^{\beta},\gamma}}=t^{-\frac{Q}{p}-\beta}\|f\|_{L_{p,|x|^{\beta},\gamma}},$$

$$(I_{\alpha,\gamma}f_t)(x)=t^{-\alpha}I_{\alpha,\gamma}f(tx),$$

and

$$\left\|I_{\alpha,\gamma}f_t\right\|_{L_{q,|x|^{-\lambda},\gamma}}=t^{-\alpha-\frac{Q}{q}+\lambda}\left\|I_{\alpha,\gamma}f\right\|_{L_{q,|x|^{-\lambda},\gamma}}.$$

From the boundedness of $I_{\alpha,\gamma}$, we have

$$\left\|I_{\alpha,\gamma}f\right\|_{L_{q,|x|^{-\lambda},\gamma}}\leqslant C\|f\|_{L_{p,|x|^{\beta},\gamma}},$$

where C does not depend on f. Then we get

$$\begin{aligned}\left\|I_{\alpha,\gamma}f\right\|_{L_{q,|x|^{-\lambda},\gamma}}&=t^{\alpha+Q/q-\lambda}\left\|I_{\alpha,\gamma}f_t\right\|_{L_{q,|x|^{-\lambda},\gamma}}\\&\leqslant Ct^{\alpha+Q/q-\lambda}\|f_t\|_{L_{p,|x|^{\beta},\gamma}}\\&=Ct^{\alpha+Q/q-\lambda-Q/p-\beta}\|f\|_{L_{p,|x|^{\beta},\gamma}}.\end{aligned}$$

If $1/p-1/q<(\alpha-\beta-\lambda)/Q$, then for all $f\in L_{p,|x|^{\beta},\gamma}$ we have $\left\|I_{\alpha,\gamma}f\right\|_{L_{q,|x|^{-\lambda},\gamma}}=0$ as $t\to 0$.

If $1/p-1/q>(\alpha-\beta-\lambda)/Q$, then for all $f\in L_{p,|x|^{\beta},\gamma}$ we have $\left\|I_{\alpha,\gamma}f\right\|_{L_{q,|x|^{-\lambda},\gamma}}=0$ as $t\to\infty$.

Therefore we obtain the equality $1/p-1/q=(\alpha-\beta-\lambda)/Q$.

2) The proof of necessity for the case 2) is similar to that of the case 1); therefore we omit it.

3) Let $f\in L_{p,|x|^{\beta},\gamma}$, $1<p=Q/(\alpha-\beta-\lambda)$. For given $t>0$ we denote

$$f_1(x)=f(x)\chi_{B_{2t}}(x),\quad f_2(x)=f(x)-f_1(x),\tag{11}$$

where $\chi_{B_{2t}}$ is the characteristic function of the set B_{2t}. Then

$$\widetilde{I}_{\alpha,\gamma}f(x)=\widetilde{I}_{\alpha,\gamma}f_1(x)+\widetilde{I}_{\alpha,\gamma}f_2(x)=F_1(x)+F_2(x),$$

where

$$F_1(x)=\int_{B_{2t}}\left(T^y|x|^{\alpha-Q}-|y|^{\alpha-Q}\chi_{\complement_{B_1}}(y)\right)f(y)(y')^{\gamma}dy,$$

and

$$F_2(x) = \int_{\complement_{B_{2t}}} \left(T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{\complement_{B_1}}(y) \right) f(y)(y')^\gamma dy.$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1 = -\int_{B_{2t}\setminus B_{\min\{1,2t\}}} |y|^{\alpha-Q} f(y)(y')^\gamma dy$$

is finite.

Note also that

$$\begin{aligned} F_1(x) - a_1 &= \int_{B_{2t}} T^y|x|^{\alpha-Q} f(y)(y')^\gamma dy - \int_{B_{2t}\setminus B_{\min\{1,2t\}}} |y|^{\alpha-Q} f(y)(y')^\gamma dy \\ &\qquad + \int_{B_{2t}\setminus B_{\min\{1,2t\}}} |y|^{\alpha-Q} f(y)(y')^\gamma dy \\ &= \int_{\mathbb{R}^n_{k,+}} T^y|x|^{\alpha-Q} f_1(y)(y')^\gamma dy = I_{\alpha,\gamma} f_1(x). \end{aligned}$$

Therefore

$$\begin{aligned} |F_1(x) - a_1| &\leqslant \int_{\mathbb{R}^n_{k,+}} |y|^{\alpha-Q} |T^y f_1(x)| (y')^\gamma dy \\ &= \int_{B(x,2t)} |y|^{\alpha-Q} |T^y f(x)| (y')^\gamma dy. \end{aligned}$$

