Trends in Mathematics

Michael Ruzhansky
Yeol Je Cho
Praveen Agarwal
Iván Area
Editors

Advances in Real and Complex Analysis with Applications

Birkhäuser

Birkhäuser

Trends in Mathematics

Trends in Mathematics is a series devoted to the publication of volumes arising from conferences and lecture series focusing on a particular topic from any area of mathematics. Its aim is to make current developments available to the community as rapidly as possible without compromise to quality and to archive these for reference.

Proposals for volumes can be submitted using the Online Book Project Submission Form at our website www.birkhauser-science.com.

Material submitted for publication must be screened and prepared as follows:

All contributions should undergo a reviewing process similar to that carried out by journals and be checked for correct use of language which, as a rule, is English. Articles without proofs, or which do not contain any significantly new results, should be rejected. High quality survey papers, however, are welcome.

We expect the organizers to deliver manuscripts in a form that is essentially ready for direct reproduction. Any version of TeX is acceptable, but the entire collection of files must be in one particular dialect of TeX and unified according to simple instructions available from Birkhäuser.

Furthermore, in order to guarantee the timely appearance of the proceedings it is essential that the final version of the entire material be submitted no later than one year after the conference.

More information about this series at http://www.springer.com/series/4961

Michael Ruzhansky · Yeol Je Cho
Praveen Agarwal · Iván Area
Editors

Advances in Real and Complex Analysis with Applications

Birkhäuser

Editors
Michael Ruzhansky
Department of Mathematics
Imperial College London
London
UK

Praveen Agarwal
Department of Mathematics
Anand International College of Engineering
Jaipur, Rajasthan
India

Yeol Je Cho
Department of Mathematics Education and the RINS
Gyeonsang National University
Jinju
Korea (Republic of)

Iván Area
Departamento de Matemática Aplicada II
E.E. Aeronáutica e do Espazo, Universidade de Vigo
Ourense
Spain

ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-981-10-4336-9 ISBN 978-981-10-4337-6 (eBook)
DOI 10.1007/978-981-10-4337-6

Library of Congress Control Number: 2017945222

Mathematics Subject Classification (2010): 11A15, 30C45, 42A38, 44A10, 46G20, 46E50

Printed on acid-free paper

This book is published under the trade name Birkhäuser
The registered company is Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore
(www.birkhauser-science.com)

Contents

Certain Image Formulae and Fractional Kinetic Equations Involving Extended Hypergeometric Functions 1
Krunal B. Kachhia, Praveen Agarwal and Jyotindra C. Prajapati

The Compact Approximation Property for Weighted Spaces of Holomorphic Mappings 33
Manjul Gupta and Deepika Baweja

Bloch Mappings on Bounded Symmetric Domains 49
Tatsuhiro Honda

Certain Class of Meromorphically Multivalent Functions Defined by a Differential Operator 71
Ghazi S. Khammash and Praveen Agarwal

Bivariate Symmetric Discrete Orthogonal Polynomials 87
Y. Guemo Tefo, Iván Area and M. Foupouagnigni

New and Extended Applications of the Natural and Sumudu Transforms: Fractional Diffusion and Stokes Fluid Flow Realms 107
Fethi Bin Muhammed Belgacem, Rathinavel Silambarasan, Hammouch Zakia and Toufik Mekkaoui

On Uncertain-Fractional Modeling: The Future Way of Modeling Real-World Problems 121
Abdon Atangana and Ilknur Koca

Quadratic Reciprocity and Some "Non-differentiable" Functions 145
Kalyan Chakraborty and Azizul Hoque

Survey on Metric Fixed Point Theory and Applications 183
Yeol Je Cho

Sums of Finite Products of Euler Functions 243
Taekyun Kim, Dae San Kim, Gwan Woo Jang and Jongkyum Kwon

On a New Extension of Caputo Fractional Derivative Operator 261
İ.O. Kıymaz, P. Agarwal, S. Jain and A. Çetinkaya

An Extension of the Shannon Wavelets for Numerical Solution of Integro-Differential Equations 277
Maryam Attary

Inverse Source Problem for Multi-term Fractional Mixed Type Equation 289
E.T. Karimov, S. Kerbal and N. Al-Salti

About the Editors

Michael Ruzhansky is a Professor at the Department of Mathematics, Imperial College London. He has published over 100 research articles in several leading international journals. He has also published five books and memoirs and nine edited volumes. He has researched topics related to pseudo-differential operators, harmonic analysis, and partial differential equations. More recently, he has worked on boundary value problems and their applications. He has been on the editorial board of many respected international journals and served as the President of the International Society of Analysis, Applications, and Computations (ISAAC) in the period 2009–2013.

Yeol Je Cho is a Professor at the Department of Mathematics Education, Gyeongsang National University, Korea, and Fellow of the Korean Academy of Science and Technology. He has published over 350 research papers in respected international journals, mainly in the field of fixed point theory, variational inequality problems, equilibrium problems, optimization problems, stability of functional equations, and inequality theory. He has been on the editorial boards of several leading journals, such as *Fixed Point Theory and Applications*, *Journal of Nonlinear Science and Applications*, and 15 more international journal of mathematics and applications. He has also published a number of remarkable books on fixed point theory, stability of functional equations, numerical analysis, and degree theory.

Praveen Agarwal is a Professor at the Department of Mathematics, Anand International College of Engineering, Jaipur, India. He has published over 150 articles related to special functions and fractional calculus in leading mathematics journals, such as *Applied Mathematics and Computation*, the *Journal of Nonlinear Science and Applications*, and the *Journal of Inequalities and Applications*. Recently, he has focused on partial differential equations and fractional differential equations. He has been on the editorial boards of several journals, including SpringerPlus and the Maejo International Journal of Science and Technology. He has been involved in a number of conferences and has also won numerous international research grants such as Research in Group by International Centre for Mathematical Sciences (ICMS) and World Academy of Sciences (TWAS) Visiting Scientist fellowship and many more.

Iván Area is a Professor at the Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo Universidade de Vigo, Spain. He has published over 100 articles related to orthogonal polynomials and special functions in leading international journals. His recent research has also focused on fractional analysis and bioinformatics. He has been elected General Secretary of the International Center for Pure and Applied Mathematics (CIMPA), a nonprofit organization whose aim is to promote mathematics in developing countries.

Certain Image Formulae and Fractional Kinetic Equations Involving Extended Hypergeometric Functions

Krunal B. Kachhia, Praveen Agarwal and Jyotindra C. Prajapati

Abstract In this chapter, our aim is to establish certain new image formulae of generalized hypergeometric functions by using the operators of fractional calculus. Some new image formulae are obtained by applying specific integral transforms on resulting image formulae. We also acquired generalization of fractional kinetic equations involving extended hypergeometric functions.

Keywords Generalized Gauss hypergeometric function · Fractional derivative operators · Integral transforms · Fractional kinetic equation · Mittag–Leffler function

2010 AMS Math. Subject Classification 26A33 · 33B15 · 33C15 · 33C20 · 33C99 · 44A10 · 33E20

1 Introduction

Fractional calculus is one of the generalizations of classical calculus, and it has been used successfully in various fields of science and technology. Many applications of fractional calculus can be found in other diverse fields, etc. (See [15, 17, 19–22, 35]).

Integral transforms and fractional integral formulae involving well-known special functions are interesting in themselves and play significant roles in their diverse

K.B. Kachhia
Faculty of Applied Sciences, Department of Mathematical Sciences, Charotar University of Science and Technology (CHARUSAT), Changa 388421, Gujarat, India
e-mail: krunalmaths@hotmail.com

P. Agarwal (✉)
Department of Mathematics, Anand International College of Engineering, Jaipur 303012, Rajasthan, India
e-mail: goyal.praveen2011@gmail.com

J.C. Prajapati
Department of Mathematics, Marwadi University, Rajkot-Morbi Road, Rajkot 360003, Gujarat, India
e-mail: jyotindra.prajapati@marwadieducation.edu.in

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_1

applications. Certain new integral transforms and fractional integral formulae for the generalized hypergeometric type function which has recently been introduced by various authors [29–31].

Fractional kinetic equations gained remarkable significance due to their applications in astrophysics and mathematical physics. The extension and generalization of fractional kinetic equations involving many fractional operators were found [5, 18, 25, 32, 36, 38, 41, 42].

1.1 Extended Hypergeometric Function

Luo et al. [24] introduced the following extended generalized hypergeometric function ${}_pF_q^{(\delta,\xi;\kappa,\mu)}$ and obtained its various properties: The extended generalized hypergeometric function ${}_pF_q^{(\delta,\xi;\kappa,\mu)}$ is defined by

$$
{}_pF_q^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}; z; \omega\right] := \sum_{n=0}^{\infty} \Theta\left(n/p,q\right)\frac{z^n}{n!} \tag{1}
$$

$$
\left(\min\{\Re(\delta),\ \Re(\xi),\ \Re(\omega)\} > 0,\ \ \min\left\{\Re(\kappa),\ \Re(\mu)\right\} \geq 0\right),
$$

whose coefficient $\Theta\left(n/p,q\right)$ is determined by

$$
\Theta\left(n/p,q\right) = \begin{cases} (a_1)_n \prod\limits_{j=1}^{q} \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(a_{j+1}+n,b_j-a_{j+1})}{B(a_{j+1},b_j-a_{j+1})} \\ \qquad (p=q+1; \Re(b_j) > \Re(a_{j+1}) > 0; |z| < 1), \\ \prod\limits_{j=1}^{q} \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(a_j+n,b_j-a_j)}{B(a_j,b_j-a_j)} \\ \qquad (p=q; \Re(b_j) > \Re(a_j) > 0; z \in \mathbb{C}), \\ \prod\limits_{i=1}^{r} \frac{1}{(b_i)_n} \prod\limits_{j=1}^{p} \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(a_j+n,b_{r+j}-a_j)}{B(a_j,b_r+j-a_j)} \\ \qquad (r=q-p, p<q; \Re(b_{r+j}) > \Re(a_j) > 0; z \in \mathbb{C}). \end{cases}
$$

Here, the generalized beta function $B_\omega^{(\delta,\xi;\kappa,\mu)}(x,y)$ is defined by Luo et al. [24]

$$
B_\omega^{(\delta,\xi;\kappa,\mu)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\delta;\xi;-\frac{\gamma}{t^k(1-t)^\mu}\right)dt \tag{2}
$$

$$
(\min\{\Re(\omega),\ \Re(\kappa)\} \geq 0,\ \min\{\Re(x),\ \Re(y),\ \Re(\delta), \Re(\xi),\ \Re(k),\ \Re(\mu)\} > 0)
$$

and the beta function $B(\delta,\ \xi)$ may be recalled as follows (Srivastava and Choi [9]):

$$B(\delta,\ \xi) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\delta) > 0;\ \Re(\xi) > 0) \\ \\ \dfrac{\Gamma(\delta)\,\Gamma(\xi)}{\Gamma(\delta+\xi)} & (\delta,\ \xi \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \tag{3}$$

The special case of the function (1), when $\omega = 0$ is seen to reduce to the generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters, is defined by Rainville [6] and Srivastava and Choi [9]

$${}_pF_q\left[\begin{matrix} a_1,\ \ldots,\ a_p; \\ b_1,\ \ldots,\ b_q; \end{matrix}\ z\right] == \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!},$$

where in terms of the gamma function $\Gamma(z)$ (Srivastava and Choi [9]) whose Euler's integral is given by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\,dt \quad (\Re(z) > 0),$$

the widely used Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\ \nu \in \mathbb{C})$ is defined, in general, by Srivastava and Manocha [8], Srivastava and Choi [9]

$$\begin{aligned} (\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad \left(\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \\ &= \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \end{aligned}$$

The special case of the function (2) when $\gamma = 0$ would reduce immediately to the familiar classical beta function $B(x, y)$ (Srivastava and Choi [9]).

It is also noted that for $p = 2$ and $q = 1$ the definitions in (1) would reduce immediately to the extended hypergeometric type function defined as follows (Luo et al. [24]):

$${}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; z; \omega\right] := \sum_{n=0}^{\infty} (a)_n \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \tag{4}$$

$$(\Re(\omega) > 0,\ \Re(\kappa) \geq 0, \mu \geq 0;\ \ \min\{\Re(\delta), \Re(\xi)\} > 0;\ \Re(c) > \Re(b) > 0,\ |z| < 1).$$

The various properties of extended hypergeometric functions are studied by some authors in [11, 23, 28].

The present investigation requires the concept of Hadamard product which can be used to decompose a newly emerged function into two known functions.

Let

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) := \sum_{n=0}^{\infty} b_n z^n$$

be two power series whose radii of convergence are given by R_f and R_g, respectively. Then, their Hadamard product (Pohlen [40]) is the power series defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n.$$

The radius of convergence R of the Hadamard product series $(f * g)(z)$ satisfies $R_f \cdot R_g \leq R$. If, in particular, one of the power series defines an entire function, then the Hadamard product series defines an entire function, too.

Consider the function ${}_pF_{r+p}^{(\delta,\xi;\kappa,\mu)}[z;\omega]$ one of whose Hadamard products can, for example, be given as follows:

$$\begin{aligned}&{}_pF_{r+p}^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}x_1,\ldots,x_p\\ y_1,\ldots,y_{r+p}\end{matrix};z;\omega\right]\\ &= {}_1F_r\left[\begin{matrix}1;\\ y_1,\ldots,y_r;\end{matrix}z\right] * {}_pF_p^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}x_1,\ldots,x_p\\ y_{r+1},\ldots,y_{r+p}\end{matrix};z;\omega\right] \quad (|z|<\infty),\end{aligned}$$

where ${}_1F_r$ is a special case of the generalized hypergeometric functions ${}_pF_q$ (Srivastava and Choi [9]).

1.2 Fractional Calculus

Appell hypergeometric function F_3 in two variables (see Appell and Kampé de Fériet [33] and Srivastava and Karlsson [12]) is defined by

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x;y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} \quad (\max\{|x|,|y|\}<1). \tag{5}$$

Let $\alpha,\alpha',\beta,\beta',\gamma \in \mathbb{C}$, $\mathbb{C}$ being the set of complex numbers and $x>0$. Then for $\Re(\gamma)>0$, the generalized fractional integral operators involving the Appell hypergeometric function F_3 as a kernel are defined as follows (Saigo and Maeda [27]):

$$(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt \tag{6}$$

and

$$(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}f)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,dt \tag{7}$$

Then, the generalized fractional derivative operators of a function $f(x)$ are defined as follows (Saigo and Maeda [27]);

$$(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f)(x)=(I_{0,x}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f)(x)=\left(\frac{d}{dx}\right)^k(I_{0,x}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}f)(x) \tag{8}$$

and

$$(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}f)(x)=(I_{x,\infty}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f)(x)=\left(-\frac{d}{dx}\right)^k(I_{x,\infty}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}f)(x), \tag{9}$$

$(\Re(\gamma)>0;k=[\Re(\gamma)]+1)$.

