Journal of Mathematical Sciences, Vol. 170, No. 4, 2010

BOUNDEDNESS OF MAXIMAL, POTENTIAL TYPE, AND SINGULAR INTEGRAL OPERATORS IN THE GENERALIZED VARIABLE EXPONENT MORREY TYPE SPACES

V. S. Guliyev

Ahi Evran University
40100 Kirsehir, Turkey

Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences
9, F. Agaev St., Baku AZ1141, Azerbaijan
vagif@guliyev.com

J. J. Hasanov

Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences
9, F. Agaev St., Baku AZ1141, Azerbaijan
hasanovjavanshir@yahoo.com.tr

S. G. Samko *

Universidade do Algarve
Campus de Gambelas, Faro 8000, Portugal
ssamko@ualg.pt

UDC 517.9

We consider generalized Morrey type spaces $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ *with variable exponents* $p(x)$, $\theta(r)$ *and a general function* $\omega(x,r)$ *defining a Morrey type norm. In the case of bounded sets* $\Omega \subset \mathbb{R}^n$, *we prove the boundedness of the Hardy–Littlewood maximal operator and Calderón–Zygmund singular integral operators with standard kernel. We prove a Sobolev–Adams type embedding theorem* $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega) \to \mathscr{M}^{q(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$ *for the potential type operator* $I^{\alpha(\cdot)}$ *of variable order. In all the cases, we do not impose any monotonicity type conditions on* $\omega(x,r)$ *with respect to* r. *Bibliography*: 40 *titles.*

1 Introduction

In the study of local properties of solutions of partial differential equations, Morrey spaces $\mathscr{M}^{p,\lambda}(\Omega)$, together with weighted Lebesgue spaces, play an important role (cf. [1, 2]). These spaces were introduced by Morrey [3] in 1938. The norm in a Morrey space is defined by

$$\|f\|_{\mathscr{M}^{p,\lambda}(\Omega)} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\widetilde{B}(x,r))},$$

* To whom the correspondence should be addressed.

Translated from *Problems in Mathematical Analysis* **50**, September 2010, pp. 3–20

where $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, $0 \leqslant \lambda < n$, $1 \leqslant p < \infty$, and $\Omega \subseteq \mathbb{R}^n$ is an open set.

During last two decades, there has been an increasing interest in the study of spaces with variable exponents and operators with variable parameters in such spaces (cf., for example, [4]–[7]).

The Morrey spaces $\mathscr{M}^{p(\cdot),\lambda(\cdot)}(\Omega)$ with variable exponents were introduced and studied in the Euclidean setting [8, 9] and in the setting of metric measure spaces [10].

In particular, in the case of bounded sets, there were proved [8] the boundedness of the maximal operator in variable exponent Morrey spaces $\mathscr{M}^{p(\cdot),\lambda(\cdot)}(\Omega)$ provided that $p(\cdot)$ satisfies the log-condition and a Sobolev–Adams type theorem $\mathscr{M}^{p(\cdot),\lambda(\cdot)} \to \mathscr{M}^{q(\cdot),\lambda(\cdot)}$ for potential type operators provided that $p(\cdot)$ and $\lambda(\cdot)$ satisfy the log-condition and

$$\inf_{x\in\Omega} \alpha(x) > 0, \quad \sup_{x\in\Omega} [\lambda(x) + \alpha(x)p(x)] < n.$$

In the case of constant α, there was also proved a boundedness theorem in the limiting case

$$p(x) = \frac{n-\lambda(x)}{\alpha}$$

when the potential type operator I^α acts from $\mathscr{M}^{p(\cdot),\lambda(\cdot)}$ into BMO.

In [9], the maximal operator and potential type operators were considered in more general spaces, but under more restrictive conditions on $p(x)$. Hästö [11] used his new "local-to-global" approach to extend the result of [8] about the maximal operator to the whole space $\mathbb{R}^n$.

The boundedness of the maximal operator and singular integral operators in the variable exponent Morrey spaces $\mathscr{M}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [10].

For constant p and λ the results on the boundedness of potential type operators and classical Calderón–Zygmund singular integral operators go back to [12] and [13] respectively, whereas the boundedness of the maximal operator in the Euclidean setting was proved in [14] (cf., for example, [15]–[20] for further results for constant p and λ).

The generalized variable exponent Morrey spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{\mathscr{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L_{p(\cdot)}(\widetilde{B}(x,r))}, \quad 1 \leqslant p < \infty,$$

were introduced in [21]; here,

$$\inf_{x\in\Omega, r>0} \omega(x,r) > 0. \tag{1.1}$$

Owing to the last assumption, the space is nontrivial. Hereinafter, $\omega(x,r)$ is a nonnegative measurable function on $\Omega \times (0,\ell)$, $\ell = \operatorname{diam} \Omega$. In [21], the maximal, singular, and potential type operators were studied in the case of bounded sets, $\ell < \infty$.

We introduce the generalized Morrey type spaces $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ with variable exponent. In comparison with the usual definition of Morrey norm, the L^∞-norm in r is replaced by the L^θ-norm. Such spaces with constant p were first introduced in [22, 23] (cf. [16], [24]–[30] for further development).

The case of spaces

$$\mathscr{L}^{p,\lambda,\theta}(\mathbb{R}^n) := \left\{ f : \|f\|_{p.\lambda,\theta} = \sup_{x\in\Omega} \left[\int_0^\infty \Big(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \Big)^{\frac{\theta}{p}} dr \right]^{\frac{1}{\theta}} < \infty \right\} \tag{1.2}$$

goes back to [31], where such spaces first appeared. In [31, p. 44], a theorem on mapping properties of the Riesz potential operator in such spaces was given (cf. also a reformulation of this result in [19, Theorem 3]).

In the spaces $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ with bounded open sets $\Omega \subset \mathbb{R}^n$, we consider the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy,$$

the potential type operator

$$I^{\alpha(x)}f(x) = \int\limits_{\Omega} |x-y|^{\alpha(x)-n} f(y)dy, \quad 0 < \alpha(x) < n,$$

the fractional maximal operator

$$M^{\alpha(x)}f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha(x)}{n}-1} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy, \quad 0 \leqslant \alpha(x) < n,$$

of variable order $\alpha(x)$, and the Calderón–Zygmund type singular operator

$$Tf(x) = \int\limits_{\Omega} K(x,y)f(y)\,dy,$$

where $K(x,y)$ is a "standard singular kernel," i.e., a continuous function on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ such that

$$|K(x,y)| \leqslant C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$

$$|K(x,y) - K(x,z)| \leqslant C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0 \quad \text{if} \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leqslant C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0 \quad \text{if} \quad |x-y| > 2|x-\xi|.$$

We find a condition on $\omega(x,r)$ for the boundedness of the maximal operator M and the singular integral operator T in the generalized Morrey space $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ with variable exponent $p(x)$ provided that $p(x)$ satisfies the log-condition. For potential type operators, under the same log-condition and the assumptions

$$\inf_{x\in\Omega} \alpha(x) > 0, \quad \sup_{x\in\Omega} \alpha(x)p(x) < n,$$

we also find the condition on $\omega(x,r)$ and $q(x)$ for the validity of the mapping theorem,

$$\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega) \to \mathscr{M}^{q(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega),$$

which recovers the Sobolev–Adams type result [8] in the case of the classical Morrey spaces with variable exponents, when

$$\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$$

and

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n - \lambda(x)}.$$

The paper is organized as follows. In Section 2, we recall some facts concerning Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and Morrey spaces $\mathscr{M}^{p(\cdot),\lambda(\cdot)}(\Omega)$ with variable exponents. Section 3 contains necessary information about generalized Morrey spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ with variable exponent. In Section 4, we introduce a new type of generalized Morrey spaces $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ with variable exponents. In this section, we also prove some embeddings (Subsection 4.1) and formulate the main results (Subsections 4.2–4.4) which are proved in Section 5.