Further, for $x \in B_t$, $y \in B(x,2t)$ we have

$$|y| \leqslant |x| + |x-y| < 3t.$$

Consequently, for all$x \in B_t$ we have

$$|F_1(x) - a_1| \leqslant \int_{B_{3t}} |y|^{\alpha-Q} |T^y f(x)| (y')^\gamma dy. \tag{12}$$

By Theorem C and inequality (12), for $(\alpha - \beta - \lambda)p = Q$ we have

$$\begin{aligned} t^{-Q-\lambda} &\int_{B_t} |T^z F_1(x) - a_1| (z')^\gamma dz \\ &\leqslant C t^{-Q-\lambda} \int_{B_t} T^z \left(\int_{B_{3t}} |y|^{\alpha-Q} T^y |f(x)| (y')^\gamma dy \right) (z')^\gamma dz \\ &\leqslant C t^{\alpha-Q-\lambda} \cdot t^{Q/p'} \left(\int_{B_t} T^z \left(M_\gamma(f(x)) \right)^p (z')^\gamma dz \right)^{1/p} \\ &\leqslant C t^\beta \left(\int_{B_t} T^z \left(M_\gamma(f(x)) \right)^p (z')^\gamma dz \right)^{1/p} \end{aligned}$$

$$\leqslant C\left(\int_{B_t}|z|^{\beta p}\,T^z\left(M_\gamma(f(x))\right)^p(z')^\gamma dz\right)^{1/p}$$
$$=C\left(\int_{\mathbb{R}^n_{k,+}}T^z\left(\chi_{B_t}|x|^{\beta p}\right)\ \left(M_\gamma(f(x))\right)^p(z')^\gamma dz\right)^{1/p}$$
$$=C\left(\int_{\mathbb{R}^n_{k,+}}|z|^{\beta p}\left(M_\gamma(f(x))\right)^p(z')^\gamma dz\right)^{1/p}$$
$$\leqslant C\|f\|_{L_{p,|x|^\beta,\gamma}}. \tag{13}$$

Denote

$$a_2=\int_{B_{\max\{1,2t\}}\setminus B_{2t}}|y|^{\alpha-Q}f(y)(y')^\gamma dy$$

and estimate $|F_2(x)-a_2|$ for $x\in B_t$:

$$|F_2(x)-a_2|\leqslant\int_{\complement_{B_{2t}}}|f(y)|\left|T^y|x|^{\alpha-Q}-|y|^{\alpha-Q}\right|{y_n}^\gamma dy.$$

Applying Lemma 1 and Hölder's inequality we get

$$\begin{aligned}|F_2(x)-a_2|&\leqslant 2^{Q-\alpha+1}|x|\int_{\complement_{B_{2t}}}|f(y)||y|^{\alpha-Q-1}{y_n}^\gamma dy\\
&\leqslant 2^{Q-\alpha+1}|x|\left(\int_{\complement_{B_t}}|y|^{\beta p}|f(y)|^p{y_n}^\gamma dy\right)^{1/p}\left(\int_{\complement_{B_t}}|y|^{(-\beta+\alpha-Q-1)p'}{y_n}^\gamma dy\right)^{1/p'}\\
&\leqslant C|x|t^{\alpha-\beta-1-Q/p}\|f\|_{L_{p,|x|^\beta,\gamma}}\\
&\leqslant C|x|t^{\lambda-1}\|f\|_{L_{p,|x|^\beta,\gamma}}\\
&\leqslant C|x|^\lambda\|f\|_{L_{p,|x|^\beta,\gamma}}.\end{aligned}$$

Note that if $|x|\leqslant t$ and $|z|\leqslant 2t$, then $T^z|x|\leqslant|x|+|z|\leqslant 3t$. Thus for $(\alpha-\beta-\lambda)p=Q$ we obtain

$$|T^zF_2(x)-a_2|\leqslant T^z|F_2(x)-a_2|\leqslant C|x|^\lambda\|f\|_{L_{p,|x|^\beta,\gamma}}. \tag{14}$$

Denote

$$a_f=a_1+a_2=\int_{B_{\max\{1,2t\}}}|y|^{\alpha-Q}f(y)(y')^\gamma dy.$$

Finally, from (13) and (14) we have

$$\sup_{x,t}t^{-Q-\lambda}\int_{B_t}\left|T^y\widetilde{I}_{\alpha,\gamma}f(x)-a_f\right|(y')^\gamma dy\leqslant C\|f\|_{L_{p,|x|^\beta,\gamma}}.$$

Thus

$$\left\|\widetilde{I}_{\alpha,\gamma}f\right\|_{BMO_{|x|^{-\lambda},\gamma}} \leqslant 2C\sup_{x,t} t^{-Q-\lambda}\int_{B_t}\left|T^y\widetilde{I}_{\alpha,\gamma}f(x)-a_f\right|(y')^\gamma dy \leqslant C\|f\|_{L_{p,|x|^\beta,\gamma}}.$$

Thus Theorem 3 is proved. □

If we take $p=q$, $\beta=0$ or $p=q$, $\lambda=0$ in Theorem 3, then we get the following

COROLLARY 1. *1) Let* $0<\alpha<Q/p$, $1<p<\infty$, *then* $I_{\alpha,\gamma}$ *is bounded from* $L_{p,\gamma}$ *to* $L_{p,|x|^{-\alpha},\gamma}$.
2) Let $0<\alpha<Q/p'$, $1<p<\infty$, *then* $I_{\alpha,\gamma}$ *is bounded from* $L_{p,|x|^\alpha,\gamma}$ *to* $L_{p,\gamma}$.