The Appell function (5) in (8) and (9) satisfies a system of two partial differential equations of the second order and reduces to the Gauss hypergeometric function ${}_2F_1$ as follows (see Appell and Kampé de Fériet [33] and Srivastava and Karlsson [12]):

$$F_3(\alpha,\gamma-\alpha,\beta,\gamma-\beta;\gamma;x;y)={}_2F_1(\alpha,\beta;\gamma;x+y-xy)$$

Further, it is easy to see that

$$F_3(\alpha,0,\beta,\beta';\gamma;x;y)={}_2F_1(\alpha,\beta;\gamma;x) \tag{10}$$

and

$$F_3(0,\alpha',\beta,\beta';\gamma;x;y)={}_2F_1(\alpha',\beta';\gamma;y)$$

In view of the reduction formula (10), the general operators (6) and (7) reduce to Saigo operators

$$(I_{0,x}^{\alpha,\beta,\gamma}f)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x}\right)f(t)\,dt$$

and

$$(I_{x,\infty}^{\alpha,\beta,\gamma}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty (x-t)^{\alpha-1}t^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\right)f(t)\,dt,$$

Then, the left-sided Saigo fractional derivative operator can be defined as Saigo [26] and Srivastava and Saigo [10]

$$(D_{0+}^{\alpha,\beta,\gamma} f)(x) = (I_{0,x}^{-\alpha,-\beta,\alpha+\gamma} f) = \left(\frac{d}{dx}\right)^n \{(I_{0,x}^{-\alpha+\gamma,-\beta-\gamma,\alpha+\gamma-n} f)(x)\} \tag{11}$$

and

$$(D_{0-}^{\alpha,\beta,\gamma} f)(x) = (I_{x,\infty}^{-\alpha,-\beta,\alpha+\gamma} f) = \left(-\frac{d}{dx}\right)^n \{(I_{x,\infty}^{-\alpha+\gamma,-\beta-\gamma,\alpha+\gamma-n} f)(x)\}, \tag{12}$$

$(\Re(\alpha) \geq 0; n = [\Re(\alpha)] + 1)$. If we take $\alpha = 0$, then (8) and (9) reduce to Saigo fractional derivative operators defined by (11) and (12), respectively.

If we set $\beta = -\alpha$, then operators (11) and (12) reduce to Riemann–Liouville fractional derivative operator and Weyl fractional derivative operator as follows (Kilbas et al. [1])

$$(D_{0+}^{\alpha,-\alpha,\gamma} f)(x) = ({}^{RL}D_{0+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt\right\} \tag{13}$$

and

$$(D_{0-}^{\alpha,-\alpha,\gamma} f)(x) = ({}^{W}D_{0-}^{\alpha} f)(x) = \left(-\frac{d}{dx}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha-n+1}} dt\right\}, \tag{14}$$

$(x > 0; n = [\Re(\alpha)] + 1; \Re(\alpha) \geq 0)$. Again, if $\beta = 0$, (11) and (12) reduce to left-sided Erdélyi–Kober fractional differential operator and right-sided Erdélyi–Kober fractional differential operator and are defined below (Kilbas et al. [1])

$$(D_{0+}^{\alpha,0,\gamma} f)(x) = ({}^{EK}D_{0+}^{\alpha,\gamma} f)(x) = x^{\gamma} \left(\frac{d}{dx}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{t^{\alpha+\gamma} f(t)}{(x-t)^{\alpha-n+1}} dt\right\} \tag{15}$$

and

$$(D_{0-}^{\alpha,0,\gamma} f)(x) = ({}^{EK}D_{0-}^{\alpha,\gamma} f)(x) = x^{\alpha+\gamma} \left(-\frac{d}{dx}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)} \int_x^\infty \frac{t^{-\gamma} f(t)}{(t-x)^{\alpha-n+1}} dt\right\}, \tag{16}$$

$(x > 0; n = [\Re(\alpha)] + 1; \Re(\alpha) \geq 0)$.

The generalized integration for a power function is given by Saigo and Maeda [27], Saxena and Saigo [37]:

$$(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1})(x)=\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}x^{\rho-\alpha-\alpha'+\gamma-1},\quad (17)$$

where $\Re(\gamma)>0, \Re(\rho)>\max\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\}$ and

$$(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1})(x)=\frac{\Gamma(1-\rho-\gamma+\alpha+\alpha')\Gamma(1-\rho+\alpha+\beta'-\gamma)\Gamma(1-\rho-\beta)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\alpha'+\beta'-\gamma)\Gamma(1-\rho+\alpha-\beta)}x^{\rho-\alpha-\alpha'+\gamma-1}\quad (18)$$

where $\Re(\gamma)>0, \Re(\rho)<1+\min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$.

1.3 Certain Basic Tools

The beta transforms of $f(z)$ are defined as Sneddon [16]

$$B\{f(z):a,b\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)\,dz \quad (19)$$

The pathway type transforms (P_ν-transforms) of a function $f(z)$ of a real variable z denoted by $P_\nu[f(z);s]$ are a function $F(s)$ of complex variable s, valid under certain conditions on $f(z)$ along with the condition $\nu>1$, and are defined by Kumar [4]

$$P_\nu[f(z);s]=F(s)=\int_0^\infty [1+(\nu-1)s]^{-\frac{z}{\nu-1}}f(z)\,dz. \quad (20)$$

For $\rho\in\mathbb{C}, \Re(\rho)>0$ and $\nu>1$, the P_ν-transform of power function is given by Kumar [4]

$$P_\nu[z^{\rho-1};s]=\left\{\frac{\nu-1}{\ln[1+(\nu-1)s]}\right\}^\rho\Gamma(\rho) \quad (21)$$

Furthermore, upon letting $\nu\to1$ in (20), the P_ν-transform is reduced to classical Laplace transform of a function $f(z)$ (Sneddon [16]) is given by

$$L\{f(z):s\}=\int_0^\infty e^{-sz}f(z)\,dz. \quad (22)$$

Agarwal et al. [34] obtained solution of fractional volterra integral equation and nonhomogeneous time fractional heat equation using integral transform of pathway type.

The Whittaker function (Mathai et al. [2, p. 55]) is defined by

$$W_{\lambda,m}(z)=\sum_{m,-m}\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-\lambda-m)}M_{\lambda,m}(z)$$

where the summation symbol indicates that the expression following it, a similar expression with m replaced by $-m$, is to be added and

$$M_{\lambda,m}(z)=z^{m+\frac{1}{2}}e^{-\frac{z}{2}}{}_1F_1\left(\frac{1}{2}-\lambda+m;2m+1;z\right).$$

We shall use the following formula (Mathai et al. [2])

$$\int_0^\infty t^{\rho-1}e^{-\frac{t}{2}}W_{\lambda,m}(t)\,dz=\frac{\Gamma(\rho+m+\frac{1}{2})\Gamma(\rho-m+\frac{1}{2})}{\Gamma(\rho-\lambda+\frac{1}{2})} \tag{23}$$

Two-parameter Mittag–Leffler function (Wiman [3]) is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{24}$$

$(\alpha,\beta\in\mathbb{C};\Re(\alpha)>0,\Re(\beta)>0)$.

1.4 Fractional Kinetic Equations

If an arbitrary reaction is characterized by a time dependent $N=N(t)$, then it is possible to calculate the rate of change of $\frac{dN}{dt}$ by mathematical equation

$$\frac{dN}{dt}=-d+p,$$

where d is the destruction rate and p is the production rate of N.

Haubold and Mathai [7] established a functional differential equation between the rate of change of reaction, the destruction rate and the production rate as follows:

$$\frac{dN}{dt}=-d(N_t)+p(N_t), \tag{25}$$

where $N=N(t)$ is the rate of reaction, $d(N(t))$ is the rate of destruction, $p(N_t)$ is the rate of production, and N_t denotes the function defined by $N_t(t^*)=N(t)-t^*$, $t^*>0$.

A special case of (25), when spatial fluctuations or homogeneities in the quantity $N(t)$ are neglected, is given by the following differential equation (Haubold and Mathai [7] and Kourganoff [43]):

$$\frac{dN_i}{dt} = -c_i N_i(t), \tag{26}$$

where initial condition $N_t(t = 0) = N_0$ is the number of density of species i at time $t = 0$, $c_i > 0$. Solution of standard kinetic equation (26) is given by Kourganoff [43] as

$$N_i(t) = N_0 e^{-c_i t}.$$

If we decline the index i and integrate standard kinetic equation (26), we have

$$N(t) - N_0 = -c_0\, {}_0D_t{}^{-1} N(t),$$

where ${}_0D_t{}^{-1}$ is standard integral operator.

Haubold and Mathai [7] obtained the fractional generalization of the standard kinetic equation (26) as

$$N(t) - N_0 = -c_0^{\nu}\, {}_0D_t{}^{-\nu} N(t), \tag{27}$$

where ${}_0D_t{}^{-\nu}$ is Riemann–Liouville fractional integral operator defined as Samko et al. [39]

$${}_0D_t{}^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u)\, du, \;\; t > 0, \Re(\nu) > 0.$$

The Laplace transform (22) of the Riemann–Liouville fractional integral operator is given by [14]

$$L\{{}_0D_t{}^{-p} f(t); s\} = s^{-p} L\{f(t); s\}. \tag{28}$$

Solution of Eq. (27) is given by Haubold and Mathai [7]

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (c_0 t)^{\nu k}.$$

2 Image Formulae Associated with Fractional Derivative Operators

In this section, we establish certain fractional derivative formulae for the extended generalized hypergeometric function (4).

2.1 Saigo–Maeda Fractional Derivative Operators

Theorem 2.1 *Let* $x > 0$*, the parameters* $\alpha, \alpha', \beta, \beta', \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$*,* $\Re(\omega) > 0$*,* $\Re(\kappa) \geq 0$*,* $\mu \geq 0$*,* $\min\{\Re(\delta), \Re(\xi)\} > 0$*,* $\Re(c) > \Re(b) > 0$*, and* $\Re(\rho) > 0$ *be such that*

$$\Re(\rho) > \max\{0, \Re(\gamma - \alpha - \alpha' - \beta'), \Re(\beta - \alpha)\}.$$

Then, the following formula holds:

$$\begin{aligned}&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix}; zt; \omega\right]\right)(x) =\\ &x^{\rho+\alpha+\alpha'-\gamma-1}\frac{\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}\\ &\times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix}; zx; \omega\right] * {}_3F_3\left[\begin{matrix}\rho, \rho-\beta+\alpha, \rho-\gamma+\alpha+\alpha'+\beta'\\ \rho-\beta, \rho-\gamma+\alpha+\beta', \rho-\gamma+\alpha+\alpha'\end{matrix}; zx\right]\end{aligned} \tag{29}$$

Proof Applying (4) and using (8), we obtain

$$\begin{aligned}&\left(I_{0,x}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix}; zt; \omega\right]\right)(x) =\\ &\sum_{k=0}^{\infty}(a)_k\,\frac{\mathcal{B}_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{\mathcal{B}(b, c-b)}\frac{z^k}{k!}\left(I_{0,x}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} t^{\rho+k-1}\right)(x)\end{aligned}$$

Now using (17), we get

$$\begin{aligned}&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix}; zt; \omega\right]\right)(x) =\\ &\sum_{k=0}^{\infty}(a)_k\,\frac{\mathcal{B}_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{\mathcal{B}(b, c-b)}\frac{z^k x^{\rho+k+\alpha+\alpha'-\gamma-1}}{k!}\\ &\times\frac{\Gamma(\rho+k)\Gamma(\rho+k-\beta+\alpha)\Gamma(\rho+k-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho+k-\beta)\Gamma(\rho+k-\gamma+\alpha+\beta')\Gamma(\rho+k-\gamma+\alpha+\alpha')}\end{aligned}$$

Hence,

$$\begin{aligned}&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix}; zt; \omega\right]\right)(x) =\\ &x^{\rho+\alpha+\alpha'-\gamma-1}\frac{\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}\\ &\times\sum_{k=0}^{\infty}(a)_k\frac{\mathcal{B}_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{\mathcal{B}(b, c-b)}\frac{(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k}\frac{(zx)^k}{k!}\end{aligned}$$

which, in view of Hadamard product series and (4), gives the right-hand side of (29).

Theorem 2.2 *Let $x > 0$, the parameters $\alpha, \alpha', \beta, \beta', \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\rho) < 1 + \min\{0, \Re(\beta'), \Re(\gamma - \alpha - \alpha'), \Re(\gamma - \alpha' - \beta)\}.$$

Then, the following formula holds:

$$\begin{aligned}
&\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a, & b \\ & c\end{matrix}; \frac{z}{t}; \omega\right]\right)(x) \\
&= x^{\rho+\alpha+\alpha'-\gamma+1} \frac{\Gamma(1-\rho+\gamma-\alpha-\alpha')\Gamma(1-\rho-\alpha'-\beta+\gamma)\Gamma(1-\rho+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\rho-\alpha'+\beta')} \\
&\times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a, & b \\ & c\end{matrix}; \frac{z}{x}; \omega\right] * {}_3F_3\left[\begin{matrix} 1-\rho+\gamma-\alpha-\alpha', 1-\rho-\alpha'-\beta+\gamma, 1-\rho+\beta' \\ 1-\rho, 1-\rho-\alpha-\alpha'-\beta+\gamma, 1-\rho-\alpha'+\beta' \end{matrix}; \frac{z}{x}\right]
\end{aligned} \tag{30}$$

Proof Using (4) and using (9), we obtain

$$\left(I_{x,\infty}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a, & b \\ & c\end{matrix}; \frac{z}{t}; \omega\right]\right)(x) =$$
$$\sum_{k=0}^{\infty} (a)_k \frac{\mathcal{B}_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{\mathcal{B}(b, c-b)} \frac{z^k}{k!} \left(I_{x,\infty}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} t^{\rho-k-1}\right)(x)$$

Using (18), we get

$$\begin{aligned}
&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a, & b \\ & c\end{matrix}; \frac{z}{t}; \omega\right]\right)(x) = x^{\rho+\alpha+\alpha'-\gamma-1} \\
&\times \frac{\Gamma(1-\rho+\gamma-\alpha-\alpha')\Gamma(1-\rho-\alpha'-\beta+\gamma)\Gamma(1-\rho+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\rho-\alpha'+\beta')} \\
&\times \sum_{k=0}^{\infty} (a)_k \frac{\mathcal{B}_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{\mathcal{B}(b, c-b)} \frac{(1-\rho+\gamma-\alpha-\alpha')_k(1-\rho-\alpha'-\beta+\gamma)_k(1-\rho+\beta')_k}{(1-\rho)_k(1-\rho-\alpha-\alpha'-\beta+\gamma)_k(1-\rho-\alpha'+\beta')_k} \frac{(\frac{z}{x})^k}{k!}
\end{aligned}$$

which, in view of Hadamard product series and (4), gives the right-hand side of (30).

The following theorems are due to Srivastava et al. in [13] for $p = 2$ and $q = 1$.

2.2 Saigo Fractional Derivative Operators

Theorem 2.3 *Let $x > 0$, the parameters $\alpha, \beta, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) > -\min\{0, \alpha+\beta+\gamma\}.$$

Then, the following Saigo fractional derivative formula holds:

$$\left(D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a,\ b \\ c \end{matrix}; tz; \omega\right]\right)(x) = x^{\rho+\beta-1}\frac{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\gamma)}{\Gamma(\rho+\beta)\Gamma(\rho+\gamma)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a,\ b \\ c \end{matrix}; xz; \omega\right] * {}_2F_2\left[\begin{matrix} \rho, \rho+\alpha+\beta+\gamma \\ \rho+\beta, \rho+\gamma \end{matrix}; xz\right] \tag{31}$$

Theorem 2.4 *Let* $x > 0$, *the parameters* $\alpha, \beta, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, *and* $\Re(\rho) > 0$ *be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(-\beta-\gamma), \Re(\alpha+\gamma)\}.$$

Then, the following Saigo fractional derivative formula holds:

$$\left(D_{0-}^{\alpha,\beta,\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a,\ b \\ c \end{matrix}; \frac{z}{t}; \omega\right]\right)(x) = x^{\rho+\beta-1}\frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho-\beta+\gamma)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a,\ b \\ c \end{matrix}; \frac{z}{x}; \omega\right] * {}_2F_2\left[\begin{matrix} 1-\rho-\beta, 1-\rho+\alpha+\gamma \\ 1-\rho, 1-\rho-\beta+\gamma \end{matrix}; \frac{z}{x}\right]$$

Further, replacing β by $-\alpha$ in Theorems 2.3 and 2.4 and making use of relations (13) and (14) gives Riemann–Liouville fractional derivative formula of generalized Gauss hypergeometric function given in (4) given by the following corollaries (Srivastava et al. [13]).