Note that we do not impose any monotonicity type conditions on $\omega(x,r)$, which was possible due to the usage of the results of our previous paper [21]. We assume that the variable exponent $p(x)$ is log-continuous, whereas for the variable exponent $\theta(r)$ no log-Hölder condition is used; only in some reformulations of the results for the case of power functions ω, we impose a log-type decay condition on $\theta(r)$ at the point $r = 0$.

Notation

$\mathbb{R}^n$ is the n-dimensional Euclidean space,

$\Omega \subseteq \mathbb{R}^n$ is an open set,

$\ell = \operatorname{diam} \Omega$,

$\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$,

$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\})$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$,

c, C, c_1, c_2 etc are various absolute positive constants.

An open set Ω is assumed to be bounded throughout the paper.

2 Preliminaries. Variable Exponent Lebesgue Spaces $L^{p(\cdot)}(\Omega)$ and Morrey Spaces $\mathscr{M}^{p(\cdot),\lambda(\cdot)}(\Omega)$

Let $p(\cdot)$ be a measurable function on Ω with the values in $[1,\infty)$. We suppose that

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty, \tag{2.1}$$

where

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty.$$

We denote by $L^{p(\cdot)}(\Omega)$ the space of measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int\limits_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

This space equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\Big\{\eta > 0 : \ I_{p(\cdot)}\Big(\frac{f}{\eta}\Big) \leqslant 1\Big\}$$

is a Banach space. The conjugate exponent p' is defined by the formula

$$p'(\cdot) = \frac{p(x)}{p(x) - 1}, \quad x \in \Omega.$$

Basic facts concerning Lebesgue spaces with variable exponents can be found in [32, 33].

In the case $\Omega = (0, \ell)$, we consider variable exponent Lebesgue spaces $L^{\theta(\cdot)}(0, \ell)$ admitting the value $\theta(t) = \infty$. In this case, we suppose that $\theta(t)$ is bounded outside the set

$$E_\infty(\theta) = \{t \in (0, \ell) : \theta(t) = \infty\}$$

and the norm is introduced in the standard way:

$$\|\varphi\|_{\theta(\cdot)} = \|\varphi\|_{L^{\theta(\cdot)}((0,\ell)\setminus E_\infty(\theta))} + \sup_{t \in E_\infty(\theta)} |\varphi(t)|.$$

We denote by $\mathbb{P}(0, \ell)$ the set of measurable exponents $\theta(t)$ with the values in $[1, \infty]$ such that

$$\theta \in L^\infty((0, \ell)\setminus E_\infty(\theta)).$$

Definition 2.1. Denote by $\mathscr{P}^{\log} = \mathscr{P}^{\log}(\Omega)$ the class of functions that are defined on Ω and satisfy the log-condition

$$|p(x) - p(y)| \leqslant \frac{A}{-\ln |x - y|}, \quad |x - y| \leqslant \frac{1}{2}, \quad x, y \in \Omega, \tag{2.2}$$

where $A = A(p) > 0$ is independent of x and y. If $\Omega = (0, \ell)$, we denote by $\mathscr{P}_0(0, \ell)$ the set of bounded measurable functions θ on $(0, \ell)$ with the values in $[1, \infty)$ such that there exists the limit

$$\theta(0) = \lim_{t \to 0} \theta(t)$$

and

$$|\theta(t) - \theta(0)| \leqslant \frac{A}{\ln(1/t)}, \quad 0 < t \leqslant \frac{1}{2}.$$

We write $\theta \in \mathscr{M}_0(0, \ell)$ if there exists a constant $c \in \mathbb{R}^1$ such that $c + \theta(t) \in \mathscr{P}_0(0, \ell)$.

The following theorem was proved in [34] under the condition that the maximal operator is bounded in $L^{p(\cdot)}(\Omega)$. As is known, this condition may be omitted due to the result of Diening [35].

Theorem 2.1. *Suppose that $p, \alpha \in \mathscr{P}^{\log}(\Omega)$ satisfy (2.1) and the following conditions:*

$$\inf_{x \in \mathbb{R}^n} \alpha(x) > 0, \quad \sup_{x \in \mathbb{R}^n} \alpha(x) p(x) < n. \tag{2.3}$$

Then $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$, where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}. \tag{2.4}$$

Singular integral operators in spaces with variable exponents were studied in [36].

We will use the following estimate (cf. the corollary to Lemma 2.22 in [37]).

Lemma 2.1. *Suppose that p, $1 \leqslant p(x) \leqslant p_+ < \infty$, satisfies (2.2) and*

$$\sup_{x\in\Omega} \nu(x) < \infty, \quad \inf_{x\in\Omega} [n + \nu(x)p(x)] > 0.$$

Then

$$\||x - \cdot|^{\nu(x)} \chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leqslant C r^{\nu(x)+\frac{n}{p(x)}}, \quad x \in \Omega, \quad 0 < r < \ell = \operatorname{diam} \Omega, \tag{2.5}$$

where C is independent of x and r.

It can be shown that the dependence of the constant C in (2.5) on Ω can be expressed as

$$C = C_0 \ell^{n\left(\frac{1}{p_-} - \frac{1}{p_+}\right)},$$

where C_0 is independent of Ω.

For the weighted Hardy type operator

$$\overline{H}_{v,w} f(t) = v(t) \int_t^\ell f(\xi) w(\xi) d\xi, \quad t \in (0, \ell),$$

the following assertion was proved in [38].

Lemma 2.2. *Let $\theta(t)$ and $r(t)$ be measurable functions on $I = (0, \ell)$ such that*

$$1 < \inf_{t\in(0,\ell)} \theta(t), \quad \sup_{t\in(0,\ell)} r(t) < \infty, \quad \theta(t) \leqslant r(t), \quad t \in (0, \ell). \tag{2.6}$$

If

$$\sup_{0<t<\ell} \int_0^t v(\xi)^{r(\xi)} \left(\int_t^\ell w^{[\widetilde{\theta}(\xi)]'}(r) dr \right)^{\frac{r(\xi)}{[\widetilde{\theta}(\xi)]'}} d\xi < \infty,$$

where

$$\widetilde{\theta}(\xi) = \inf_{s\in(\xi,\ell)} \theta(s),$$

then the operator $\overline{H}_{v,w}$ is bounded from $L^{\theta(\cdot)}(0, \ell)$ to $L^{r(\cdot)}(0, \ell)$.

Note that there are no assumptions on the continuity of $\theta(\cdot)$ and $r(\cdot)$ in Lemma 2.2.

Remark 2.1. For power weights

$$v(t) = t^{\alpha(t)}, \quad w(\xi) = \frac{1}{\xi^{1+\beta(\xi)}}$$

Lemma 2.2 means that, under the assumption (2.6), the condition

$$\sup_{t\in(0,\ell)} \int_0^t \left[\frac{\xi^{\alpha(\xi)}}{t^{\frac{1}{\theta(\xi)} - \frac{\beta(\xi)}{1+\beta(\xi)}}} \right]^{r(\xi)} d\xi < \infty$$

guarantees the validity of the Hardy inequality

$$\left\| t^{\alpha(t)} \int\limits_t^{\ell} \frac{f(\xi)\,d\xi}{\xi^{1+\beta(\xi)}} \right\|_{r(\cdot)} \leqslant \|f\|_{\theta(\cdot)}, \tag{2.7}$$

where no continuity assumptions are imposed on $\theta(t)$, $r(t)$, and $\alpha(t)$.