Proof of Theorem 4. By the definition of the weighted B-Besov spaces it suffices to show that

$$\|T^yI_{\alpha,\gamma}f-I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} \leqslant C\|T^yf-f\|_{L_{p,|x|^\beta,\gamma}}.$$

It is easy to see that T^y commutes with $I_{\alpha,\gamma}$, i.e., $T^yI_{\alpha,\gamma}f=I_{\alpha,\gamma}(T^yf)$. Hence we obtain

$$|T^yI_{\alpha,\gamma}f-I_{\alpha,\gamma}f|=|I_{\alpha,\gamma}(T^yf)-I_{\alpha,\gamma}f|\leqslant I_{\alpha,\gamma}(|T^yf-f|).$$

Taking $L_{q,|x|^{-\lambda},\gamma}$-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of $I_{\alpha,\gamma}$ from $L_{p,|x|^\beta,\gamma}$ to $L_{q,|x|^{-\lambda},\gamma}$. □

From Theorem 4 we get the following result on the boundedness of $I_{\alpha,\gamma}$ on the B-Besov spaces $B^s_{p\theta,\gamma}\equiv B^s_{p\theta,1,\gamma}$.

COROLLARY 2. *Let* $0<\alpha<Q$, $1<p<Q/\alpha$, $1/p-1/q=\alpha/Q$, $1\leqslant\theta\leqslant\infty$ *and* $0<s<1$. *Then the operator* $I_{\alpha,\gamma}$ *is bounded from* $B^s_{p\theta,\gamma}$ *to* $B^s_{q\theta,\gamma}$. *More precisely, there is a constant* $C>0$ *such that*

$$\|I_{\alpha,\gamma}f\|_{B^s_{q\theta,\gamma}}\leqslant C\|f\|_{B^s_{p\theta,\gamma}}$$

holds for all $f\in B^s_{p\theta,\gamma}$.

Acknowledgements

The authors are thankful to the referee for his/her valuable comments and suggestions.

REFERENCES

[1] I.A. ALIEV, B. RUBIN, *Wavelet-like transforms for admissible semi-groups; inversion formulas for potentials and Radon transforms,* J. Fourier Anal. Appl., **11**, 3 (2005), 333–352.

[2] G. ALTENBURG, *Eine Realisierung der Theorie der abstrakten Besov-Raume* $B^s_q(A)$ $s>0$, $(1\leqslant q\leqslant\infty)$ *und der Lebesgue-Raume* $H^s_{p,\gamma}$ *auf der Grundlage Besselscher Differential-operatoren,* Z. Anal. Anwendungen, **3**, 1 (1984), 43–63.

[3] M. ASSAL, H. BEN ABDALLAH, *Generalized weighted Besov spaces on the Bessel hypergroup,* J. Funct. Spaces Appl., **4**, 1 (2006), 91–111.

[4] R.R. COIFMAN, G. WEISS, *Analyse harmonique non commutative sur certains expaces homogenes*, Lecture Notes in Math., 242, Springer- Verlag, Berlin, 1971.

[5] D.I. CRUZ-BAEZ AND J. RODRIGUEZ, *Spaces of distributions of Besov and Lizorkin-Triebel type for the Fourier-Bessel transform,* J. Math. Anal. Appl., **259** (2001), 51–63.

[6] D.E. EDMUNDS, V.M. KOKILASHVILI AND A. MESKHI, *Boundedness and compactness of Hardy-type operators on Banach function spaces defined on measure space,* Proc. A. Razmadze Math. Inst., **117** (1998), 7–30.

[7] D.E. EDMUNDS, V.M. KOKILASHVILI AND A. MESKHI, *Bounded and Compact Integral Operators*, Dordrecht: Kluwer 2002.

[8] A.D. GADJIEV, I.A. ALIEV, *On classes of operators of potential types, generated by a generalized shift,* Reports of enlarged Session of the Seminars of I.N. Vekua Inst. of Applied Mathematics, Tbilisi. **3**, 2 (1988), 21–24. (Russian)

[9] A.D. GADJIEV AND V.S. GULIYEV, *The Stein-Weiss type inequality for fractional integrals, associated with the Laplace-Bessel differential operator,* Fract. Calc. Appl. Anal., **11**, 1 (2008), 77–90.