2.3 Riemaan–Liouville Fractional Derivative Operator

Corollary 2.5 *Let* $x > 0$, *the parameters* $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, $\Re(\alpha) \geq 0$, *and* $\Re(\rho) > 0$. *Then, the following Riemann–Liouville fractional derivative formula holds:*

$$\left({}^{RL}D_{0+}^{\alpha} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a,\ b \\ c \end{matrix}; zt; \omega\right]\right)(x) = x^{\rho-\alpha-1}\frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a,\ b \\ c \end{matrix}; zx; \omega\right] * {}_1F_1\left[\begin{matrix} \rho \\ \rho-\alpha \end{matrix}; zx\right] \tag{32}$$

2.4 Weyl Fractional Derivative Operator

Corollary 2.6 *Let* $x > 0$, *the parameters* $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, *and* $\Re(\rho) > 0$ *be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(\alpha)\}.$$

Then, the following Weyl fractional derivative formula holds:

$$\left({}^{W}D_{0-}^{\alpha}\, t^{\rho-1}{}_{2}F_{1}^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{z}{t};\omega\right]\right)(x)=x^{\rho-\alpha-1}\frac{\Gamma(1-\rho+\alpha)}{\Gamma(1-\rho)}$$
$$\times\,{}_{2}F_{1}^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{z}{x};\omega\right]*{}_{1}F_{1}\left[\begin{matrix}1-\rho+\alpha\\ 1-\rho\end{matrix};\frac{z}{x}\right] \tag{33}$$

Upon setting $\beta=0$ in Theorems 2.3 and 2.4, we can deduce the following corollaries (Srivastava et al. [13]).

2.5 Erdélyi–Kober Fractional Derivative Operators

Corollary 2.7 *Let* $x>0$, *the parameters* $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, *and* $\Re(\rho)>0$ *be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)>-\min\{0,\Re(\gamma)\}.$$

Then, the following Erdélyi–Kober fractional derivative formula holds:

$$\left({}^{EK}D_{0+}^{\alpha,\gamma}\, t^{\rho-1}{}_{2}F_{1}^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};zt;\omega\right]\right)(x)=x^{\rho-1}\frac{\Gamma(\rho+\alpha+\gamma)}{\Gamma(\rho+\gamma)}$$
$$\times\,{}_{2}F_{1}^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};zx;\omega\right]*{}_{1}F_{1}\left[\begin{matrix}\rho+\alpha+\gamma\\ \rho+\gamma\end{matrix};zx\right] \tag{34}$$

Corollary 2.8 *Let* $x>0$, *the parameters* $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, *and* $\Re(\rho)>0$ *be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)<1+\min\{\Re(-\gamma),\Re(\alpha+\gamma)\}.$$

Then, the following Erdélyi–Kober fractional derivative formula holds:

$$\left({}^{EK}D_{0-}^{\alpha,\gamma}\, t^{\rho-1}{}_{2}F_{1}^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{z}{t};\omega\right]\right)(x)=x^{\rho-1}\frac{\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho+\gamma)}$$
$$\times\,{}_{2}F_{1}^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{z}{x};\omega\right]*{}_{1}F_{1}\left[\begin{matrix}1-\rho+\alpha+\gamma\\ 1-\rho+\gamma\end{matrix};\frac{z}{x}\right] \tag{35}$$

3 Image Formulae Associated with Integral Transforms

In this section, we prove certain theorems, which exhibit the connection between beta transforms, pathway transforms, Laplace transforms, and Whittaker transforms with the results obtained in previous section.

3.1 Beta Transforms

Theorem 3.1 *Let* $x > 0$*, the parameters* $\alpha, \alpha', \beta, \beta', \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$*,* $\Re(\omega) > 0$*,* $\Re(\kappa) \geq 0$*,* $\mu \geq 0$*,* $\min\{\Re(\delta), \Re(\xi)\} > 0$*,* $\Re(c) > \Re(b) > 0$*, and* $\Re(\rho) > 0$ *be such that*

$$\Re(\rho) > \max\{0, \Re(\gamma - \alpha - \alpha' - \beta'), \Re(\beta - \alpha)\}.$$

Then, the following beta transform (19) *formula holds:*

$$B\left\{\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} l+m, & b \\ & c \end{matrix}; tz; \omega\right]\right)(x) : l, m\right\} =$$
$$x^{\rho+\alpha+\alpha'-\gamma-1} \frac{B(l,m)\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} l, & b \\ & c \end{matrix}; x; \omega\right] * {}_3F_3\left[\begin{matrix} \rho, \rho-\beta+\alpha, \rho-\gamma+\alpha+\alpha'+\beta' \\ \rho-\beta, \rho-\gamma+\alpha+\beta', \rho-\gamma+\alpha+\alpha' \end{matrix}; x\right] \tag{36}$$

Proof Let $\mathcal{L}$ be the left-hand side of (36) and applying (19) to (36). We get

$$\mathcal{L} = \int_0^1 z^{l-1}(1-z)^{m-1}\left\{\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} l+m, & b \\ & c \end{matrix}; tz; \omega\right]\right)(x) : l, m\right\} dz$$

Use of (29) gives

$$\mathcal{L} = \int_0^1 z^{l-1}(1-z)^{m-1} \frac{x^{\rho+\alpha+\alpha'-\gamma-1}\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times \sum_{k=0}^{\infty} (l+m)_k \frac{\mathcal{B}_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{\mathcal{B}(b, c-b)} \frac{(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k} \frac{(xz)^k}{k!} dz$$

By changing the order of integration and summation which may be verified under the conditions, and using the classical beta function (3), we obtain

$$\mathcal{L} = B(l,m)\frac{x^{\rho+\alpha+\alpha'-\gamma-1}\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times\sum_{k=0}^{\infty}(l)_k\,\frac{\mathcal{B}_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{\mathcal{B}(b,c-b)}\frac{(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k}\frac{x^k}{k!}dz$$

which, in view of (4), is seen to lead to the right-hand side of (36).

Theorem 3.2 *Let* $x>0$*, the parameters* $\alpha,\alpha',\beta,\beta',\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$*,* $\Re(\omega)>0$*,* $\Re(\kappa)\geq 0$*,* $\mu\geq 0$*,* $\min\{\Re(\delta),\Re(\xi)\}>0$*,* $\Re(c)>\Re(b)>0$*, and* $\Re(\rho)>0$ *be such that*

$$\Re(\rho)<1+\min\{0,\Re(\beta'),\Re(\gamma-\alpha-\alpha'),\Re(\gamma-\alpha'-\beta)\}.$$

Then, the following beta transform (19) *formula holds:*

$$B\left\{\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l+m,\ b\\ c\end{matrix};\frac{z}{t};\omega\right]\right)(x):l,m\right\}=$$
$$x^{\rho+\alpha+\alpha'-\gamma+1}\frac{B(l,m)\Gamma(1-\rho+\gamma-\alpha-\alpha')\Gamma(1-\rho-\alpha'-\beta+\gamma)\Gamma(1-\rho+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\rho-\alpha'+\beta')}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l,\ b\\ c\end{matrix};\frac{1}{x};\omega\right]*{}_3F_3\left[\begin{matrix}1-\rho+\gamma-\alpha-\alpha',1-\rho-\alpha'-\beta+\gamma,1-\rho+\beta'\\ 1-\rho,1-\rho-\alpha-\alpha'+\beta+\gamma,1-\rho-\alpha'+\beta'\end{matrix};\frac{1}{x}\right]$$

Theorem 3.3 *Let* $x>0$*, the parameters* $\alpha,\beta,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$*,* $\Re(\omega)>0$*,* $\Re(\kappa)\geq 0$*,* $\mu\geq 0$*,* $\min\{\Re(\delta),\Re(\xi)\}>0$*,* $\Re(c)>\Re(b)>0$*, and* $\Re(\rho)>0$ *be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)>-\min\{0,\alpha+\beta+\gamma\}.$$

Then, the following beta transform formula holds:

$$B\left\{\left(D_{0+}^{\alpha,\beta,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l+m,\ b\\ c\end{matrix};tz;\omega\right]\right)(x)\right\}$$
$$=x^{\rho+\beta-1}B(l,m)\frac{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\gamma)}{\Gamma(\rho+\beta)\Gamma(\rho+\gamma)}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l,\ b\\ c\end{matrix};xz;\omega\right]*{}_2F_2\left[\begin{matrix}\rho,\rho+\alpha+\beta+\gamma\\ \rho+\beta,\rho+\gamma\end{matrix};xz\right]$$

Theorem 3.4 *Let* $x>0$*, the parameters* $\alpha,\beta,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$*,* $\Re(\omega)>0$*,* $\Re(\kappa)\geq 0$*,* $\mu\geq 0$*,* $\min\{\Re(\delta),\Re(\xi)\}>0$*,* $\Re(c)>\Re(b)>0$*, and* $\Re(\rho)>0$ *be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)<1+\min\{\Re(-\beta-\gamma),\Re(\alpha+\gamma)\}$$

Then, the following beta transform formula holds:

$$
\begin{aligned}
&B\left\{\left(D_{0-}^{\alpha,\beta,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l+m,\ b\\ c\end{matrix};\frac{t}{z};\omega\right]\right)(x)\right\}\\
&=B(l,m)x^{\rho+\beta-1}\frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho-\beta+\gamma)}\\
&\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l,\ b\\ c\end{matrix};\frac{z}{x};\omega\right]*{}_2F_2\left[\begin{matrix}1-\rho-\beta,1-\rho+\alpha+\gamma\\ 1-\rho,1-\rho-\beta+\gamma\end{matrix};\frac{z}{x}\right]
\end{aligned}
$$

Corollary 3.5 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, $\Re(\alpha)\geq 0$, and $\Re(\rho)>0$. Then, the following beta transform fractional derivative formula holds:*

$$
\begin{aligned}
&B\left\{\left({}^{RL}D_{0+}^{\alpha}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l+m,\ b\\ c\end{matrix};zt;\omega\right]\right)(x)\right\}=x^{\rho-\alpha-1}B(l,m)\frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\\
&\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l,\ b\\ c\end{matrix};zx;\omega\right]*{}_1F_1\left[\begin{matrix}\rho\\ \rho-\alpha\end{matrix};zx\right]
\end{aligned}
$$

Corollary 3.6 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)<1+\min\{\Re(\alpha)\}.$$

Then, the following beta transform formula holds:

$$
\begin{aligned}
&B\left\{\left({}^{W}D_{0-}^{\alpha}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l+m,\ b\\ c\end{matrix};\frac{z}{t};\omega\right]\right)(x)\right\}=x^{\rho-\alpha-1}B(l,m)\frac{\Gamma(1-\rho+\alpha)}{\Gamma(1-\rho)}\\
&\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l,\ b\\ c\end{matrix};\frac{z}{x};\omega\right]*{}_1F_1\left[\begin{matrix}1-\rho+\alpha\\ 1-\rho\end{matrix};\frac{z}{x}\right]
\end{aligned}
$$

Corollary 3.7 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)>-\min\{0,\Re(\gamma)\}.$$

Then, the following beta transform formula holds:

$$
\begin{aligned}
&B\left\{\left({}^{EK}D_{0+}^{\alpha,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l+m,\ b\\ c\end{matrix};zt;\omega\right]\right)(x)\right\}=x^{\rho-1}B(l,m)\frac{\Gamma(\rho+\alpha+\gamma)}{\Gamma(\rho+\gamma)}\\
&\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}l,\ b\\ c\end{matrix};zx;\omega\right]*{}_1F_1\left[\begin{matrix}\rho+\alpha+\gamma\\ \rho+\gamma\end{matrix};zx\right]
\end{aligned}
$$

Corollary 3.8 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(-\gamma), \Re(\alpha+\gamma)\}.$$

Then, the following beta transform formula holds:

$$\begin{aligned}
&B\left\{\left({}^{EK}D_{0-}^{\alpha,\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} l+m,\ b \\ c \end{matrix}; \frac{z}{t}; \omega\right]\right)(x)\right\} \\
&= x^{\rho-1} B(l,m) \frac{\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho+\gamma)} \\
&\quad \times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} l,\ b \\ c \end{matrix}; \frac{z}{x}; \omega\right] * {}_1F_1\left[\begin{matrix} 1-\rho+\alpha+\gamma \\ 1-\rho+\gamma \end{matrix}; \frac{z}{x}\right]
\end{aligned}$$

3.2 Pathway Transforms

Theorem 3.9 *Let* $x > 0$*, the parameters* $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$*,* $\Re(\omega) > 0$*,* $\Re(\kappa) \geq 0$*,* $\mu \geq 0$*,* $\min\{\Re(\delta), \Re(\xi)\} > 0$*,* $\Re(c) > \Re(b) > 0$*, and* $\Re(\rho) > 0$ *be such that*

$$\Re(\rho) > \max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}.$$

Then, the following pathway transform (20) *formula holds:*

$$\begin{aligned}
&P_\nu\left[z^{l-1}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a\ \ b \\ c \end{matrix}; tz; \omega\right]\right)(x); s\right] = \\
&\frac{1}{[\xi(\nu;s)]^l} \frac{x^{\rho+\alpha+\alpha'-\gamma-1}\Gamma(l)\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')} \\
&\times {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a,\ b \\ c \end{matrix}; \frac{x}{\xi(\nu;s)}; \omega\right] * {}_4F_3\left[\begin{matrix} l, \rho, \rho-\beta+\alpha, \rho-\gamma+\alpha+\alpha'+\beta' \\ \rho-\beta, \rho-\gamma+\alpha+\beta', \rho-\gamma+\alpha+\alpha' \end{matrix}; \frac{x}{\xi(\nu;s)}\right]
\end{aligned} \tag{37}$$

where $\xi(\nu;s) = \frac{\ln[1+(\nu-1)s]}{\nu-1}$

Proof Let $\mathcal{L}$ be the left-hand side of (37). Using definition of the P_ν-transform (20) and (17), we obtain

$$\begin{aligned}
\mathcal{L} &= \frac{x^{\rho+\alpha+\alpha'-\gamma-1}\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')} \\
&\times \left\{\sum_{k=0}^{\infty} (a)_k \frac{\mathcal{B}_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{\mathcal{B}(b, c-b)} \frac{(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k} \frac{x^k}{k!}\right. \\
&\left.\int_0^\infty [1+(\nu-1)s]^{-\frac{z}{\nu-1}} z^{l+k-1} dz\right\}
\end{aligned}$$

Here, making use of the result (21), we obtain

$$\mathcal{L}=\left\{\frac{\nu-1}{\ln[1+(\nu-1)s]}\right\}^{l}\frac{x^{\rho+\alpha+\alpha'-\gamma-1}\Gamma(l)\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times\sum_{k=0}^{\infty}(a)_k\,\frac{\mathcal{B}_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{\mathcal{B}(b,c-b)}\,\frac{(l)_k(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k}\,\frac{\left(\frac{x(\nu-1)}{\ln[1+(\nu-1)s]}\right)^k}{k!}dz$$

which, upon using Hadamard product series and (4), leads to the right-hand side of (37).