In the case $\alpha, \beta \in \mathscr{M}_0(0,\ell)$, $\theta, r \in \mathscr{P}_0(0,\ell)$, where

$$\frac{1}{r(0)} = \frac{1}{\theta(0)} + \beta(0) - \alpha(0),$$

the following necessary and sufficient conditions for the validity of the Hardy inequality (2.7) are known [39]:

$$\beta(0) > -\frac{1}{\theta(0)}, \quad \beta(0) \leqslant \alpha(0) < \frac{1}{\theta(0)} + \beta(0).$$

Let $\lambda(x)$ be a measurable function on Ω with the values in $[0,n]$. The *variable exponent Morrey space* $\mathscr{M}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined [8] as the set of measurable functions f on Ω with the finite norm

$$\|f\|_{\mathscr{M}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

3 Preliminaries. Generalized Variable Exponent Morrey Spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$

The generalized Morrey space $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ was introduced in [21] as the space of functions equipped with the norm

$$\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

In Subsection 3.2, we recall some results obtained in [21] for the spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$. The case of a constant p is treated in Subsection 3.1.

3.1 Generalized Morrey spaces with constant exponent p

Sufficient conditions on ω_1 and ω_2 for the boundedness of a singular operator T from $\mathscr{M}^{p,\omega_1}(\mathbb{R}^n)$ to $\mathscr{M}^{p,\omega_2}(\mathbb{R}^n)$ were obtained in [22, 23, 26, 27]. In particular, monotonicity type conditions, together with some integral conditions, were imposed on ω_1 and ω_2 in [26, 27], whereas no monotonicity conditions were required in [22, 23].

The following statement containing the results of [26, 27] was proved in [22] (cf. also [23, 28]).

Theorem 3.1 (cf. [22]). *Let $1 < p < \infty$, and let $\omega_1(x,r)$, $\omega_2(x,r)$ be positive measurable functions such that*

$$\int\limits_r^{\infty} \omega_1(x,t)\frac{dt}{t} \leqslant c_1\,\omega_2(x,r),$$

where $c_1 > 0$ is independent of $x \in \mathbb{R}^n$ and $t > 0$. Then the operators M and T are bounded from $\mathscr{M}^{p,\omega_1}(\mathbb{R}^n)$ to $\mathscr{M}^{p,\omega_2}(\mathbb{R}^n)$.

Theorem 3.2 (cf. [22]). *Suppose that $0 < \alpha < n$, $1 < p < \infty$, $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$. Let $\omega_1(x,r)$ and $\omega_2(x,r)$ be positive measurable functions such that*

$$\int_r^\infty t^\alpha \omega_1(x,t)\frac{dt}{t} \leqslant c_1\, \omega_2(x,r).$$

Then the operators M^α and I^α are bounded from $\mathscr{M}^{p,\omega_1}(\mathbb{R}^n)$ to $\mathscr{M}^{q,\omega_2}(\mathbb{R}^n)$.

Theorem 3.3 (cf. [28]). *Suppose that $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$. Let $\omega(x,t)$ be such that*

$$\int_t^\infty \omega(x,r)\frac{dr}{r} \leqslant C\, \omega(x,t)$$

and

$$t^\alpha \omega(x,t) + \int_t^\infty r^\alpha\, \omega(x,r)\frac{dr}{r} \leqslant C\omega(x,t)^{\frac{p}{q}},$$

where $q \geqslant p$ and C is independent of $x \in \mathbb{R}^n$ and $t > 0$. Suppose that for almost all $x \in \mathbb{R}^n$ the function $w(x,r)$ satisfies the condition

there exist $a = a(x) > 0$ such that $\omega(x,\cdot) : [0,\infty] \to [a,\infty)$ is a surjection.

Then the operators M_α and I_α are bounded from $\mathscr{M}_{p,\omega}(\mathbb{R}^n)$ to $\mathscr{M}_{q,\omega^{p/q}}(\mathbb{R}^n)$.

3.2 Generalized Morrey spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ with variable exponents

The following three theorems were proved in [21].

Theorem 3.4. *Suppose that $p \in \mathscr{P}^{\log}(\Omega)$ satisfies (2.1) and $\omega_1(x,r)$, $\omega_2(x,r)$ satisfy the condition*

$$\int_r^\ell \omega_1(x,t)\frac{dt}{t} \leqslant C\, \omega_2(x,r), \tag{3.1}$$

where C is independent of x and t. Then the operators M and T are bounded from $\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathscr{M}^{p(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 3.5. *Suppose that $p, q \in \mathscr{P}^{\log}(\Omega)$ satisfy (2.1), $\alpha(x)$, $q(x)$ satisfy (2.3), (2.4), and $\omega_1(x,r)$, $\omega_2(x,r)$ satisfy the condition*

$$\int_r^\ell t^{\alpha(x)}\omega_1(x,t)\frac{dt}{t} \leqslant C\, \omega_2(x,r),$$

where C is independent of x and r. Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathscr{M}^{q(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 3.6. *Suppose that* $p \in \mathscr{P}^{\log}(\Omega)$ *satisfies* (2.1), $\alpha(x)$ *satisfies* (2.3), *and* $\omega(x,t)$ *satisfies* (3.1) *and the conditions*

$$\omega(x,r) \leqslant \frac{C}{r^{\alpha(x)/(1-p(x)/q(x))}},$$

$$\int_r^\ell t^{\alpha(x)-1}\,\omega(x,t)dt \leqslant C\omega(x,r)^{\frac{p(x)}{q(x)}},$$

where $q(x) > p(x)$ *and* C *is independent of* $x \in \Omega$ *and* $r \in (0,\ell]$. *Suppose also that for almost all* $x \in \Omega$ *the function* $w(x,r)$ *satisfies the condition*

$$\text{there exist } a = a(x) > 0 \text{ such that } \omega(x,\cdot) : [0,\ell] \to [a,\infty) \text{ is a surjection.}$$

Then the operators $M^{\alpha(\cdot)}$ *and* $I^{\alpha(\cdot)}$ *are bounded from* $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ *to* $\mathscr{M}^{q(\cdot),\omega(\cdot)}(\Omega)$.

4 The Main Results

We introduce generalized spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$.

Definition 4.1. Let $\omega(x,r) : \Omega \times (0,\ell) \to [0,\infty)$ and $\theta(r) : (0,\ell) \to [1,\infty]$ be measurable functions. The *space* $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ is the set of functions with the finite norm

$$\|f\|_{\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)} = \sup_{x\in\Omega} \left\| \frac{\omega(x,r)}{r^{\frac{n}{p(x)}}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \right\|_{L^{\theta(\cdot)}(0,\ell)}.$$

If $\theta(r) \equiv \infty$, then $\mathscr{M}^{p(\cdot),\infty,\omega(\cdot)}(\Omega)$ is the space of functions with the finite norm

$$\sup_{x\in\Omega, r\in(0,\ell)} \omega(x,r) r^{-\frac{n}{p(x)}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

In the above notation, we can write

$$\mathscr{M}^{p(\cdot),\infty,\omega(\cdot)}(\Omega) = \mathscr{M}^{p(\cdot),\frac{1}{\omega(\cdot)}}(\Omega).$$

Note that we impose the standard local log-condition on the exponent $p(x)$ to obtain statements on the maximal, singular, and potential type operators. However, the local log-condition is not required for the variable exponent $\theta(r)$ when we integrate the means

$$\int_{B(x,r)} |f(y)|^p dy$$

over balls with radius r. In some statements (for example, Theorems 4.1, 4.3–4.5), there are no log decay type conditions on $\theta(r)$ as $r \to 0$, whereas such a condition is imposed, for example, in Theorem 4.2.