[10] E.V. GULIYEV, *On limiting case of the Stein-Weiss type inequality for the B-Riesz potentials,* Fract. Calc. Appl. Anal., **12**, 1 (2009), 1–10.

[11] E.V. GULIYEV, *Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator,* Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **26**, 1 (2006), 71–80.

[12] V.S. GULIYEV, *Sobolev's theorem for Riesz B-potentials,* Dokl. Akad. Nauk, **358**, 4 (1998), 450–451 (in Russian).

[13] V.S. GULIYEV, *On maximal function and fractional integral, associated the Bessel differential operator,* Math. Inequal. Appl., **6**, 2 (2003), 317–330.

[14] V. S. GULIYEV, A. SERBETCI AND I. EKINCIOGLU, *Necessary and sufficient conditions for the boundedness of rough B-fractional integral operators in the Lorentz spaces,* J. Math. Anal. Appl., **336**, 1 (2007), 425–437.

[15] V.S. GULIEV, A. SERBETCI, AND I. EKINCIOGLU, *On boundedness of the generalized B-potential integral operators in the Lorentz spaces,* Integral Transforms Spec. Funct., **18**, 12 (2007), 885–895.

[16] V.S. GULIYEV, A.SERBETCI, Z.V. SAFAROV, *Nikolskii-Besov spaces associated with the Fourier-Bessel transforms,* Transactions of NAS of Azerbaijan. Embedding theorems, Harmonic analysis, **13** (2007), 194–211.

[17] I.A. KIPRIYANOV, *Fourier-Bessel transformations and imbedding theorems,* Trudy Math. Inst. Steklov, **89** (1967), 130–213.

[18] I.A. KIPRIYANOV, V.I. KONONENKO, *The fundamental solutions for B-elliptic equations,* Differentsialnye Uravneniya, **3**, 1 (1967), 114–129.

[19] I.A. KIPRIYANOV, L.A. IVANOV, *Obtaining fundamental solutions for homogeneous equations with singularities in several variables,* Trudy Sem. Soboleva, Inst. Mat., Novosibirsk, **1** (1983), 55–77.

[20] V.M. KOKILASHVILI, A. KUFNER, *Fractional integrals on spaces of homogeneous type,* Comment. Math. Univ. Carolinae, **30**, 3 (1989), 511–523.

[21] V. KOKILASHVILI, A. MESKHI, *On some weighted inequalities for fractional integrals on nonhomogeneous spaces,* Zeitschrift für Analysis und ihre Anwendungen, **24**, 4 (2005), 871–885.

[22] B.M. LEVITAN, *Bessel function expansions in series and Fourier integrals,* Uspekhi Mat. Nauk **6**, 2 (42) (1951), 102–143. (Russian)

[23] L.N. LYAKHOV, *Multipliers of the Mixed Fourier-Bessel Transformation,* Proc. V.A. Steklov Inst. Math., **214** (1997), 234–249.

[24] R.A. MACIAS, C. SEGOVIA, *A Well Behaved Quasi-distance for Spaces of Homogeneous Type,* Trabajos de Matematica, Instituto Argentino de Matematica, Buenos Aires, **32** (1981), 1–17.

[25] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function,* Trans. Amer. Math. Soc., **165** (1972), 207–226.

[26] B. MUCKENHOUPT AND E.M. STEIN, *Classical expansions and their relation to conjugate harmonic functions,* Trans. Amer. Math. Soc., **118** (1965), 17–92.

[27] E.M. STEIN AND G. WEISS, *Fractional integrals on n-dimensional Euclidean spaces,* J. Math. Mech., **7**, 4 (1958), 503–514.

[28] K. STEMPAK, *The Littlewood-Paley theory for the Fourier-Bessel transform,* Mathematical Institute of Wroclaw (Poland), Preprint no. **45** (1985).

[29] K. STEMPAK, *Almost everywhere summability of Laguerre series*, Studia Math., **100**, 2 (1991), 129–147.

[30] K. TRIMÈCHE, *Generalized Harmonic Analysis and Wavelet Packets*, Gordon and Breach Science Publishers, Australia, 2001.

(Received May 13, 2010)

A. D. Gadjiev
Institute of Mathematics and Mechanics
Baku
Azerbaijan
e-mail: akif_gadjiev@mail.az

V. S. Guliyev
Ahi Evran University
Department of Mathematics
Kırşehir
Turkey
e-mail: vagif@guliyev.com

A. Serbetci
Ankara University
Department of Mathematics, Ankara
Turkey
e-mail: serbetci@science.ankara.edu.tr

E. V. Guliyev
Institute of Mathematics and Mechanics
Baku
Azerbaijan
e-mail: emin@guliyev.com