Theorem 3.10 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\rho)<1+\min\{0,\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}.$$

Then, the following pathway transform (20) formula holds:

$$P_{\nu}\left[z^{l-1}\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a\ \ b\\ c\end{matrix};\frac{z}{t};\omega\right]\right)(x);s\right]=$$
$$\frac{1}{[\xi(\nu;s)]^{l}}\,\frac{x^{\rho+\alpha+\alpha'-\gamma+1}\Gamma(l)\Gamma(1-\rho+\gamma-\alpha-\alpha')\Gamma(1-\rho-\alpha'-\beta+\gamma)\Gamma(1-\rho+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\rho-\alpha'+\beta')}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{x\,\xi(\nu;s)};\omega\right]*{}_4F_3\left[\begin{matrix}l,1-\rho+\gamma-\alpha-\alpha',1-\rho-\alpha'-\beta+\gamma,1-\rho+\beta'\\ 1-\rho,1-\rho-\alpha-\alpha'-\beta+\gamma,1-\rho-\alpha'+\beta'\end{matrix};\frac{1}{x\,\xi(\nu;s)}\right]$$

Theorem 3.11 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\alpha)\geq 0 \textit{ and } \Re(\rho)>-\min\{0,\alpha+\beta+\gamma\}.$$

Then, the following pathway transform formula holds:

$$P_{\nu}\left[z^{l-1}\left(D_{0+}^{\alpha,\beta,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]=$$
$$\frac{1}{[\xi(\nu;s)]^{l}}\,\frac{x^{\rho+\beta-1}\Gamma(l)\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\gamma)}{\Gamma(\rho+\beta)\Gamma(\rho+\gamma)}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{x}{\xi(\nu;s)};\omega\right]*{}_3F_2\left[\begin{matrix}l,\rho,\rho+\alpha+\beta+\gamma\\ \rho+\beta,\rho+\gamma\end{matrix};\frac{x}{\xi(\nu;s)}\right]$$

Theorem 3.12 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\alpha)\geq 0 \textit{ and } \Re(\rho)<1+\min\{\Re(-\beta-\gamma),\Re(\alpha+\gamma)\}$$

Then, the following pathway transform formula holds:

$$P_\nu\left[z^{l-1}\left(D_{0-}^{\alpha,\beta,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{t}{z};\omega\right]\right)(x);s\right]=$$
$$\frac{1}{[\xi(\nu;s)]^l}x^{\rho+\beta-1}\frac{\Gamma(l)\Gamma(1-\rho-\beta)\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho-\beta+\gamma)}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{x\ \xi(\nu;s)};\omega\right]*{}_3F_2\left[\begin{matrix}l,1-\rho-\beta,1-\rho+\alpha+\gamma\\ 1-\rho,1-\rho-\beta+\gamma\end{matrix};\frac{1}{x\ \xi(\nu;s)}\right]$$

Corollary 3.13 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, $\Re(\alpha)\geq 0$, and $\Re(\rho)>0$. Then, the following pathway transform formula holds:*

$$P_\nu\left[z^{l-1}\left({}^{RL}D_{0+}^{\alpha}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]=$$
$$\frac{1}{[\xi(\nu;s)]^l}\frac{x^{\rho-\alpha-1}\Gamma(l)\Gamma(\rho)}{\Gamma(\rho-\alpha)}\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{x}{\xi(\nu;s)};\omega\right]*{}_2F_1\left[\begin{matrix}l,\rho\\ \rho-\alpha\end{matrix};\frac{x}{\xi(\nu;s)}\right]$$

Corollary 3.14 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)<1+\min\{\Re(\alpha)\}.$$

Then, the following pathway transform formula holds:

$$P_\nu\left[z^{l-1}\left({}^{W}D_{0-}^{\alpha}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{t}{z};\omega\right]\right)(x);s\right]$$
$$=\frac{1}{[\xi(\nu;s)]^l}x^{\rho-\alpha-1}\frac{\Gamma(l)\Gamma(1-\rho+\alpha)}{\Gamma(1-\rho)}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{x\ \xi(\nu;s)};\omega\right]*{}_2F_1\left[\begin{matrix}l,1-\rho+\alpha\\ 1-\rho\end{matrix};\frac{1}{x\ \xi(\nu;s)}\right]$$

Corollary 3.15 *Let $x>0$, the parameters $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, and $\Re(\rho)>0$ be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)>-\min\{0,\Re(\gamma)\}.$$

Then, the following pathway transform formula holds:

$$P_\nu\left[z^{l-1}\left({}^{EK}D_{0+}^{\alpha,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]$$
$$=\frac{1}{[\xi(\nu;s)]^l}\frac{x^{\rho-1}\Gamma(l)\Gamma(\rho+\alpha+\gamma)}{\Gamma(\rho+\gamma)}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{x}{\xi(\nu;s)};\omega\right]*{}_2F_1\left[\begin{matrix}l,\rho+\alpha+\gamma\\ \rho+\gamma\end{matrix};\frac{x}{\xi(\nu;s)}\right]$$

Corollary 3.16 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(-\gamma), \Re(\alpha+\gamma)\}.$$

Then, the following pathway transform formula holds:

$$\begin{aligned}
&P_\nu\left[z^{l-1}\left({}^{EK}D_{0-}^{\alpha,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]\\
&=\frac{1}{[\xi(\nu;s)]^l}x^{\rho-1}\frac{\Gamma(l)\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho+\gamma)}\\
&\quad\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{x\ \xi(\nu;s)};\omega\right]*{}_2F_1\left[\begin{matrix}l,1-\rho+\alpha+\gamma\\ 1-\rho+\gamma\end{matrix};\frac{1}{x\ \xi(\nu;s)}\right]
\end{aligned}$$

It is interesting to observe that for taking $\nu \to 1$ in the P_ν-transform defined by (20) reduces to the well-known Laplace transform (22). In fact, we have an interesting Laplace transform asserted by the following corollaries.

3.3 Laplace Transforms

Corollary 3.17 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\rho) > \max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}.$$

Then, the following Laplace transform formula holds:

$$\begin{aligned}
&L\left[z^{l-1}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]=\\
&\frac{x^{\rho+\alpha+\alpha'-\gamma-1}}{s^l}\frac{\Gamma(l)\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}\\
&\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{x}{s};\omega\right]*{}_4F_3\left[\begin{matrix}l,\rho,\rho-\beta+\alpha,\rho-\gamma+\alpha+\alpha'+\beta'\\ \rho-\beta,\rho-\gamma+\alpha+\beta',\rho-\gamma+\alpha+\alpha'\end{matrix};\frac{x}{s}\right]
\end{aligned}$$

Corollary 3.18 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\rho) < 1 + \min\{0, \Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}.$$

Then, the following Laplace transform formula holds:

$$L\left[z^{l-1}\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a\ b\\ c\end{matrix};\frac{z}{t};\omega\right]\right)(x);s\right]=$$
$$\frac{x^{\rho+\alpha+\alpha'-\gamma+1}}{s^l}\frac{\Gamma(l)\Gamma(1-\rho+\gamma-\alpha-\alpha')\Gamma(1-\rho-\alpha'-\beta+\gamma)\Gamma(1-\rho+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\rho-\alpha'+\beta')}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{xs};\omega\right]*{}_4F_3\left[\begin{matrix}l,1-\rho+\gamma-\alpha-\alpha',1-\rho-\alpha'-\beta+\gamma,1-\rho+\beta'\\ 1-\rho,1-\rho-\alpha-\alpha'-\beta+\gamma,1-\rho-\alpha'+\beta'\end{matrix};\frac{1}{xs}\right]$$

Corollary 3.19 *Let* $x>0$, *the parameters* $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, *and* $\Re(\rho)>0$ *be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)>-\min\{0,\alpha+\beta+\gamma\}.$$

Then, the following Laplace transform formula holds:

$$L\left[z^{l-1}\left(D_{0+}^{\alpha,\beta,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]$$
$$=\frac{x^{\rho+\beta-1}}{s^l}\frac{\Gamma(l)\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\gamma)}{\Gamma(\rho+\beta)\Gamma(\rho+\gamma)}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{x}{s};\omega\right]*{}_3F_2\left[\begin{matrix}l,\rho,\rho+\alpha+\beta+\gamma\\ \rho+\beta,\rho+\gamma\end{matrix};\frac{x}{s}\right]$$

Corollary 3.20 *Let* $x>0$, *the parameters* $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, *and* $\Re(\rho)>0$ *be such that*

$$\Re(\alpha)\geq 0 \text{ and } \Re(\rho)<1+\min\{\Re(-\beta-\gamma),\Re(\alpha+\gamma)\}$$

Then, the following Laplace transform formula holds:

$$L\left[z^{l-1}\left(D_{0-}^{\alpha,\beta,\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{t}{z};\omega\right]\right)(x);s\right]=$$
$$\frac{x^{\rho+\beta-1}}{s^l}\frac{\Gamma(l)\Gamma(1-\rho-\beta)\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho-\beta+\gamma)}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{xs};\omega\right]*{}_3F_2\left[\begin{matrix}l,1-\rho-\beta,1-\rho+\alpha+\gamma\\ 1-\rho,1-\rho-\beta+\gamma\end{matrix};\frac{1}{xs}\right]$$

Corollary 3.21 *Let* $x>0$, *the parameters* $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$, $\Re(\omega)>0$, $\Re(\kappa)\geq 0$, $\mu\geq 0$, $\min\{\Re(\delta),\Re(\xi)\}>0$, $\Re(c)>\Re(b)>0$, $\Re(\alpha)\geq 0$, *and* $\Re(\rho)>0$. *Then, the following Laplace transform formula holds:*

$$L\left[z^{l-1}\left({}^{RL}D_{0+}^{\alpha}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]=$$
$$\frac{x^{\rho-\alpha-1}}{s^l}\frac{\Gamma(l)\Gamma(\rho)}{\Gamma(\rho-\alpha)}\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{x}{s};\omega\right]*{}_2F_1\left[\begin{matrix}l,\rho\\ \rho-\alpha\end{matrix};\frac{x}{s}\right]$$

Corollary 3.22 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(\alpha)\}.$$

Then, the following Laplace transform formula holds:

$$L\left[z^{l-1}\left({}^{W}D^{\alpha}_{0-}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{t}{z};\omega\right]\right)(x);s\right]=$$
$$\frac{x^{\rho-\alpha-1}}{s^l}\frac{\Gamma(l)\Gamma(1-\rho+\alpha)}{\Gamma(1-\rho)}\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{xs};\omega\right]*{}_2F_1\left[\begin{matrix}l,1-\rho+\alpha\\ 1-\rho\end{matrix};\frac{1}{xs}\right]$$

Corollary 3.23 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) > -\min\{0, \Re(\gamma)\}.$$

Then, the following Laplace transform formula holds:

$$L\left[z^{l-1}\left({}^{EK}D^{\alpha,\gamma}_{0+}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};tz;\omega\right]\right)(x);s\right]=$$
$$\frac{x^{\rho-1}}{s^l}\frac{\Gamma(l)\Gamma(\rho+\alpha+\gamma)}{\Gamma(\rho+\gamma)}\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{x}{s};\omega\right]*{}_2F_1\left[\begin{matrix}l,\rho+\alpha+\gamma\\ \rho+\gamma\end{matrix};\frac{x}{s}\right]$$

Corollary 3.24 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(-\gamma), \Re(\alpha+\gamma)\}.$$

Then, the following Laplace transform formula holds:

$$L\left[z^{l-1}\left({}^{EK}D^{\alpha,\gamma}_{0-}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{t}{z};\omega\right]\right)(x);s\right]=$$
$$\frac{x^{\rho-1}}{s^l}\frac{\Gamma(l)\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho+\gamma)}\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1}{xs};\omega\right]*{}_2F_1\left[\begin{matrix}l,1-\rho+\alpha+\gamma\\ 1-\rho+\gamma\end{matrix};\frac{1}{xs}\right]$$

3.4 Whittekar Transforms

Theorem 3.25 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\rho) > \max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}.$$

Then, the following formula holds:

$$\int_0^\infty z^{\sigma-1}e^{-\frac{\eta z}{2}}W_{\lambda,m}(\eta z)\left\{\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a\ b\\ c\end{matrix};\varepsilon zt;\omega\right]\right)(x)\right\}dz$$
$$=\frac{x^{\rho+\alpha+\alpha'-\gamma-1}}{\eta^\sigma}\frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})}\frac{\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a\ b\\ c\end{matrix};\frac{\varepsilon x}{\eta};\omega\right]*{}_5F_4\left[\begin{matrix}\sigma+m+\frac{1}{2},\sigma-m+\frac{1}{2},\rho,\rho-\beta+\alpha,\rho-\gamma+\alpha+\alpha'+\beta'\\ \sigma-\lambda+\frac{1}{2},\rho-\beta,\rho-\gamma+\alpha+\beta',\rho-\gamma+\alpha+\alpha'\end{matrix};\frac{\varepsilon x}{\eta}\right] \tag{38}$$

Proof For convenience, let the left-hand side of (38) be denoted by $\mathcal{L}$. Applying (29)–(38) and changing the order of integration and summation, we get

$$\mathcal{L}=x^{\rho+\alpha+\alpha'-\gamma-1}\frac{\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times\left\{\sum_{k=0}^\infty(a)_k\frac{\mathcal{B}_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{\mathcal{B}(b,c-b)}\frac{(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k}\frac{(\varepsilon x)^k}{k!}\right.$$
$$\left.\int_0^\infty z^{k+\sigma-1}e^{-\frac{\eta z}{2}}W_{\lambda,m}(\eta z)\,dz\right\}$$

substituting $\eta z=\nu$, we get

$$\mathcal{L}=x^{\rho+\alpha+\alpha'-\gamma-1}\frac{\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times\left\{\sum_{k=0}^\infty(a)_k\frac{\mathcal{B}_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{\mathcal{B}(b,c-b)}\frac{(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k}\frac{(\varepsilon x)^k}{\eta^{\sigma+k}k!}\right.$$
$$\left.\int_0^\infty \nu^{k+\sigma-1}e^{-\frac{\nu}{2}}W_{\lambda,m}(\nu)\,d\nu\right\}$$

use of (23) gives

$$\mathcal{L}=\frac{x^{\rho+\alpha+\alpha'-\gamma-1}}{\eta^\sigma}\frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})}\frac{\Gamma(\rho)\Gamma(\rho-\beta+\alpha)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\beta')\Gamma(\rho-\gamma+\alpha+\alpha')}$$
$$\times\left\{\sum_{k=0}^\infty(a)_k\frac{\mathcal{B}_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{\mathcal{B}(b,c-b)}\frac{(\rho)_k(\rho-\beta+\alpha)_k(\rho-\gamma+\alpha+\alpha'+\beta')_k}{(\rho-\beta)_k(\rho-\gamma+\alpha+\beta')_k(\rho-\gamma+\alpha+\alpha')_k}\frac{(\varepsilon x)^k}{\eta^k k!}\right.$$
$$\left.\frac{(\sigma+m+\frac{1}{2})_k(\sigma-m+\frac{1}{2})_k}{(\sigma-\lambda+\frac{1}{2})_k}\right\}$$

In view of (4), we arrive at the desired result (38).