Throughout the paper, we assume that $\omega(x,r)$ satisfies the condition

$$\sup_{x\in\Omega} \|\omega(x,\cdot)\|_{L^{\theta(\cdot)}(0,\ell)} < \infty. \tag{4.1}$$

Then the space $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ contains bounded functions (cf. Lemma 4.1) and thereby is nonempty.

The fact that $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ is a Banach space can be proved in a standard way.

4.1 Embeddings $L^\infty(\Omega) \hookrightarrow \mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ and $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$

Lemma 4.1. *Suppose that $p \in \mathscr{P}^{\log}(\Omega)$ and $\theta \in \mathbb{P}(0,\ell)$. Then the condition (4.1) is sufficient for the embedding*

$$L^\infty(\Omega) \hookrightarrow \mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega). \tag{4.2}$$

If Ω is such that

$$\inf_{x\in\Omega} |\Omega \cap B(x,r)| \geqslant cr^n, \tag{4.3}$$

then the condition (4.1) is also necessary.

Proof. For $f(x) \equiv 1$ we have

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} = \|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)},$$

so that

$$\|f\|_{\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)} \leqslant \sup_{x\in\Omega} \|\omega(x,\cdot)\|_{L^{\theta(\cdot)}(0,\ell)} < \infty$$

in view of (2.5) and (4.1).

To prove the necessity, we note that

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} = \|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)} \geqslant c\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\Omega)} \geqslant Cr^{\frac{n}{p(x)}},$$

where the last inequality is easily checked by the definition of the norm:

$$\int\limits_\Omega \left(\frac{\chi_{B(x,r)}(y)}{\lambda |B(x,r)|^{\frac{1}{p(x)}}} \right)^{p(y)} dy \geqslant 1$$

for some $\lambda > 0$, which is valid for $p \in \mathscr{P}^{\log}(\Omega)$. □

Corollary 4.1. *If Ω satisfies the assumption (4.3), then the condition*

$$\sup_{x\in\Omega} \int\limits_0^\ell \omega(x,r)dr < \infty$$

is necessary for the embedding (4.2).

The following lemma provides us with a condition on $\omega(x,r)$ that guarantees the embedding

$$\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega). \tag{4.4}$$

Lemma 4.2. *Let p be a bounded measurable function with the values in $[1,\infty)$, and let $\theta \in \mathbb{P}(0,\ell)$. If there exists $\delta \in (0,\ell)$ such that*

$$\inf_{x\in\Omega} \|\omega(x,\cdot)\|_{L^{\theta(\cdot)}(\delta,\ell)} > 0, \tag{4.5}$$

then the embedding (4.4) holds.

Proof. We have

$$\begin{aligned} \left\| \frac{\omega(x,r)}{r^{\frac{n}{p(x)}}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \right\|_{L^{\theta(\cdot)}(0,\ell)} &\geqslant \left\| \frac{\omega(x,r)}{r^{\frac{n}{p(x)}}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \right\|_{L^{\theta(\cdot)}(\delta,\ell)} \\ &\geqslant C\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,\delta))} \|\omega(x,r)\|_{L^{\theta(\cdot)}(\delta,\ell)} . \end{aligned}$$

Hence

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,\delta))} \leqslant C \frac{\|f\|_{\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)}}{\|\omega(x,r)\|_{L^{\theta(\cdot)}(\delta,\ell)}} \leqslant C\|f\|_{\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)}$$

for all $x \in \Omega$. For given δ there exists a finite set of balls $B(x,r)$ covering Ω, which yields (4.4). $\square$

It is convenient to introduce the following definition.

Definition 4.2. For given $\delta \in (0,\ell)$ we denote by $\mathscr{W}(\delta,\ell)$ the set of pairs (θ,ω) satisfying the condition (4.5).

Thus, for $p \in \mathscr{P}^{\log}(\Omega)$ the embeddings

$$L^\infty(\Omega) \hookrightarrow \mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \tag{4.6}$$

hold if (4.1) is satisfied (which implies the left embedding) and there exists $\delta \in (0,\ell)$ such that $(\theta,\omega) \in \mathscr{W}(\delta,\ell)$ (which implies the right embedding).

4.2 Maximal operator

The following assertion extends the result obtained in [8] to the generalized Morrey spaces $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$.

Theorem 4.1. *Let $p \in \mathscr{P}^{\log}(\Omega)$ satisfy (2.1), and let*

$$\begin{aligned} &1 < \theta_1^- \leqslant \theta_1(t) \leqslant \theta_1^+ < \infty, \quad 0 < t < \ell, \\ &1 \leqslant \theta_2^- \leqslant \theta_2(t) \leqslant \theta_2^+ < \infty, \quad 0 < t < \ell. \end{aligned} \tag{4.7}$$

Assume that there exists $\delta > 0$ such that

$$\theta_1(t) \leqslant \theta_2(t), \quad t \in (0,\delta), \tag{4.8}$$

and

$$(\theta_1, \omega_1) \in \mathscr{W}(\delta, \ell).$$

If

$$\sup_{x\in\Omega,\, 0<t<\delta} \int\limits_0^t \omega_2(x,\xi)^{\theta_2(\xi)} \left(\int\limits_t^\delta \frac{dr}{[r\omega_1(x,r)]^{[\widetilde{\theta}_1(\xi)]'}} \right)^{\frac{\theta_2(\xi)}{[\widetilde{\theta}_1(\xi)]'}} d\xi < \infty, \tag{4.9}$$

where

$$\widetilde{\theta}_1(\xi) = \inf_{s\in(\xi,\ell)} \theta_1(s),$$

then the operator M *is bounded from* $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ *to* $\mathscr{M}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$*.*

Remark 4.1. Note that the condition (4.9) is imposed only in a neighborhood $(0,\delta)$, where δ can be arbitrarily small. No log conditions on $\theta_1(r)$ and $\theta_2(r)$ or even log decay type conditions are imposed at the point $r = 0$.

Corollary 4.2. *Under the assumptions of Theorem* 4.1*, the following embedding holds*:

$$\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega) \hookrightarrow \mathscr{M}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega).$$

Corollary 4.3. *If* $\omega_1(x,r) = \omega_2(x,r) = r^{\beta(x)}$*,* $\theta_1(r) = \theta_2(r) =: \theta(r)$*, and*

$$\inf_{x\in\Omega} \beta(x) > -\frac{1}{\inf\limits_{t\in(\delta,\ell)} \theta(t)}, \tag{4.10}$$

then (4.9) *takes the form*

$$\sup_{x\in\Omega,\, 0<t<\delta} \int\limits_0^t \left(\frac{\xi}{t}\right)^{\beta(x)\theta(\xi)} \frac{d\xi}{t^{\frac{\theta(\xi)}{\widetilde{\theta}(\xi)}}} < \infty \tag{4.11}$$

(*there is no log-conditions on* $\theta(\xi)$ *and* $\beta(x)$)*. In particular, if*

$$\theta(t) \equiv \theta = \text{const}\,, \quad 1 < \theta < \infty,$$

then the conditions

$$p \in \mathscr{P}^{\log}(\Omega), \quad \inf_{x\in\Omega} \beta(x) > -\frac{1}{\theta}$$

are sufficient for the boundedness of the maximal operator M *in the space* $\mathscr{M}^{p(\cdot),\theta,r^{\alpha(x)}}(\Omega)$*.*

In the following assertion, we show that for the power functions

$$\omega_1(x,r) = r^{\beta(r)}, \quad \omega_2(x,r) = r^{\gamma(r)}, \tag{4.12}$$

where $\beta, \gamma \in \mathscr{M}_0(0,\ell)$, the conditions on $\theta_1(\cdot)$ and $\theta_2(\cdot)$ can be simplified (cf. (4.14)–(4.13) and (4.11)) provided that $\theta_1(\cdot)$ and $\theta_2(\cdot)$ satisfy a log decay type condition as $r \to 0$.