Theorem 3.26 *Let* $x>0$*, the parameters* $\alpha,\gamma,\rho,\delta,\xi,\omega\in\mathbb{C}$*,* $\Re(\omega)>0$*,* $\Re(\kappa)\geq 0$*,* $\mu\geq 0$*,* $\min\{\Re(\delta),\Re(\xi)\}>0$*,* $\Re(c)>\Re(b)>0$*, and* $\Re(\rho)>0$ *be such* *that*

$$\Re(\rho) < 1 + \min\{0, \Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)\}.$$

Then, the following formula holds:

$$\int_0^\infty z^{\sigma-1} e^{-\frac{\eta z}{2}} W_{\lambda,m}(\eta z) \left\{ \left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \ b \\ c \end{matrix}; \frac{\varepsilon z}{t}; \omega \right] \right)(x) \right\} dz$$
$$= \frac{x^{\rho+\alpha+\alpha'-\gamma+1}}{\eta^\sigma} \frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})} \frac{\Gamma(1-\rho+\gamma-\alpha-\alpha')\Gamma(1-\rho-\alpha'-\beta+\gamma)\Gamma(1-\rho+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\alpha-\alpha'+\beta+\gamma)\Gamma(1-\rho-\alpha'+\beta')}$$
$$\times \left\{ {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \ b \\ c \end{matrix}; \frac{\varepsilon}{x\eta}; \omega \right] * \right.$$
$$\left. {}_5F_4 \left[\begin{matrix} \sigma+m+\frac{1}{2}, \sigma-m+\frac{1}{2}, 1-\rho+\gamma-\alpha-\alpha', 1-\rho-\alpha'-\beta+\gamma, 1-\rho+\beta' \\ \sigma-\lambda+\frac{1}{2}, 1-\rho, 1-\rho-\alpha-\alpha'+\beta+\gamma, 1-\rho-\alpha'+\beta' \end{matrix}; \frac{\varepsilon}{x\eta} \right] \right\}$$

Theorem 3.27 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) > -\min\{0, \alpha + \beta + \gamma\}.$$

Then, the following formula holds:

$$\int_0^\infty z^{\sigma-1} e^{-\frac{\eta z}{2}} W_{\lambda,m}(\eta z) \left\{ \left(D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \ b \\ c \end{matrix}; \varepsilon z t; \omega \right] \right)(x) \right\} dz$$
$$= \frac{x^{\rho+\beta-1}}{\eta^\sigma} \frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})} \frac{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\gamma)}{\Gamma(\rho+\beta)\Gamma(\rho+\gamma)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \ b \\ c \end{matrix}; \frac{\varepsilon x}{\eta}; \omega \right] * {}_4F_3 \left[\begin{matrix} \sigma+m+\frac{1}{2}, \sigma-m+\frac{1}{2}, \rho, \rho+\alpha+\beta+\gamma \\ \sigma-\lambda+\frac{1}{2}, \rho+\beta, \rho+\gamma \end{matrix}; \frac{\varepsilon x}{\eta} \right]$$

Theorem 3.28 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(-\beta-\gamma), \Re(\alpha+\gamma)\}.$$

Then, the following formula holds:

$$\int_0^\infty z^{\sigma-1} e^{-\frac{\eta z}{2}} W_{\lambda,m}(\eta z) \left\{ \left(D_{0-}^{\alpha,\beta,\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \ b \\ c \end{matrix}; \frac{\varepsilon z}{t}; \omega \right] \right)(x) \right\} dz$$
$$= \frac{x^{\rho+\beta-1}}{\eta^\sigma} \frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})} \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho-\beta+\gamma)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \ b \\ c \end{matrix}; \frac{\varepsilon}{x\eta}; \omega \right] * {}_4F_3 \left[\begin{matrix} \sigma+m+\frac{1}{2}, \sigma-m+\frac{1}{2}, 1-\rho-\beta, 1-\rho+\alpha+\gamma \\ \sigma-\lambda+\frac{1}{2}, 1-\rho, 1-\rho-\beta+\gamma \end{matrix}; \frac{\varepsilon}{x\eta} \right]$$

Corollary 3.29 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, $\Re(\alpha) \geq 0$, and $\Re(\rho) > 0$. Then, the following formula holds:*

$$\int_0^\infty z^{\sigma-1} e^{-\frac{\eta z}{2}} W_{\lambda,m}(\eta z) \left\{ \left({}^{RL}D_{0+}^{\alpha} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \; b \\ c \end{matrix}; \varepsilon z t; \omega \right] \right)(x) \right\} dz$$
$$= \frac{x^{\rho-\alpha-1}}{\eta^\sigma} \frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})} \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \; b \\ c \end{matrix}; \frac{\varepsilon x}{\eta}; \omega \right] * {}_3F_2 \left[\begin{matrix} \sigma+m+\frac{1}{2}, \sigma-m+\frac{1}{2}, \rho \\ \sigma-\lambda+\frac{1}{2}, \rho-\alpha \end{matrix}; \frac{\varepsilon x}{\eta} \right]$$

Corollary 3.30 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(\alpha)\}.$$

Then, the following formula holds:

$$\int_0^\infty z^{\sigma-1} e^{-\frac{\eta z}{2}} W_{\lambda,m}(\eta z) \left\{ \left({}^{W}D_{0-}^{\alpha} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \; b \\ c \end{matrix}; \frac{\varepsilon z}{t}; \omega \right] \right)(x) \right\} dz$$
$$= \frac{x^{\rho-\alpha-1}}{\eta^\sigma} \frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})} \frac{\Gamma(1-\rho+\alpha)}{\Gamma(1-\rho)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \; b \\ c \end{matrix}; \frac{\varepsilon}{x\eta}; \omega \right] * {}_3F_2 \left[\begin{matrix} \sigma+m+\frac{1}{2}, \sigma-m+\frac{1}{2}, 1-\rho+\alpha \\ \sigma-\lambda+\frac{1}{2}, 1-\rho \end{matrix}; \frac{\varepsilon}{x\eta} \right]$$

Corollary 3.31 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) > -\min\{0, \Re(\gamma)\}.$$

Then, the following formula holds:

$$\int_0^\infty z^{\sigma-1} e^{-\frac{\eta z}{2}} W_{\lambda,m}(\eta z) \left\{ \left({}^{EK}D_{0+}^{\alpha,\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \; b \\ c \end{matrix}; \varepsilon z t; \omega \right] \right)(x) \right\} dz$$
$$= \frac{x^{\rho-1}}{\eta^\sigma} \frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})} \frac{\Gamma(\rho+\alpha+\gamma)}{\Gamma(\rho+\gamma)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a \; b \\ c \end{matrix}; \frac{\varepsilon x}{\eta}; \omega \right] * {}_3F_2 \left[\begin{matrix} \sigma+m+\frac{1}{2}, \sigma-m+\frac{1}{2}, \rho+\alpha+\gamma \\ \sigma-\lambda+\frac{1}{2}, \rho+\gamma \end{matrix}; \frac{\varepsilon x}{\eta} \right]$$

Corollary 3.32 *Let $x > 0$, the parameters $\alpha, \gamma, \rho, \delta, \xi, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$, $\min\{\Re(\delta), \Re(\xi)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(\rho) > 0$ be such that*

$$\Re(\alpha) \geq 0 \text{ and } \Re(\rho) < 1 + \min\{\Re(-\gamma), \Re(\alpha+\gamma)\}.$$

Then, the following formula holds:

$$\int_0^\infty z^{\sigma-1} e^{-\frac{\eta z}{2}} W_{\lambda,m}(\eta z) \left\{ \left({}^{EK}D_{0-}^{\alpha,\gamma} t^{\rho-1} {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a\ b \\ c \end{matrix}; \frac{\varepsilon z}{t}; \omega \right] \right)(x) \right\} dz$$
$$= \frac{x^{\rho-1}}{\eta^\sigma} \frac{\Gamma(\sigma+m+\frac{1}{2})\Gamma(\sigma-m+\frac{1}{2})}{\Gamma(\sigma-\lambda+\frac{1}{2})} \frac{\Gamma(1-\rho+\alpha+\gamma)}{\Gamma(1-\rho+\gamma)}$$
$$\times {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a\ b \\ c \end{matrix}; \frac{\varepsilon}{x\eta}; \omega \right] * {}_3F_2 \left[\begin{matrix} \sigma+m+\frac{1}{2}, \sigma-m+\frac{1}{2}, 1-\rho+\alpha+\gamma \\ \sigma-\lambda+\frac{1}{2}, 1-\rho+\gamma \end{matrix}; \frac{\varepsilon}{x\eta} \right]$$

4 Fractional Kinetic Equations Involving Extended Hypergeometric Function

In this section, generalized fractional kinetic equation involving extended hypergeometric function is established as Theorems.

Theorem 4.1 *If $d > 0$, $p > 0$, $\Re(\omega) > 0$, $\Re(\kappa) \geq 0$, $\mu \geq 0$; $\min\{\Re(\delta), \Re(\xi)\} > 0$; $\Re(c) > \Re(b) > 0$, $|\frac{d^p}{s^p}| < 1$, then the solution of equation*

$$N(t) - N_0\, {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a\ b \\ c \end{matrix}; t; \omega \right] = -d^p {}_0D_t^{-p} N(t) \tag{39}$$

is given by

$$N(t) = N_0 \sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} t^k E_{p,k+1}(-d^p t^p)$$

where $E_{\alpha,\beta}(x)$ is the generalized Mittag–Leffler function given by (24).

Proof Applying Laplace transform (22) on (39), we have

$$L\{N(t); s\} = N_0\, L \left\{ {}_2F_1^{(\delta,\xi;\kappa,\mu)} \left[\begin{matrix} a\ b \\ c \end{matrix}; t; \omega \right]; s \right\} - d^p L\{{}_0N_t{}^{-p} f(t); s\}$$

use of (28) gives

$$L\{N(t);s\} = N_0\left(\int_0^\infty e^{-st}\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{t^k}{k!}\,dt\right) - d^p s^{-p} L\{N(t);s\}$$

therefore,

$$(1+d^p s^{-p})L\{N(t);s\} = N_0\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{1}{k!}\int_0^\infty e^{-st}t^k\,dt$$

this gives,

$$L\{N(t);s\} = N_0\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{1}{s^{k+1}}\frac{1}{1+d^p s^{-p}}$$

After simplification of above equation, we get

$$L\{N(t);s\} = N_0\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\left\{\sum_{r=0}^\infty (-1)^r d^{pr} s^{-(pr+k+1)}\right\} \tag{40}$$

Taking inverse Laplace transform of (40) and using $L^{-1}\{s^{-p};t\} = \frac{t^{p-1}}{\Gamma(p)}$, we obtain

$$\begin{aligned}
N(t) &= N_0\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)} L^{-1}\left\{\sum_{r=0}^\infty (-1)^r d^{pr} s^{-(pr+k+1)}\right\}\\
&= N_0\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\left\{\sum_{r=0}^\infty (-1)^r d^{pr}\frac{t^{pr+k}}{\Gamma(pr+k+1)}\right\}\\
&= N_0\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)} t^k\left\{\sum_{r=0}^\infty \frac{(-dt)^{pr}}{\Gamma(pr+k+1)}\right\}
\end{aligned}$$

hence,

$$N(t) = N_0\sum_{k=0}^\infty (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)} t^k E_{p,k+1}(-d^p t^p)$$

Theorem 4.2 *If* $d>0$, $p>0$, $\Re(\omega)>0$, $\Re(\kappa)\ge 0$, $\mu\ge 0$; $\min\{\Re(\delta),\Re(\xi)\}>0$; $\Re(c)>\Re(b)>0$, $|\frac{d^p}{s^p}|<1$, *then the solution of equation*

$$N(t) - N_0\,{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a\ b \\ c\end{matrix}; d^p t^p;\omega\right] = -d^p\,{}_0D_t^{-p}N(t) \tag{41}$$

is given by

$$N(t)=N_0\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{(d^pt^p)^k}{k!}\Gamma(pk+1)E_{p,pk+1}(-d^pt^p) \tag{42}$$

Proof Applying Laplace transform (22) on (41), we have

$$L\{N(t);s\}=N_0\ L\left\{{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix}a\ b\\ c\end{matrix};d^pt^p;\omega\right];s\right\}-d^pL\{{}_0N_t^{-p}f(t);s\}$$

using (28), we get

$$L\{N(t);s\}=N_0\left(\int_0^{\infty}e^{-st}\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{(d^pt^p)^k}{k!}\,dt\right)-d^ps^{-p}L\{N(t);s\}$$

therefore,

$$(1+d^ps^{-p})L\{N(t);s\}=N_0\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{d^{pk}}{k!}\int_0^{\infty}e^{-st}t^{pk}\,dt$$

hence,

$$L\{N(t);s\}=N_0\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{d^{pk}}{k!}\frac{\Gamma(pk+1)}{s^{pk+1}}\frac{1}{1+d^ps^{-p}}$$

finally,

$$L\{N(t);s\}=N_0\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{d^{pk}}{k!}\Gamma(pk+1)\left\{\sum_{r=0}^{\infty}(-1)^rd^{pr}s^{-(pr+pk+1)}\right\} \tag{43}$$

Taking inverse Laplace transform of (43) and using $L^{-1}\{s^{-p};t\}=\frac{t^{p-1}}{\Gamma(p)}$, we obtain

$$\begin{aligned}N(t)&=N_0\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{d^{pk}}{k!}\Gamma(pk+1)L^{-1}\left\{\sum_{r=0}^{\infty}(-1)^rd^{pr}s^{-(pr+pk+1)}\right\}\\&=N_0\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{d^{pk}}{k!}\Gamma(pk+1)\left\{\sum_{r=0}^{\infty}(-1)^rd^{pr}\frac{t^{pr+pk}}{\Gamma(pr+pk+1)}\right\}\\&=N_0\sum_{k=0}^{\infty}(a)_k\frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k,c-b)}{B(b,c-b)}\frac{(t^pd^p)^k}{k!}\Gamma(pk+1)\left\{\sum_{r=0}^{\infty}\frac{(-dt)^{pr}}{\Gamma(pr+pk+1)}\right\}\end{aligned}$$

In view of definition of Mittag–Leffler function (24), we obtain desired result (42).