Theorem 4.2. *Suppose that* $p \in \mathscr{P}^{\log}(\Omega)$ *satisfies* (2.1), $\theta_1, \theta_2 \in \mathscr{P}_0(0,\ell)$, *and* ω_1, ω_2 *have the form* (4.12). *Then the maximal operator* M *is bounded from* $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ *to* $\mathscr{M}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$ *provided that*

$$\gamma(0) > -\frac{1}{\theta_1(0)}, \quad -\frac{1}{\theta_1(0)} < \beta(0) \leqslant \gamma(0) \tag{4.13}$$

and

$$\frac{1}{\theta_2(0)} = \frac{1}{\theta_1(0)} + \beta(0) - \gamma(0). \tag{4.14}$$

Theorem 4.3. *Let* $p \in \mathscr{P}^{\log}(\Omega)$ *satisfy* (2.1), *and let* $\theta \in \mathbb{P}(0,\ell)$. *Assume that there exists* $\delta > 0$ *such that* $(\theta, \omega_1) \in \mathscr{W}(\delta,\ell)$ *and*

$$\sup_{x\in\Omega}\left\{ \int\limits_{(0,\delta)\setminus E_\infty(\theta)} \left[\omega_2(x,t)\int\limits_t^\delta \frac{dr}{r\omega_1(x,r)}\right]^{\theta(t)} dt + \sup_{t\in(0,\delta)\cap E_\infty(\theta)} \omega_2(x,t)\int\limits_t^\delta \frac{dr}{r\omega_1(x,r)}\right\} < \infty. \tag{4.15}$$

Then the operator M *is bounded from* $\mathscr{M}^{p(\cdot),\infty,\omega_1(\cdot)}(\Omega)$ *into* $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 4.3 recovers Theorem 3.4 with constant θ.

4.3 Potential type operators

We begin with a Spanne type result (with respect to $q(\cdot)$) when $\theta_1(\cdot)$ is bounded. This result was proved in [22] (cf. also [23]) for constant $p(x)$, $q(x)$, $\alpha(x)$, and $\theta(t)$.

Theorem 4.4. *Suppose that* $p, \alpha \in \mathscr{P}^{\log}(\Omega)$ *satisfy* (2.1), (2.3) *and*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}.$$

Let θ_1 *and* θ_2 *satisfy* (4.7). *If there exists* $\delta \in (0,\ell)$ *such that* $(\theta_1, \omega_1) \in \mathscr{W}(\delta,\ell)$, (4.8) *is satisfied, and*

$$\sup_{x\in\Omega,\,0<t<\delta} \int\limits_0^t \omega_2(x,\xi)^{\theta_2(\xi)} \left(\int\limits_t^\delta \left(\frac{r^{\alpha(x)-1}}{\omega_1(x,r)}\right)^{[\widetilde{\theta}_1(\xi)]'} dr\right)^{\frac{\theta_2(\xi)}{[\widetilde{\theta}_1(\xi)]'}} d\xi < \infty \tag{4.16}$$

on $(0,\delta)$, *then* $M^{\alpha(\cdot)}$ *and* $I^{\alpha(\cdot)}$ *are bounded from* $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ *to* $\mathscr{M}^{q(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$.

The following assertion is also a Spanne type result (with respect to $q(\cdot)$) in the case $\theta_1(r) \equiv \infty$. It recovers Theorem 3.5 with $\theta(t) \equiv \infty$.

Theorem 4.5. *Suppose that* $p, \alpha \in \mathscr{P}^{\log}(\Omega)$ *satisfy the conditions* (2.1) *and* (2.3),

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n},$$

and $\theta \in \mathbb{P}(0,\ell)$. *Assume that there exists* $\delta > 0$ *such that* $(\theta, \omega_1) \in \mathscr{W}(\delta,\ell)$ *and*

$$\sup_{x\in\Omega}\left\{\int\limits_{(0,\delta)\setminus E_\infty(\theta)}\left[\omega_2(x,t)\int\limits_t^\delta\frac{r^{\alpha(x)-1}dr}{\omega_1(x,r)}\right]^{\theta(t)}dt+\sup_{t\in(0,\delta)\cap E_\infty(\theta)}\omega_2(x,t)\int\limits_t^\delta\frac{r^{\alpha(x)-1}dr}{\omega_1(x,r)}\right\}<\infty. \tag{4.17}$$

Then $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathscr{M}^{p(\cdot),\infty,\omega_1(\cdot)}(\Omega)$ into $\mathscr{M}^{q(\cdot),\theta(\cdot),\omega_2(\cdot)}(\Omega)$.

Like Theorem 4.2, the following theorem asserts that for $\omega_1(x,r)$ and $\omega_2(x,r)$ of the form (4.12) the boundedness conditions can be written in a very simple form provided that $\theta_1,\theta_2\in\mathscr{P}_0(0,\ell)$. We set

$$\alpha(x)=\alpha=\text{const}$$

(for variable $\alpha(x)$ we must deal with spaces for which $\theta(t)$ may depend also on x; this case is not considered in this paper).

Theorem 4.6. *Suppose that $p\in\mathscr{P}^{\log}(\Omega)$ satisfies (2.1), $\theta_1,\theta_2\in\mathscr{P}_0(0,\ell)$, and ω_1, ω_2 have the form (4.12). Then the potential type operator $I^\alpha, 0<\alpha<n$, is bounded from $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathscr{M}^{q(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$ with*

$$\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}$$

provided that

$$\gamma(0)>-\frac{1}{\theta_1(0)},\quad \alpha-\frac{1}{\theta_1(0)}<\beta(0)\leqslant\alpha+\gamma(0) \tag{4.18}$$

and

$$\frac{1}{\theta_2(0)}=\frac{1}{\theta_1(0)}+\beta(0)-\gamma(0)-\alpha. \tag{4.19}$$

The following assertion is an Adams type result (with respect to $q(\cdot)$) for bounded $\theta_1(\cdot)$. It was proved in [28] for constant $p(x)$, $q(x)$, $\alpha(x)$, and $\theta(t)$.

Theorem 4.7. *Suppose that $p,\alpha\in\mathscr{P}^{\log}(\Omega)$ satisfy (2.1) and (2.3). Assume that $q(x)>p(x)$ on Ω, θ_1, θ_2 satisfy (4.7), and $\omega(x,r)$ satisfies the condition*

$$\left\|\frac{t^{\alpha(x)-1}}{\omega_1(x,t)}\right\|_{L^{\theta_1'(\cdot)}(r,\ell)}\leqslant Cr^{-\frac{\alpha(x)p(x)}{q(x)-p(x)}}, \tag{4.20}$$

where C is independent of $x\in\Omega$ and $r\in(0,\ell]$. If there exists $\delta\in(0,\ell)$ such that $(\theta_1,\omega_1)\in\mathscr{W}(\delta,\ell)$ and the conditions (4.8) and (4.9) hold on $(0,\delta)$, then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathscr{M}^{q(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$.