Theorem 4.3 *If* $d > 0, \eta > 0, d \neq \eta, p > 0, \Re(\omega) > 0, \Re(\kappa) \geq 0, \mu \geq 0;$ min $\{\Re(\delta), \Re(\xi)\} > 0;$ $\Re(c) > \Re(b) > 0,$ $|\frac{\eta^p}{s^p}| < 1$, *then the solution of equation*

$$N(t) - N_0 \, {}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a \; b \\ c \end{matrix}; d^p t^p; \omega\right] = -\eta^p \, {}_0D_t^{-p} N(t) \tag{44}$$

is given by

$$N(t) = N_0 \sum_{k=0}^{\infty} (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{(d^p t^p)^k}{k!} \Gamma(pk+1) E_{p,pk+1}(-\eta^p t^p) \tag{45}$$

Proof Applying Laplace transform (22) on (44), we have

$$L\{N(t); s\} = N_0 \; L\left\{{}_2F_1^{(\delta,\xi;\kappa,\mu)}\left[\begin{matrix} a \; b \\ c \end{matrix}; d^p t^p; \omega\right]; s\right\} - \eta^p L\{{}_0N_t{}^{-p} f(t); s\}$$

using (28), we obtain

$$L\{N(t); s\} = N_0 \left(\int_0^\infty e^{-st} \sum_{k=0}^{\infty} (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{(d^p t^p)^k}{k!} \, dt \right) - \eta^p s^{-p} L\{N(t); s\}$$

therefore,

$$(1 + \eta^p s^{-p}) L\{N(t); s\} = N_0 \sum_{k=0}^{\infty} (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{d^{pk}}{k!} \int_0^\infty e^{-st} t^{pk} \, dt$$

This leads to

$$L\{N(t); s\} = N_0 \sum_{k=0}^{\infty} (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{d^{pk}}{k!} \frac{\Gamma(pk+1)}{s^{pk+1}} \frac{1}{1 + \eta^p s^{-p}}$$

finally,

$$L\{N(t); s\} = N_0 \sum_{k=0}^{\infty} (a)_k \frac{B_\omega^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{d^{pk}}{k!} \Gamma(pk+1) \left\{ \sum_{r=0}^{\infty} (-1)^r \eta^{pr} s^{-(pr+pk+1)} \right\} \tag{46}$$

Taking inverse Laplace transform of (46) and using $L^{-1}\{s^{-p}; t\} = \frac{t^{p-1}}{\Gamma(p)}$, we obtain

$$
\begin{aligned}
N(t) &= N_0 \sum_{k=0}^{\infty} (a)_k \frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{d^{pk}}{k!} \Gamma(pk+1) L^{-1} \left\{ \sum_{r=0}^{\infty} (-1)^r \eta^{pr} s^{-(pr+pk+1)} \right\} \\
&= N_0 \sum_{k=0}^{\infty} (a)_k \frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{d^{pk}}{k!} \Gamma(pk+1) \left\{ \sum_{r=0}^{\infty} (-1)^r \eta^{pr} \frac{t^{pr+pk}}{\Gamma(pr+pk+1)} \right\} \\
&= N_0 \sum_{k=0}^{\infty} (a)_k \frac{B_{\omega}^{(\delta,\xi;\kappa,\mu)}(b+k, c-b)}{B(b, c-b)} \frac{(t^p d^p)^k}{k!} \Gamma(pk+1) \left\{ \sum_{r=0}^{\infty} \frac{(-\eta t)^{pr}}{\Gamma(pr+pk+1)} \right\}
\end{aligned}
$$

In view of definition of Mittag–Leffler function (24), we obtain desired result (45).

Remark 4.1 It is interesting to observe that for $\kappa = \mu = 1$ in Theorems 2.1, 2.2, 3.1, 3.2, 3.25, 3.26 and Corollaries 3.17, 3.18, we obtain results given by Agarwal et al. [31].

References

1. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies (Elsevier Science, Amsterdam, 2006)
2. A.M. Mathai, R.K. Saxena, H.J. Haubold, *The H-Function Theory and Applications* (Springer, New York, 2010)
3. A. Wiman, Uber de fundamental theorie der funktionen $E_\alpha(x)$. Acta Mathematica **29**(1), 191–201 (1905)
4. D. Kumar, Solution of fractional kinetic equation by a class of integral transform of pathway type. J. Math. Phys. **54**, 043509 (2013). doi:10.1063/1.4800768
5. D. Kumar, S.D. Purohit, A. Secer, A. Atangana, On generalized fractional kinetic equation involving generalized Bessel functions of the first kind. Math. Probl. Eng. **2015**, Article ID 289387, 7 pp. (2015)
6. E.D. Rainville, *Special Functions* (Macmillan Company, New York, 1960) (Reprinted by Chelsea Publishing Company, Bronx, 1971)
7. H.J. Haubold, A.M. Mathai, The fractional kinetic equation and thermonuclear functions. Astrophys. Space Sci. **273**(1), 53–63 (2000)
8. H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester) (Wiley, New York, 1984)
9. H.M. Srivastava, J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals* (Elsevier Science, Amsterdam, 2012)
10. H.M. Srivastava, M. Saigo, Multiplication of fractional calculus operators and boundary value problems involving the Euler-Darboux equation. J. Math. Anal. Appl. **121**, 325–369 (1987)
11. H.M. Srivastava, P. Agarwal, S. Jain, Generating functions for the generalized Gauss hypergeometric functions. Appl. Math. Comput. **247**, 348–352 (2014)
12. H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester) (Wiley, New York, 1985)
13. H.M. Srivastava, R. Agarwal, S. Jain, Integral transform and fractional derivative formulas involving the extended generalized hypergeometric functions and probability distributions. Math. Methods Appl. Sci. (Article in press)
14. H.M. Srivastava, R.K. Saxena, Operators of fractional integration and their applications. Appl. Math. Comput. **118**, 1–52 (2001)
15. I.J. Podulbuny, *Fractional Differential Equations* (Academic Press, New York, 1999)

16. I.N. Sneddon, *The Use of Integral Transform* (Tata McGraw-Hill, New Delhi, 1979)
17. I.S. Jeses, J.A.T. Machado, Fractional control of heat diffusion systems. Nonlinear Dyn. **54**(3), 263–282 (2008)
18. J. Choi, D. Kumar, Solutions of generalized fractional kinetic equations involving Aleph functions. Math. Commun. **20**, 113–123 (2015)
19. J.C. Prajapati, K.B. Kachhia, Fractional modeling of temperature distribution and heat flux in the semi infinite solid. J. Fract. Calc. Appl. **5**(2), 38–43 (2014)
20. J.C. Prajapati, K.B. Kachhia, S.P. Kosta, Fractional calculus approach to study temperature distribution within a spinning satellite. Alex. Eng. J. **55**, 2345–2350 (2016)
21. K.B. Kachhia, J.C. Prajapati, Solution of fractional partial differential equation aries in study of heat transfer through diathermanous material. J. Interdiscip. Math. **18**(1–2), 125–132 (2015)
22. K.B. Kachhia, J.C. Prajapati, On generalized fractional kinetic equations involving generalized Lommel-Wright functions. Alex. Eng. J. **55**, 2953–2957 (2016)
23. K.B. Kachhia, J. Choi, J.C. Prajapati, S.D. Purohit, Some integral transforms involving extended generalized Gauss hypergeometric functions. Commun. Korean Math. Soc. **31**(4), 779–790 (2016)
24. M.J. Luo, G.V. Milovanovic, P. Agarwal, Some results on the extended beta and extended hypergeometric functions. Appl. Math. Comput. **248**, 631–651 (2014)
25. M.J. Luo, R.K. Raina, On certain classes of fractional kinetic equations. Filomat **28**(10), 2077–2090 (2014)
26. M. Saigo, A remark on integrals operators involving the Gauss hypergeometric functions. Math. Rep. Kyushu Univ. **11**, 135–143 (1978)
27. M. Saigo, N. Maeda, More generalization of fractional calculus, in *Proceedings of the 2nd International Workshop on Transform Methods and Special Functions, Verna, 1996, IMI-BAS. Sofia* (1998), pp. 386–400
28. P. Agarwal, Certain properties of the generalized Gauss hypergeometric functions. Appl. Math. Inf. Sci. **8**(5), 2315–2320 (2014)
29. P. Agarwal, J. Choi, Fractional calculus operators and their image formulae. J. Korean Math. Soc. **53**(5), 1183–1210 (2016)
30. P. Agarwal, J. Choi, K.B. Kachhia, J.C. Prajapati, H. Zhou, Some integral transforms and fractional integral formulas for the extended hypergeometric functions. Commun. Korean Math. Soc. **31**(3), 591–601 (2016)
31. P. Agarwal, M. Chand, E.T. Karimov, Certain image formulae of generalized hypergeometric functions. Appl. Math. Comput. **266**, 763–772 (2015)
32. P. Agarwal, M. Chand, G. Singh, Certain fractional kinetic equations involving the product of generalized k-Bessel function. Alex. Eng. J. (Article in press)
33. P. Appell, J. Kampé de Fériet, *Fonctions Hypergétriques et Hypersphériques, Polynômes d'Hermite* (Gauthier-Villars, Paris, 1926)
34. R. Agarwal, S. Jain, R.P. Agarwal, Solution of fractional volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type. Prog. Fract. Differ. Appl. **1**(3), 145–155 (2015)
35. R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000)
36. R.K. Saxena, A.M. Mathai, H.J. Haubold, On generalized fractional kinetic equation. Phys. A **344**, 657–664 (2004)
37. R.K. Saxena, M. Saigo, Generalised fractional calculus of the H-function associated with the Appell function. J. Fract. Calc. **19**, 89–104 (2001)
38. R.K. Saxena, S.L. Kalla, On the solution of certain fractional kinetic equations. Appl. Math. Comput. **199**, 504–511 (2008)
39. S.G. Samko, A. Kilbas, O. Marichev, *Fractional Integral and Derivatives. Theory and Applications* (Gordon and Breach Science Publishers, New York, 1990)
40. T. Pohlen, The Hadamard product and universal power series. Dissertation, Universität Trier (2009)
41. V.B.L. Chaurasia, S.C. Pandey, On the new computable solution of the generalized fractional kinetic equations involving the generalized function for fractional calculus and related functions. Astrophys. Space Sci. **317**, 213–219 (2008)

42. V.B.L. Chaurasia, S.C. Pandey, Computable extensions of generalized fractional kinetic equations in astrophysics. Res. Astron. Astrophys. **10**(1), 22–32 (2010)
43. V. Kourganoff, *Introduction to the Physics of Stellar Interiors* (D. Reidel Publishing Company, Dordrecht, 1973)

The Compact Approximation Property for Weighted Spaces of Holomorphic Mappings

Manjul Gupta and Deepika Baweja

Abstract In this paper, we examine the compact approximation property for the weighted spaces of holomorphic functions. We show that a Banach space E has the compact approximation property if and only if the predual $\mathcal{G}_v(U)$ of the space $H_v(U)$ consisting of all holomorphic mappings $f : U \to \mathbb{C}$ (complex plane) with $\sup_{x \in U} v(x)\|f(x)\| < \infty$ has the compact approximation property, where v is a radial weight defined on a balanced open subset U of E such that $H_v(U)$ contains all the polynomials. We have also studied the compact approximation property for the weighted (LB)-space $VH(E)$ of holomorphic mappings and its predual $VG(E)$ for a countable decreasing family V of radial rapidly decreasing weights on E.

Keywords Weighted spaces of holomorphic mappings · Approximation property · Compact approximation property

2010 AMS Math. Subject Classification Primary 46G20 · 46E50 · Secondary 46B28

1 Introduction

The approximation property plays a vital role in the structural study of Banach spaces and appeared for the first time in the book by Banach [4]. A systematic study of this concept was taken up by Grothendieck [26] in the year 1955 who considered the approximation property, bounded approximation property, and the basis property. At present, we have several variants of this property such as metric approximation property, compact approximation property, strong approximation property,

M. Gupta (✉) · D. Baweja
Department of Mathematics and Statistics, IIT Kanpur,
Kanpur 208016, India
e-mail: manjul@iitk.ac.in

D. Baweja
e-mail: dbaweja@iitk.ac.in

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_2

p-approximation property, and ideal approximation property, cf. [8, 20–22, 34–36, 45, 50], etc. As the identity operator on the space is approximated by linear operators having simpler representation in the study of the approximation property, there are three standard tools for studying approximation property for spaces of holomorphic mappings: ϵ-products, linearization, and $\mathcal{S}$-absolute decompositions. The notion of ϵ-products for locally convex spaces X and Y written as $X\epsilon Y$ introduced by L. Schwartz is defined as the space $\mathcal{L}_e(Y_c'; X)$ of all continuous linear operators from Y_c' to X, endowed with the topology of uniform convergence on equicontinuous subsets of Y, where Y_c' is the topological dual of Y equipped with the topology of uniform convergence on compact subsets of Y. Using the method of ϵ-products, the study of the approximation property for spaces of holomorphic mappings was initiated by Aron and Schottenloher in their pioneer work [2] and was further carried out in [14–19, 27, 29, 41, 49]. Through linearization results, one identifies a given class of holomorphic functions defined on an open subset U of a Banach space E with values in a Banach space F, with the space of continuous linear mappings from a certain Banach space G to F, i.e., a holomorphic mapping is being identified with a linear operator and so one can pursue the study of the approximation property for spaces of holomorphic mappings by this method. The first linearization theorem for such spaces was obtained by Mazet [38] in the year 1984. Almost six years later, J. Mujica obtained a linearization theorem for $\mathcal{H}^\infty(U; F)$, the space of bounded holomorphic mappings defined on an open subset U of a Banach space E with values in F; indeed, the space $\mathcal{H}^\infty(U; F)$ is being identified with $\mathcal{L}(\mathcal{G}^\infty(U); F)$ where $\mathcal{G}^\infty(U)$ is the predual of $\mathcal{H}^\infty(U)$. Using this linearization theorem, Mujica proved several results characterizing the approximation property for E in terms of the approximation property for $\mathcal{H}^\infty(U)$ and $\mathcal{G}^\infty(U)$. This study has further been continued by E. Caliskan in [15–19]. The study of the approximation property for a locally convex X having Schauder decomposition is characterized through the approximation property for the subspaces forming its Schauder decomposition; indeed, if a sequence $\{X_n\}_{n\geq 1}$ forms an $\mathcal{S}$-absolute decomposition for a locally convex space X, then X has the approximation property if and only if each X_n has the approximation property. As the sequence of spaces of m-homogenous polynomials forms an $\mathcal{S}$-absolute decomposition for their parent space, this method has been proved to be useful in such a study.

Weighted spaces of holomorphic functions defined on an open subset of a finite or infinite dimensional Banach space have been studied widely in the literature by several mathematicians. Whereas for the results in the finite dimensional case, we attribute to the contributions of K.D. Bierstedt, J. Bonet, A. Galbis, W.H. Summers, R.G. Meise, Rubel, and Shields [9–13, 46] etc., the infinite dimensional case was introduced by Garcia, Maestre, and Rueda in [25] and further investigated by Beltran [6, 7] Jorda [32], Rueda [47], etc. Though Mujica and Caliskan considered the approximation property for spaces of bounded holomorphic mappings, we initiated this study for weighted spaces in our work [27–29]. In the present article, we consider the compact approximation property for such spaces.

In Sect. 3, we show that a Banach space E has the compact approximation property if and only if the predual $\mathcal{G}_v(U)$ has the compact approximation property for a radial weight v defined on a balanced open subset U of E such that $H_v(U)$ contains all the polynomials. Also, it has been shown that E has the compact approximation property if and only if each weighted holomorphic mapping can be approximated by such a map with relatively compact range.

In Sect. 4, we introduce a locally convex topology $\tau'_{\mathcal{M}}$ and prove a characterization for the $\tau'_{\mathcal{M}}$-denseness of weighted spaces of holomorphic mappings with relatively compact range in $\mathcal{H}_v(U; F)$.

Finally, in the last section, we study the approximation properties for the weighted (LB)-spaces $VH(E)$ defined corresponding to a countable decreasing family V of radial rapidly decreasing weights and its predual $VG(E)$; indeed, it is proved that E has the approximation property if and only if $VG(E)$ has the approximation property. Also, this result holds for the compact approximation property for suitably restricted family V of weights.

2 Preliminaries

Throughout this paper, E and F denote complex Banach spaces with closed unit balls B_E and B_F, respectively. The symbols X' and X^*_b, respectively, stand for the algebraic and strong topological dual of a locally convex space X. The notation X^* is used for X^*_b in case of a normed space X. The symbols $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{C}$ are, respectively, used for the set of natural numbers, $\mathbb{N} \cup \{0\}$, and the complex plane.