4.4 Singular operators

Consider the maximal singular operator

$$T_*f(x)=\sup_{\varepsilon>0}|T_\varepsilon f(x)|,$$

where

$$T_\varepsilon f(x) = \int\limits_{|x-y|\geqslant\varepsilon} K(x,y) f(y)dy.$$

The following assertion was proved in [22] (cf. also [23]) for constant $p(x)$ and $\theta(t)$.

Theorem 4.8. *Suppose that $p \in \mathscr{P}^{\log}(\Omega)$ satisfies (2.1) and $\theta_1, \theta_2 \in \mathbb{P}(0,\ell)$. Assume that there exists $\delta > 0$ such that $(\theta_1, \omega_1) \in \mathscr{W}(\delta, \ell)$ and*

$$\sup_{x\in\Omega,\, 0<t<\delta} \int\limits_0^t \omega_2(x,\xi)^{\theta_2(\xi)} \left(\int\limits_t^\delta \frac{dr}{[r\omega_1(x,r)]^{\widetilde{\theta}_1'(\xi)}} dr \right)^{\frac{\theta_2(\xi)}{\theta_1'(\xi)}} d\xi < \infty.$$

Then T and T_* are bounded from $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathscr{M}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$.

The following assertion recovers the result of Theorem 3.4 in the case $\theta_2(t) = \infty$.

Theorem 4.9. *Under the assumptions of Theorem 4.3, the operators T and T_* are bounded from $\mathscr{M}^{p(\cdot),\infty,\omega_1(\cdot)}(\Omega)$ into $\mathscr{M}^{p(\cdot),\theta(\cdot),\omega_2}(\Omega)$.*

Theorem 4.10. *Suppose that $p \in \mathscr{P}^{\log}(\Omega)$ satisfies (2.1), $\theta_1, \theta_2 \in \mathscr{P}_0(0,\ell)$, and ω_1, ω_2 have the form (4.12). Then T and T_* are bounded from $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathscr{M}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$ provided that the conditions (4.13) and (4.14) are satisfied.*

5 Proofs of the Main Results

To shorten the notation, we write

$$X_i^p := \mathscr{M}^{p(\cdot),\theta_i(\cdot),\omega_i(\cdot)}, \quad X_i^q := \mathscr{M}^{q(\cdot),\theta_i(\cdot),\omega_i(\cdot)}, \quad i = 1, 2.$$

5.1 Maximal operator

In this subsection, we use the estimate

$$\|Mf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant Ct^{\frac{n}{p(x)}} \int\limits_t^\ell r^{-\frac{n}{p(x)}-1} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} dr, \quad 0 < t < \delta, \tag{5.1}$$

for $f \in L^{p(\cdot)}(\Omega)$, where C is independent of $f, x \in \Omega$ and t (but depends on δ and increases as $\delta \to \ell$). By [21, Theorem 4.1], this estimate is valid if p belongs to $\mathscr{P}^{\log}(\Omega)$ and satisfies the condition (2.1).

Proof of Theorem 4.1. Let $f \in X_1^p(\Omega)$. We have

$$\|Mf\|_{X_1^p} \leqslant \sup_{x\in\Omega} \left\| \frac{\omega_2(x,t)}{t^{\frac{n}{p(x)}}} \|Mf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \right\|_{L^{\theta_2(\cdot)}(0,\delta)}$$

$$+\sup_{x\in\Omega}\left\|\frac{\omega_2(x,t)}{t^{\frac{n}{p(x)}}}\|Mf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\right\|_{L^{\theta_2(\cdot)}(\delta,\ell)} =: I_1 + I_2. \tag{5.2}$$

An estimate for the term I_2 directly follows from the embeddings (4.6). Indeed, by the condition (4.1) for (θ_2, ω_2),

$$I_2 \leqslant C\|Mf\|_{L^{p(\cdot)}(\Omega)}\|\omega_2(x,\cdot)\|_{L^{\theta_2(\cdot)}(\delta,\ell)} \leqslant C\|Mf\|_{L^{p(\cdot)}(\Omega)}.$$

Then

$$I_2 \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)}$$

since the maximal operator is bounded (cf. [35]) in variable exponent Lebesgue spaces if $p(x)$ satisfies the log-condition. Consequently,

$$I_2 \leqslant C\|f\|_{X_1^p}$$

by the embedding in (4.4).

To estimate the term I_1, we use (5.1) and find

$$I_1 \leqslant C\sup_{x\in\Omega}\left\|\omega_2(x,t)\int_t^\ell r^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}dr\right\|_{L^{\theta_2(\cdot)}(0,\delta)}. \tag{5.3}$$

Now, we need to estimate the one-dimensional Hardy operator in the variable exponent Lebesgue spaces. Splitting the integral with respect to r into two integrals over $(0,\delta)$ and (δ,ℓ), we estimate the integral over (δ,ℓ) as above and the integral over $(0,\delta)$ by using Lemma 2.2. This lemma can be applied in view of the embedding (4.4). Then we find

$$I_1 \leqslant C\|f\|_{X_1^p} + C\sup_{x\in\Omega}\left\|\frac{\omega_1(x,t)}{t^{\frac{n}{p(x)}}}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\right\|_{L^{\theta_1(\cdot)}(0,\delta)} \leqslant C\|f\|_{X_1^p}.$$

The theorem is proved. □

Proof of Corollary 4.2. This assertion is an immediate consequence of the inequality $f(x) \leqslant Mf(x)$. □

Proof of Corollary 4.3. It suffices to note that the inner integral in (4.9) is explicitly calculated in the case $\omega_1(x,r) = r^{\alpha(x)}$. □

Proof of Theorem 4.2 is the same as that of Theorem 4.1 with the only difference that the Hardy operator in (5.3) is now estimated by using the criterion in Remark 2.1. □

Proof of Theorem 4.3. In the expression for $\|Mf\|_{\mathscr{M}^{p(\cdot),\theta(\cdot),\omega_2(\cdot)}}$, we split the integral with respect to t in the same way as in (5.2). Hence it suffices to consider only the integral over $(0,\delta)$. We have

$$\sup_{x\in\Omega}\left\|\frac{\omega_2(x,t)}{t^{\frac{n}{p(x)}}}\|Mf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\right\|_{L^{\theta(\cdot)}(0,\delta)} \leqslant C\sup_{x\in\Omega}\left\|\omega_2(x,t)\int_t^\ell r^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}dr\right\|_{L^{\theta(\cdot)}(0,\delta)}$$

$$\leqslant C\|f\|_{\mathscr{M}^{p(\cdot),\infty,\omega_1(\cdot)}}\left(1+\sup_{x\in\Omega}\left\|\omega_2(x,t)\int\limits_t^\delta \frac{dr}{r^{\frac{n}{p(x)}+1}\omega_1(x,r)}dr\right\|_{L^{\theta(\cdot)}(0,\delta)}\right). \tag{5.4}$$

Indeed, the first inequality follows from the estimate (5.1). We split the integral with respect to r in the second line of formula (5.4) into two integrals over (δ,ℓ) and $(0,\delta)$. The first integral is easily estimated in the same way as in (5.3). Thus, the required assertion will be proved if we show that the second integral is finite. But this term is the norm in the variable exponent Lebesgue space $L^{\theta(\cdot)}$ and, as is well known, it is finite if and only if the corresponding modular is finite, which is the condition (4.15). □

5.2 Potential type operators

To prove Theorem 4.4 for potential type operators, we use the following estimate similar to (5.1):

$$\|I^{\alpha(\cdot)}f\|_{L_{q(\cdot)}(\widetilde{B}(x,t))}\leqslant Ct^{\frac{n}{q(x)}}\int\limits_t^\ell r^{-\frac{n}{q(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}dr, \quad 0<t<\delta, \tag{5.5}$$

where $0<\delta<\ell$ and

$$\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n}.$$

This estimate is valid for $f\in L^{p(\cdot)}(\Omega)$ if p and α bleong to $\mathscr{P}^{\log}(\Omega)$ and satisfy the conditions (2.1) and (2.3) (cf. [21, Theorem 5.1]), where the constant C is independent of x and t, but depends on δ. If $p(x)$, $q(x)$, $\alpha(x)$, $\theta(t)$ are constant, this estimate was proved in [22] (cf. also [23]).