For each $m \in \mathbb{N}$, $\mathcal{L}(^m E; F)$ is the Banach space of all continuous m-linear mappings from E to F endowed with the sup norm. A mapping $P : E \to F$ is a *continuous m-homogeneous polynomial* if there exists a continuous m-linear map $A \in \mathcal{L}(^m E; F)$ such that $P(x) = A(x, \dots, x), \quad x \in E$. The space of all m-homogeneous continuous polynomials from E to F is denoted by $\mathcal{P}(^m E; F)$ which is a Banach space endowed with the norm $\|P\| = \sup\limits_{\|x\| \le 1} \|P(x)\|$. For $F = \mathbb{C}$, $\mathcal{P}(^m E; \mathbb{C})$ is written as $\mathcal{P}(^m E)$. *A continuous polynomial* P is a mapping from E into F which can be represented as a sum $P = P_0 + P_1 + \cdots + P_k$ with $P_m \in \mathcal{P}(^m E; F)$ for $m = 0, 1, \dots, k$. The vector space of all continuous polynomials from E into F is denoted by $\mathcal{P}(E; F)$. A polynomial $P \in \mathcal{P}(^m E, F)$ is said to be *compact* if it takes bounded subsets of E to relatively compact subsets of F or equivalently if $P(B_E)$ is relatively compact in F. The collection of all compact m-homogenous polynomials is denoted by $\mathcal{P}_k(^m E; F)$, and for $m = 1$, we get $\mathcal{K}(E; F)$, the class of all compact linear operators from E to F.

A mapping f from E to F is said to be *weakly uniformly continuous (weakly continuous)* on bounded sets if for each bounded subset B of E (for each $x \in B$) and $\epsilon > 0$ there exists $\phi_1,\ \phi_2, \dots, \phi_n \in E^*$ such that

$$\|f(x) - f(y)\| < \epsilon$$

whenever $x, y \in B$ $(y \in B)$ with $|\phi_i(x - y)| < \delta$, for each $i = 1, 2, \ldots, n$. The space of all polynomials which are weakly uniformly continuous (weakly continuous) on bounded subsets of E is denoted by $\mathcal{P}_{wu}(E, F)(\mathcal{P}_w(E, F))$.

A mapping $f : U \to F$ is said to be *holomorphic*, if for each $\xi \in U$, there exists a ball $B(\xi, r)$ with center at ξ and radius $r > 0$, contained in U and a sequence $\{P^j f(\xi)\}_{j=0}^{\infty}$ of polynomials with $P^j f(\xi) \in \mathcal{P}(^j E; F)$, $j \in \mathbb{N}_0$ such that

$$f(x) = \sum_{j=0}^{\infty} P^j f(\xi)(x - \xi) \tag{1}$$

where the series converges uniformly for each $x \in B(\xi, r)$. The space of all holomorphic mappings from U to F is denoted by $\mathcal{H}(U; F)$. For $F = \mathbb{C}$, we write $\mathcal{H}(U)$ for $\mathcal{H}(U; \mathbb{C})$.

A weight v is a continuous and strictly positive function defined on an open subset U of a Banach space E. A weight v defined on (i) a balanced open set U is *radial* if $v(tx) = v(x)$ for all $x \in U$ and $t \in \mathbb{C}$ with $|t| = 1$ and (ii) E is said to be *rapidly decreasing* if $\sup_{x \in E} v(x)\|x\|^m < \infty$ for each $m \in \mathbb{N}_0$. Let us quote from [27] the following: The weighted space

$$\mathcal{H}_v(U; F) = \{f \in \mathcal{H}(U; F) : \|f\|_v = \sup_{x \in U} v(x)\|f(x)\| < \infty\}$$

of holomorphic functions is a Banach space endowed with the norm $\|\cdot\|_v$ with closed unit ball B_v. For $F = \mathbb{C}$, we write $\mathcal{H}_v(U) = \mathcal{H}_v(U; \mathbb{C})$.

Proposition 2.1 *Let v be a weight defined on an open subset U of a Banach space E. Then, for given $m \in \mathbb{N}$, following are equivalent:*

(a) $\mathcal{P}(^m E, F) \subset \mathcal{H}_v(U, F)$ *for each Banach space F.*
(b) $\mathcal{P}(^m E) \subset \mathcal{H}_v(U)$.

Proposition 2.2 *The topology $\tau_{\|\cdot\|_v}$ restricted to $\mathcal{P}(^m E)$ coincides with the sup norm topology.*

Since the closed unit ball B_v of $\mathcal{H}_v(U)$ is τ_0-compact, it follows by Ng's Theorem cf. [44], $\mathcal{H}_v(U)$ is a dual Banach space and its predual is defined as

$$G_v(U) = \{\phi \in \mathcal{H}_v(U)' : \phi|B_v \text{ is } \tau_0\text{-continuous }\}$$

which is endowed with the topology of uniform convergence on the set B_v.

Theorem 2.3 (Linearization Theorem) *For an open subset U of a Banach space E and a weight v on U, there exists a Banach space $\mathcal{G}_v(U)$ and a mapping $\Delta_v \in \mathcal{H}_v(U, \mathcal{G}_v(U))$ with the following property: For each Banach space F and each mapping $f \in \mathcal{H}_v(U, F)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}_v(U), F)$ such that $T_f \circ \Delta_v = f$. The correspondence Ψ between $\mathcal{H}_v(U, F)$ and $\mathcal{L}(\mathcal{G}_v(U), F)$ given by*

$$\Psi(f) = T_f$$

is an isometric isomorphism. The space $\mathcal{G}_v(U)$ is uniquely determined upto an isometric isomorphism by these properties.

A simple consequence of the above linearization theorem is

Proposition 2.4 *For a weight v defined on an open subset U of a Banach space E satisfying $\mathcal{P}(E) \subset \mathcal{H}_v(U)$, E is topologically isomorphic to a complemented subspace of $\mathcal{G}_v(U)$.*

Let us also recall the locally convex topology $\tau_{\mathcal{M}}$ on $\mathcal{H}_v(U, F)$ which is generated by the family $\{p_{\overline{\alpha},\overline{A}} : \overline{\alpha} = (\alpha_j) \in c_0^+,\ \overline{A} = (A_j),\ A_j$ being finite subset of U for each $j\}$ of semi-norms defined by

$$p_{\overline{\alpha},\overline{A}}(f) = \sup_{j \in \mathbb{N}}(\alpha_j \inf_{x \in A_j} v(x) \sup_{y \in A_j} \|f(y)\|).$$

It can be easily checked that

$$\tau_0 \leq \tau_{\mathcal{M}} \leq \tau_{\|.\|_v} \tag{2}$$

on $\mathcal{H}_v(U, F)$. For $v \equiv 1$, the space $\mathcal{H}_v(U, F) \equiv \mathcal{H}^\infty(U, F)$ and the topology $\tau_{\mathcal{M}} \equiv \tau_\gamma$ on $\mathcal{H}^\infty(U, F)$; cf. [41].

Proposition 2.5 *Let E and F be Banach spaces. For a weight v on an open subset U of E with $\mathcal{P}(E) \subset \mathcal{H}_v(U)$, $\tau_{\mathcal{M}}$ coincides with τ_0 on $\mathcal{P}(^m E; F)$ for each $m \in \mathbb{N}$.*

Proposition 2.6 *Let E and F be Banach spaces. For a radial weight v on a balanced open subset U of E with $\mathcal{P}(E) \subset \mathcal{H}_v(U)$, the space $\mathcal{P}(E; F)$ is $\tau_{\mathcal{M}}$-dense in $\mathcal{H}_v(U; F)$.*

Theorem 2.7 *Let E and F be Banach spaces, and v be a weight on an open subset U of E. Then, the mapping $\Psi : (\mathcal{H}_v(U; F), \tau_{\mathcal{M}}) \to (\mathcal{L}(\mathcal{G}_v(U); F), \tau_c)$ is a topological isomorphism.*

Let

$$\mathcal{H}_v(U) \otimes F = \{f \in \mathcal{H}_v(U, F) : f \text{ has finite dimensional range}\}$$

and

$$\mathcal{H}_v^c(U, F) = \{f \in \mathcal{H}_v(U, F) : vf \text{ has a relatively compact range}\}.$$

Then, we have

Proposition 2.8 *Let U be an open subset of a Banach space E and v be a weight on U. Then, for any Banach space F,*

(a) $f \in \mathcal{H}_v(U) \otimes F$ *if and only if* $T_f \in \mathcal{F}(\mathcal{G}_v(U); F)$, *and*
(b) $f \in \mathcal{H}_v^c(U; F)$ *if and only if* $T_f \in \mathcal{K}(\mathcal{G}_v(U); F)$.

A locally convex space X is said to have the *approximation property* if for every compact set K of X, a continuous semi-norm p on X and $\epsilon > 0$, there exists a finite rank operator $T \equiv T_{\epsilon,K}$ such that $\sup_{x \in K} p(T(x) - x) < \epsilon$ and the *compact approximation property* (CAP) if there is a compact linear operator T such that $\sup_{x \in K} p(T(x) - x) < \epsilon$.

The following is quoted from [27]

Theorem 2.9 *Let E be a Banach space and v be a radial weight on a balanced open subset U of E such that $H_v(U)$ contains all the polynomials. Then, E has the approximation property if and only if $\mathcal{G}_v(U)$ has the approximation property.*

Similar to the characterization of AP given by Grothedieck [26] and also given in [37], we have the following result from [16]

Theorem 2.10 *For a Banach space E, the following are equivalent:*

(i) *E has the compact approximation property.*
(ii) *For every Banach space F, $\overline{\mathcal{K}(E; E)}^{\tau_c} = \mathcal{L}(E; E)$.*
(iii) *For every Banach space F, $\overline{\mathcal{K}(F; E)}^{\tau_c} = \mathcal{L}(F; E)$.*
(iv) *For every Banach space F, $\overline{\mathcal{K}(E; F)}^{\tau_c} = \mathcal{K}(E; F)$.*

Using the definition of the CAP, one can easily prove

Proposition 2.11 *Let E be a Banach space with the compact approximation property. Then, each complemented subspace of E also has the compact approximation property.*

The space $\mathcal{Q}(^m E)$ defined as

$$\mathcal{Q}(^m E) = \{\phi \in \mathcal{P}(^m E)' : \phi | B_m \text{ is } \tau_0\text{-continuous}\}$$

is the predual of $\mathcal{P}(^m E)$, $m \in \mathbb{N}$, cf. [48]. It is a Banach space equipped with the topology of uniform convergence on B_m, the unit ball of $\mathcal{P}(^m E)$. Connecting the CAP for a Banach space E with the CAP for $\mathcal{Q}(^m E)$, E. Caliskan [16] proved.

Proposition 2.12 *Let E be a Banach space. Then, E the compact approximation property if and only if $\mathcal{Q}(^m E)$ has the compact approximation property for each $m \in \mathbb{N}$.*

Analogous to Proposition 2.2 in [42], we have.

Proposition 2.13 *Let E and F be Banach spaces such that E has the compact approximation property. Then, $\mathcal{P}_w(^m E; F)$ is τ_c-dense in $\mathcal{P}(^m E; F)$ for each $m \in \mathbb{N}$*

For the following, one may refer to [3], cf. also [1].

Proposition 2.14 *Let E and F be Banach spaces. Then, $\mathcal{P}_w(E; F) \subset \mathcal{P}_k(E; F)$.*

A sequence of subspaces $\{E_n\}_{n=1}^{\infty}$ of a Banach space E is called a *Schauder decomposition* of E if for each $x \in E$, there exists a unique sequence $\{x_n\}$ of vectors $x_n \in E_n$ for all n, such that

$$x = \sum_{n=1}^{\infty} x_n = \lim_{m\to\infty} u_m(x)$$

where the projection maps $\{u_m\}_{m=1}^{\infty}$ defined by $u_m(x) = \sum_{j=1}^{m} x_j,\ m \geq 1$ are continuous. Let $\mathcal{S} = \{(\alpha_n)\}_{n=1}^{\infty} : \alpha_n \in \mathbb{C},\ n \geq 1$ and $\limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \leq 1\}$. A Schauder decomposition $\{E_n\}_n$ is said to be $\mathcal{S}$-absolute if (i) for each $\beta = (\beta_j) \in \mathcal{S}$ and $x = \sum_{j=1}^{\infty} x_j \in E$, $\beta \cdot x = \sum_{j=1}^{\infty} \beta_j x_j \in E$ and (ii) if p is a continuous semi-norm on E and $\beta \in \mathcal{S}$, then $p_\beta(x) = \sum_{j=1}^{\infty} |\beta_j| p_\beta(x_j)$ defines a continuous semi-norm on E.

Following is proved in [15].

Proposition 2.15 *If $\{E_n\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition of the locally convex space E, then E has the CAP if and only if each E_n has the CAP.*

For more background and details about the theory of infinite dimensional holomorphy, Schauder decompositions, and the approximation properties, we refer to [5, 23, 24, 26, 37, 40, 43] and the reference given therein.

3 The Compact Approximation Property for $\mathcal{G}_v(U)$

This section is devoted to the study of the compact approximation property for $H_v(U)$ and its predual $G_v(U)$.

Let us begin with

Lemma 3.1 *Let v be a weight on an open subset U of a Banach space E such that $\mathcal{P}(E) \subset H_v(U)$. Then,*

$$\sup v(x)\|x\|^m < \infty$$

for each $m \in \mathbb{N}$.