To prove the Adams type result in Theorem 4.7, we use the pointwise estimate

$$|I^{\alpha(\cdot)}f(x)|\leqslant Ct^{\alpha(x)}Mf(x)+C\int\limits_t^l r^{\alpha(x)-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}dr, \quad 0<t\leqslant\delta, \tag{5.6}$$

where $0<\delta<\ell$ and $p,\alpha\in\mathscr{P}^{\log}(\Omega)$ satisfy the conditions (2.1) and (2.3) (cf. [21, Theorem 5.4]), the constant C is independent of x and t, but depends on δ. In the case where $p(x)$, $q(x)$, $\alpha(x)$, $\theta(t)$ are constant, this estimate was proved in [28].

Proof of Theorem 4.4. By the well known pointwise estimate

$$M^{\alpha(\cdot)}f(x)\leqslant C(I^{\alpha(\cdot)}|f|)(x),$$

it suffices to consider only the case of the operator $I^{\alpha(\cdot)}$. Let $f\in X_1^p(\Omega)$. As usual, we split the integral with respect to t in the expression for the norm as follows:

$$\|I^{\alpha(\cdot)}f\|_{X_2^q}\leqslant\sup_{x\in\Omega}\left(\left\|\frac{\omega_2(x,t)}{t^{\frac{n}{q(x)}}}\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\widetilde{B}(x,t))}\right\|_{L^{\theta_2(\cdot)}(0,\delta)}\right.$$
$$\left.+\left\|\frac{\omega_2(x,t)}{t^{\frac{n}{q(x)}}}\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\widetilde{B}(x,t))}\right\|_{L^{\theta_2(\cdot)}(\delta,\ell)}\right)=:\sup_{x\in\Omega}[J_1(x)+J_2(x)]. \tag{5.7}$$

To estimate $J_2(x)$, we note that

$$\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\widetilde{B}(x,t))} \leqslant \|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\Omega)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)}$$

in view of Theorem 2.1. Using the embedding (4.4), we find

$$\sup_{x\in\Omega} J_2(x) \leqslant C\|f\|_{X_1^p}. \tag{5.8}$$

To estimate $J_1(x)$, we use (5.5). We have

$$\sup_{x\in\Omega} J_1(x) \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)} + C\sup_{x\in\Omega}\left\|\omega_2(x,t)\int\limits_t^\delta r^{-\frac{n}{q(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}dr\right\|_{L^{\theta_2(\cdot)}(0,\delta)}. \tag{5.9}$$

It remains to use the embedding (4.4) for the first term and apply Lemma 2.2 for the last term, which yields

$$\sup_{x\in\Omega} J_1(x) \leqslant C\|f\|_{X_1^p} + C\sup_{x\in\Omega}\left\|\frac{\omega_1(x,t)}{t^{\frac{n}{p(x)}}}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\right\|_{L^{\theta_1(\cdot)}(0,\delta)} \leqslant C\|f\|_{X_1^p}.$$

The theorem is proved. □

Proof of Theorem 4.5. Let $f \in \mathscr{M}^{p(\cdot),\infty,\omega_1}(\Omega)$. We argue in the same way as in (5.7). Hence it suffices to estimate the term $J_1(x)$. Using (5.9) and (4.2), we find

$$\sup_{x\in\Omega} J_1(x) \leqslant C\|f\|_{\mathscr{M}^{p(\cdot),\infty,\omega_1(\cdot)}(\Omega)}\left(1+\sup_{x\in\Omega}\left\|\omega_2(x,t)\int\limits_0^\delta \frac{r^{\alpha(x)-1}}{\omega_1(x,r)}dr\right\|_{L^{\theta(\cdot)}(0,\delta)}\right).$$

Thus, we obtain the required result if we show that the $L^{\theta(\cdot)}(0,\delta)$-norm in the last expression is finite, which is equivalent to the condition (4.17). □

Proof of Theorem 4.6 is the same as that of Theorem 4.4, but, in this case, the Hardy operator in (5.9) is esitmated by means of the criterion in Remark 2.1. □

Proof of Theorem 4.7. It suffices to consider only the operator $I^{\alpha(\cdot)}$. The intial steps of the proof are exactly the same as in the case (5.7), (5.8) in the proof of Theorem 4.4. Hence it suffices to estimate $J_1(x)$. For this purpose, we use the estimate (5.6) where we first apply the Hölder inequality with the variable exponent $\theta_1(\cdot)$ to obtain

$$|I^{\alpha(\cdot)}f(x)| \leqslant Ct^{\alpha(x)}\,Mf(x) + C\left\|\frac{r^{\alpha(x)-1}}{\omega(x,r)}\right\|_{L^{\theta_1'(\cdot)}(t,\ell)}\left\|\frac{\omega(x,r)}{t^{\frac{n}{p(x)}}}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\right\|_{L^{\theta_1(\cdot)}(t,\ell)}.$$

By (4.20), we have

$$|I^{\alpha(\cdot)}f(x)| \leqslant Ct^{\alpha(x)}\,Mf(x) + Ct^{-\frac{\alpha(x)p(x)}{q(x)-p(x)}}\,\|f\|_{X_1^p}.$$

Then we choose

$$t = \left(\frac{\|f\|_{X_1^p}}{Mf(x)}\right)^{\frac{q(x)-p(x)}{\alpha(x)q(x)}},$$

where f is not identical to 0. Hence for every $x \in \Omega$

$$|I^{\alpha(\cdot)} f(x)| \leqslant C(Mf(x))^{\frac{p(x)}{q(x)}} \|f\|_{X_1^p}^{1-\frac{p(x)}{q(x)}}.$$

The required assertion follows from the boundedness of the maximal operator M in the space $\mathscr{M}^{p(\cdot),\theta_1(\cdot),\omega}(\Omega)$ because of Theorem 4.1 and the condition (4.9).

5.3 Singular operators

The estimate

$$\|Tf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant Ct^{\frac{n}{p(x)}} \int_t^\ell r^{-\frac{n}{p(x)}-1} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} dr, \quad 0 < t \leqslant \delta, \tag{5.10}$$

with $0 < \delta < \ell$ and $f \in L^{p(\cdot)}(\Omega)$, where p, α belong to $\mathscr{P}^{\log}(\Omega)$ and satisfy (2.1), (2.3), is the same as the estimate (5.1) for the maximal operator (cf. [21, Theorem 6.1]). Therefore, the proof of Theorems 4.8 and 4.9 for the operator T is the same as that of Theorems 4.1 and 4.3 with the only difference that in order to estimate the term I_2, we should use, instead of the boundedness of the maximal operator in the space $L^{p(\cdot)}(\Omega)$, the boundedness of the operator T in such spaces established in [36]. Theorem 4.10 is obtained in the known way by using the results in [39] for Hardy inequalities.

The boundedness of the operator T_* follows from the known pointwise estimate [40, p. 34]

$$T_* f(x) \leqslant c[M(Tf)(x) + Mf(x)]$$

and the corresponding theorems on the maximal operator.

Acknowledgments

The research of V. S. Guliyev and S. G. Samko was partially supported by the scientific and technological research council of Turkey (grant No. 1001-2010-2-53666 (TUBITAK Project)). The research of V. S. Guliyev was supported by 2010-Ahi Evran University Scientific Research Projects (BAP).

References

1. M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton Univ. Press, Princeton, NJ (1983).

2. A. Kufner, O. John, and S. Fučik, *Function Spaces*, Noordhoff Intern. Publish. (1977).

3. C. B. Morrey, "On the solutions of quasi-linear elliptic partial differential equations," *Am. Math. Soc.* **43**, 126–166 (1938).

4. L. Diening, P. Hasto, and A. Nekvinda, "Open problems in variable exponent Lebesgue and Sobolev spaces," In: *Function Spaces, Differential Operators and Nonlinear Analysis*, pp. 38–58. Math. Inst. Acad. Sci. Czech Republick, Praha (2005).

5. V. Kokilashvili, “On a progress in the theory of integral operators in weighted Banach function spaces,” In: *Function Spaces, Differential Operators and Nonlinear Analysis*, pp. 152–175. Math. Inst. Acad. Sci. Czech Republick, Praha (2005).

6. V. Kokilashvili and S. Samko, “Weighted boundedness of the maximal, singular and potential operators in variable exponent spaces,” In: *Analytic Methods of Analysis and Differential Equations*, pp. 139–164. Cambridge Sci. Publ. (2008).

7. S. Samko, “On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators,” *Integral Transforms Spec. Funct* **16**, No. 5-6, 461–482 (2005).

8. A. Almeida, J. J. Hasanov, and S. G. Samko, “Maximal and potential operators in variable exponent Morrey spaces,” *Georgian Math. J.* **15**, No. 2, 1–15 (2008).

9. Y. Mizuta and T. Shimomura, “Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent,” *J. Math. Soc. Japan* **60**, 583–602 (2008).

10. V. Kokilashvili and A. Meskhi, “Boundedness of maximal and singular operators in Morrey spaces with variable exponent,” *Armen. J. Math.* **1**, No. 1, 18–28 (2008).

11. P. Hästö, “Local-to-global results in variable exponent spaces,” *Math. Res. Lett.* **16**. No. 2, 263–278 (2009).

12. D. R. Adams, “A note on Riesz potentials,” *Duke Math.* **42** 765–778 (1975).

13. J. Peetre, “On the theory of $\mathscr{L}_{p,\lambda}$ spaces,” *J. Funct. Anal.* **4**, 71–87 (1969).

14. F. Chiarenza and M. Frasca, “Morrey spaces and Hardy-Littlewood maximal function,” *Rend. Math.* **7**, 273–279 (1987).

15. V. Burenkov, A. Gogatishvili, V. S. Guliyev, and R. Mustafayev, “Boundedness of the fractional maximal operator in local Morrey-type spaces,” *Compl. Variabl. Ellipt. Equat.* **55**, No. 8–10, 739–758 (2010).

16. V. I. Burenkov and V. S. Guliyev, “Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces,” *Potent. Anal.* **31**, No. 2, 1–39 (2009).

17. V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, “Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces” [in Russian], *Dokl. Ross. Akad. Nauk* **412** No 5, 585–589 (2007); English transl.: *Dokl. Math.* **75**, NO. 1, 103–107 (2007).

18. V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, “Necessary and sufficient conditions for boundedness of the fractional maximal operator in the local Morrey-type spaces,” *J. Comp. Appl. Math.* **208**, No. 1, 280–301 (2007).

19. V. I. Burenkov and V. S. Guliyev, “Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces,” *Potent. Anal.* **31**, No. 2, 1–39 (2009).

20. V. I. Burenkov, V. S. Guliyev, T. V. Tararykova, and A. Serbetci, “Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces,” *Dokl. Ross. Akad. Nauk* **422**, no. 1, 11–14 (2008); English transl.: *Dokl. Math.* **78**, No. 2, 651–654 (2008).

21. V. S. Guliyev, J. J. Hasanov, and S. G. Samko, “Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces,” *Mat. Scand.* **108** [To appear]

22. V. S. Guliyev, *Integral Operators on Function Spaces on the Homogeneous Groups and on Domains in* $\mathbb{R}^n$ [in Russian], Diss. Steklov Mat. Inst. Moscow (1994).

23. V. S. Guliyev, *Function Spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications* [in Russian], Baku (1999).

24. Eridani, H. Gunawan, and E. Nakai, "On generalized fractional integral operators," *Sci. Math. Jpn.* **60**, No.3, 539–550 (2004).

25. K. Kurata, S. Nishigaki, and S. Sugano, "Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators," *Proc. Am. Math. Soc.* **128**, No. 4, 1125–1134 (2000).

26. T. Mizuhara, "Boundedness of some classical operators on generalized Morrey spaces," In: *Harmonic Analysis*, Proc. Conf., Sendai/Jap. 1990, ICM-90 Satell. Conf. Proc., pp. 183–189 (1991).

27. E. Nakai, "Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces," *Math. Nachr.* **166**, 95–103 (1994).

28. V. S. Guliyev, "Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces," *J. Ineq. Appl.* ID 503948 (2009).

29. E. Nakai, "The Campanato, Morrey and Holder spaces on spaces of homogeneous type," *Studia Math.* **176**, 1–19 (2006).

30. L. Softova, "Singular integrals and commutators in generalized Morrey spaces," *Acta Math. Sin., Engl. Ser. 22*, No. 3, 757–766 (2006).

31. D. R. Adams, "Lectures on L^p-Potential Theory," UmeåUniversity Reports, No. 2, 1–74 (1981).

32. I. I. Sharapudinov, "Topology of the space $\mathscr{L}^{p(t)}([0,t])$" [in Russian], *Mat. Zametki* **26** 613–632 (1979); English transl.: *Math. Notes* **26**, 796–806 (1980).

33. O. Kovácǐk and J. Rákosnǐk, "On spaces $L^{p(x)}$ and $W^{k,p(x)}$," *Czech. Math. J.* **41**, No. 4, 592–618 (1991).

34. S. Samko, "Convolution and potential type operators in the space $L_{p(x)}$," *Integral Transforms Spec. Funct.* **7**, No. 3-4, 261–284 (1998).

35. L. Diening, "Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$," *Math. Inequal. Appl.* **7**, No. 2, 245–253 (2004).

36. L. Diening and M. Rüźićka, "Calderón–Zygmund operators on generalized Lebesgue spaces $L_{p(\cdot)}$ and problems related to fluid dynamics," *J. Reine Angew. Math.* **563**, 197–220 (2003).

37. S. Samko, "Convolution type operators in $L_{p(x)}$," *Integral Transforms Spec. Funct.* **7**, No. 1-2, 123–144 (1998).

38. D. E. Edmunds, V. M. Kokilashvilii, and A. Meskhi, "On the boundedness and compactness of weighted Hardy operators in spaces $L_{p(x)}$," *Georg. Math. J.* **12**, No. 1, 27–44 (2005).

39. L. Diening and S. Samko, "Hardy inequality in variable exponent Lebesgue spaces," *Fract. Calc. Appl. Anal.* **10**, No. 1, 1–18 (2007).

40. E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ (1993).

Submitted on September 2, 2010