Proof Let $m \in \mathbb{N}$. For each $x \in U$, choose $\phi_x \in E^*$ such that $\|\phi_x\| = 1$ and $\phi_x(x) = \|x\|$. Write $B = \{\phi_x^m : x \in U\}$. Then, B is a $\|.\|$-bounded subset of $\mathcal{P}(^m E)$. Hence, by Proposition 2.2, B is $\|.\|_v$-bounded. Consequently,

$$\sup_{x\in U} v(x)\|x\|^m \leq \sup_{x\in U}\sup_{y\in U} v(y)|\phi_x^m(y)| < \infty.$$

□

Theorem 3.2 *Let v be a radial weight on a balanced open subset U of a Banach space E such that $\mathcal{P}(E) \subset H_v(U)$. Then, the following assertions are equivalent:*

(i) *E has the compact approximation property.*
(ii) $\overline{\mathcal{P}_v(E,F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(U,F)$ *for each Banach space F.*
(iii) $\overline{\mathcal{P}_k(E,F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(U,F)$ *for each Banach space F.*
(iv) $\overline{\mathcal{H}_v^c(U;F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(U,F)$*, for each Banach space F.*
(v) *$\mathcal{G}_v(U)$ has the compact approximation property.*

Proof (i)$\Rightarrow$ (ii): Let $f \in \mathcal{H}_v(U;F)$ and p be a $\tau_{\mathcal{M}}$-continuous semi-norm on $\mathcal{H}_v(U,F)$. Then, there exist $P \in \mathcal{P}(E;F)$ such that $p(f-P) < \frac{\epsilon}{2}$ by Proposition 2.6. Write $P = P_0 + P_1 + \cdots P_m$, $P_j \in \mathcal{P}({}^jE,F)$, $0 \leq j \leq m$. Then, for each j, $0 \leq j \leq m$, there exist Q_j in $\mathcal{P}_w({}^jE,F)$, such that

$$p(P_j - Q_j) < \frac{\epsilon}{2m}.$$

by using Propositions 2.5 and 2.13. Write $Q = Q_0 + Q_1 + \cdots + Q_k$. Clearly, $Q \in \mathcal{P}_w(E,F)$ and $p(f-Q) < \epsilon$.
(ii) $\Rightarrow$(iii) follows by Proposition 2.14.
(iii) $\Rightarrow$ (iv): It is enough to show that $\mathcal{P}_k({}^jE;F) \subset \mathcal{H}_v^c(U;F)$ for each $j \in \mathbb{N}$. Consider $P \in \mathcal{P}_k({}^jE;F)$. By Lemma 3.1, $\sup v(x)\|x\|^j = K_j < \infty$. Hence, $v(U)P(U) \subset K_jP(B_E)$. consequently, $v(U)P(U)$ is relatively compact in F.
(iv) $\Rightarrow$ (v): Take $F = \mathcal{G}_v(U)$ in (iv). Then, by Theorem 2.3 and the hypothesis, $\Delta_v \in \overline{\mathcal{H}_v^c(U;G_v(U))}^{\tau_{\mathcal{M}}}$. Now, $\overline{\mathcal{H}_v^c(U;\mathcal{G}_v(U))}^{\tau_{\mathcal{M}}}$ can be identified with $\overline{\mathcal{K}(\mathcal{G}_v(U),\mathcal{G}_v(U))}^{\tau_c}$ via the map Ψ in view of Theorem 2.7 and Proposition 2.8(b). Since $T_{\Delta_v} \circ \Delta_v = \Delta_v$, $\Psi(\Delta_v) = I$, the identity map on $\mathcal{G}_v(U)$. Thus, $I \in \overline{\mathcal{K}(\mathcal{G}_v(U);\mathcal{G}_v(U))}^{\tau_c}$.
(v) $\Rightarrow$(i) follows by Propositions 2.4 and 2.11. □

Proposition 3.3 *For a weight v defined on an open subset U of a Banach space E, $\overline{\mathcal{K}(\mathcal{G}_v(U),F)}^{\tau_c} = \mathcal{L}(\mathcal{G}_v(U);F)$ if and only if $\overline{\mathcal{H}_v^c(U;F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(U;F)$ for each Banach space F.*

Proof Assume $\overline{\mathcal{K}(\mathcal{G}_v(U),F)}^{\tau_c} = \mathcal{L}(\mathcal{G}_v(U);F)$. Take $f \in \mathcal{H}_v(U;F)$. Then, by Theorem 2.3, $T_f \in \mathcal{L}(\mathcal{G}_v(U);F)$. By hypothesis, there exists a net $(T_\alpha) \subset \mathcal{K}(\mathcal{G}_v(U),F)$ such that $T_\alpha \xrightarrow{\tau_c} T_f$. Now, corresponding to each α, we have $f_\alpha \in \mathcal{H}_v^c(U;F)$ such that $T_{f_\alpha} = T_\alpha$ by Proposition 2.8(b). Using Theorem 2.7, we get $f_\alpha \xrightarrow{\tau_{\mathcal{M}}} f$. Hence, $\overline{\mathcal{H}_v^c(U;F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(U;F)$.

Conversely, for $T \in \mathcal{L}(\mathcal{G}_v(U),F)$, there exists $f \in \mathcal{H}_v(U,F)$ such that $T = T_f$ by Theorem 2.3. Consequently, by hypothesis, we can find a net $\{f_\alpha\} \subset \mathcal{H}_v^c(U;F)$ such that $f_\alpha \xrightarrow{\tau_{\mathcal{M}}} f$. Thus, $(T_{f_\alpha}) \subset \mathcal{K}(\mathcal{G}_v(U),F)$ by Proposition 2.8(b) and $T_\alpha \xrightarrow{\tau_c} T_f = T$ by Theorem 2.7. □

Writing $\mathcal{H}_K^\infty(V; E) \equiv \mathcal{H}_v^c(V; E)$ for $v \equiv 1$, the final result of this section characterizes the compact approximation property for the space E in terms of $\mathcal{H}_v^c(V; E)$, vis-à-vis $\mathcal{H}_K^\infty(V; E)$, as follows:

Theorem 3.4 *Let E be a Banach space. Then, for each Banach space F, the following are equivalent:*

(i) *E has the compact approximation property.*
(ii) $\overline{\mathcal{H}_v^c(V; E)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(V, E)$, *for each open subset V of F and weight v on V.*
(iii) $\overline{\mathcal{H}_K^\infty(V; E)}^{\tau_{\mathcal{M}}} = \mathcal{H}^\infty(V, E)$, *for each open subset V of F.*

Proof (i)⇒(ii): Assume that E has the compact approximation property. Then, by taking $F = \mathcal{G}_v(V)$ in Theorem 2.10(iii), $\overline{\mathcal{K}(\mathcal{G}_v(V), E)}^{\tau_c} = L(\mathcal{G}_v(V), E)$. Thus, $\overline{\mathcal{H}_v^c(V; E)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(V, E)$ by Proposition 3.2.

(iii): Follows from (ii) by taking $v \equiv 1$.

(iii)⇒(i): cf. Theorem 5 of [16]. □

4 The Topology $\tau'_{\mathcal{M}}$ on $\mathcal{H}_v(U; F)$

Analogous to the topology $\tau_{\mathcal{M}}$, we introduce another locally convex topology $\tau'_{\mathcal{M}}$ on $\mathcal{H}_v(U, F)$. It is generated by the family $\{q_{\overline{\alpha},\overline{A}} : \overline{\alpha} = (\alpha_j) \in c_0^+,\ \overline{A} = (A_j),\ A_j$ being finite subset of U for each $j\}$ of semi-norms given by

$$q_{\overline{\alpha},\overline{A}}(f) = \sup_{j\in\mathbb{N}}(\alpha_j \sup_{x\in A_j} v(x)\|f(x)\|).$$

Concerning this topology, we have

Proposition 4.1 *For a weight v on an open subset U of a Banach space E, we have:*

(i) $\tau_0 \leq \tau_{\mathcal{M}} \leq \tau'_{\mathcal{M}} \leq \tau_{\|.\|_v}$ *on* $\mathcal{H}_v(U, F)$.
(ii) $\tau'_{\mathcal{M}}|\mathcal{B} = \tau_0|\mathcal{B}$ *for any* $\|\cdot\|_v$*- bounded set* $\mathcal{B}$.

Proof (i) Clearly, $\tau_{\mathcal{M}} \leq \tau'_{\mathcal{M}}$. In view of (2), it suffices to prove $\tau'_{\mathcal{M}} \leq \tau_{\|.\|_v}$ which follows from the inequality, $q_{\overline{\alpha},\overline{A}}(f) \leq \|\overline{\alpha}\|_\infty\|f\|_v$.

(ii) The proof is analogous to the one given in [27]. However, for the sake of completeness, we outline the same. Let $\mathcal{B}$ be a bounded set in $(\mathcal{H}_v(U, F), \|\cdot\|_v)$. Then, there exists a constant $M > 0$ such that $\|f\|_v \leq M$, for every $f \in \mathcal{B}$. Consider a $\tau'_{\mathcal{M}}$ -continuous semi-norm q given by

$$q(f) = \sup_{j\in\mathbb{N}}(\alpha_j \sup_{x\in A_j} v(x)\|f(x)\|),\ f \in \mathcal{H}_v(U, F)$$

where $(\alpha_j) \in c_0^+$ and (A_j) is a sequence of finite subsets of U. Fix $\epsilon > 0$ arbitrarily. Then, there exists $m_0 \in \mathbb{N}$ such that

$$\alpha_j < \frac{\epsilon}{2M}, \quad \forall j > m_0.$$

Write $K = \bigcup_{j \leq m_0} A_j$. Then, K is a compact subset of U. Note that

$$\sup_{j \leq m_0} (\alpha_j \sup_{x \in A_j} v(x) \|(f-g)(x)\|) \leq \ L \ \|\overline{\alpha}\|_\infty p_K(f-g)$$

where $L = \sup_{x \in K} v(x)$. Thus,

$$p(f-g) < \epsilon \ \text{ whenever } p_K(f-g) < \delta$$

for $f, g \in \mathcal{B}$, where $\delta = \frac{\epsilon}{2L\|\overline{\alpha}\|_\infty}$. This completes the proof. □

For $f \in \mathcal{H}_v(U; F)$, let us define $S_n f(x) = \sum_{k=0}^{n} \frac{1}{m!} \hat{d}^m f(0)(x)$ and $C_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x)$. Then, $\|C_n(f)(x)\|_v \leq \|f\|_v$ for each $f \in \mathcal{H}_v(U; F)$ and $n \in \mathbb{N}$, cf. [27].

As a consequence of the above proposition, we derive the following result similar to Proposition 2.6

Proposition 4.2 *Let E and F be Banach spaces. For a radial weight v on a balanced open subset U of E with $\mathcal{P}(E) \subset \mathcal{H}_v(U)$, the space $\mathcal{P}(E; F)$ is $\tau'_{\mathcal{M}}$-dense in $\mathcal{H}_v(U; F)$.*

Proof Let $f \in \mathcal{H}_v(U, F)$. Then, the set $\{C_n(f) : n \in \mathbb{N}_0\}$ is a $\|\cdot\|_v$-bounded in $\mathcal{H}_v(U, F)$. As $C_n f \to f$ in $(H(U, F), \tau_0)$, the result follows by Proposition 4.1 (ii). □

Using the above proposition, we prove

Theorem 4.3 *Let v be a radial weight on a balanced open subset U of a Banach space E such that $\mathcal{P}(E) \subset \mathcal{H}_v(U)$. Then, for each Banach space F, the following are equivalent:*

(a) $v^{\frac{1}{i}-1} I_U \in \overline{\mathcal{H}_v^c(U; E)}^{\tau'_{\mathcal{M}}}$ for each $i \in \mathbb{N}$, where $I_U : U \to E$ is the inclusion mapping.
(b) $\overline{\mathcal{H}_v^c(U; F)}^{\tau'_{\mathcal{M}}} = \mathcal{H}_v(U, F)$.

Proof (a)⇒ (b): Let $f \in \mathcal{H}_v(U, F)$ and q be a $\tau'_{\mathcal{M}}$-continuous semi-norm given as

$$q(f) = \sup_{j \in \mathbb{N}} (\alpha_j \sup_{x \in A_j} v(x) \|f(x)\|)$$

where $(\alpha_j) \in c_0^+$ and (A_j) is a sequence of finite subsets of U. Then, by Proposition 4.2, there exists $P \in \mathcal{P}(E; F)$ such that

$$q(f-P) < \frac{\epsilon}{2} \tag{3}$$

Write $P = P_0 + P_1 + \cdots P_m$, $P_i \in \mathcal{P}(^i E, F)$, $0 \leq i \leq m$. Fix i, $1 \leq i \leq m$ arbitrarily. Define $K = \bigcup_{j\in\mathbb{N}} \{(\alpha_j \sup_{x\in A_j} v(x))^{\frac{1}{i}} y : y \in A_j\} \bigcup \{0\}$. Then, K is a compact subset of U, cf. [27, Proposition 4.4], and there exists a $\delta > 0$ such that

$$\|P_i(x) - P_i(y)\| < \frac{\epsilon}{2m}, \text{ for each } x \in K,\ y \in E \text{ with } \|x - y\| < \delta \tag{4}$$

Since $q_i(f) = \sup_{j\in\mathbb{N}}((\alpha_j)^{\frac{1}{i}} \sup_{x\in A_j} v(x)\|f(x)\|)$ is a $\tau'_{\mathcal{M}}$-continuous semi-norm, there exists $f_i \in \overline{\mathcal{H}^c_v(U; E)}^{\tau'_{\mathcal{M}}}$ by (a) such that

$$q_i(v^{\frac{1}{i}-1} I_U - f_i) = \sup_{j\in\mathbb{N}}((\alpha_j)^{\frac{1}{i}} \sup_{x\in A_j} v(x)\|v^{\frac{1}{i}-1}(x) - f_i(x)\|) < \delta \tag{5}$$

Let $g_i = v^{i-1} P_i \circ f_i$. Clearly $g_i \in \mathcal{H}^c_v(U; F)$. Note that

$$(\alpha_j \sup_{x\in A_j} v(x))^{\frac{1}{i}} \|x - v^{1-\frac{1}{i}}(x) f_i(x)\| \leq (\alpha_j)^{\frac{1}{i}} \sup_{x\in A_j} v(x)\|v^{\frac{1}{i}-1}(x) - f_i(x)\|.$$

Therefore, by (4) and (5), we have

$$q(P_i - g_i) = \sup_{j\in\mathbb{N}} \|P_i((\alpha_j \sup_{x\in A_j} v(x))^{\frac{1}{i}} x) - P_i(\alpha_j \sup_{x\in A_j} v(x))^{\frac{1}{i}} v^{1-\frac{1}{i}}(x) f_i(x))\| < \frac{\epsilon}{2m}.$$

Write $g = g_0 + g_1 + g_2 + \cdots + g_m$, where $g_0 = P_0$. Then, $g \in \mathcal{H}^c_v(U; F)$ with $p(f - g) < \epsilon$, thereby proving (b).

(b)⇒(a): Since $\sup_{x\in A_j} v^{\frac{1}{i}}(x)\|x\| < \infty$ for each $i \in \mathbb{N}$ by Lemma 3.1, $\|v^{\frac{1}{i}-1} I_U\|_v < \infty$. Thus, (a) follows. $\square$

Remark 4.1 (a). The above result is more general than [16, Theorem 5]; indeed, for $v \equiv 1$, $\tau_{\mathcal{M}} \equiv \tau'_{\mathcal{M}} \equiv \tau_\gamma$; (b). Since $\tau_{\mathcal{M}} \leq \tau'_{\mathcal{M}}$, $\overline{\mathcal{H}^c_v(U; F)}^{\tau_{\mathcal{M}'}} \subset \overline{\mathcal{H}^c_v(U; F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_v(U, F)$ and so the implication $(a) \Rightarrow (b)$ is true for $\tau_{\mathcal{M}}$ also. However, it would be interesting to know the non-constant weights for which $\tau_{\mathcal{M}} \equiv \tau'_{\mathcal{M}}$.

5 Weighted (LB)-Spaces and Approximation Properties

Let Λ be a directed set and $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a family of locally convex spaces such that for $\alpha \leq \beta$, $X_\alpha \subset X_\beta$, $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ and $I_{\alpha,\beta} : X_\alpha \to X_\beta$ be the continuous inclusion maps with $I_{\alpha,\beta} \circ I_\alpha = I_\beta$. In this chapter, we consider the *inductive limit* τ as the finest Hausdorff locally convex topology for which each inclusion map

$I_\alpha : X_\alpha \to X$ is continuous. We write $(X, \tau) = \varinjlim_{\alpha \in \Lambda} (X_\alpha, \tau_\alpha)$. If Λ is countable and each X_α is a Banach space, then (X, τ) is said to be an *(LB)-space*.

Let $\{(Y_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a family of locally convex spaces such that for each $\alpha \geq \beta$, $\pi_{\alpha,\beta} : Y_\alpha \to Y_\beta$ is continuous linear map and $\pi_{\alpha,\beta} \circ \pi_\beta = \pi_\alpha$, where π_α are the canonical mappings from $Y = \{(x_\alpha)_{\alpha \in \Lambda} : \pi_{\alpha,\beta}(x_\beta) = x_\alpha \text{ for each } \alpha \leq \beta\}$ to Y_α. The space Y endowed with the weakest topology on Y such that all the canonical mappings π_α are continuous is the *projective limit* of the above system and is written as $(Y, \tau) = \varprojlim_{\alpha \in \Lambda} (Y_\alpha, \tau_\alpha)$. A projective limit $(Y, \tau) = \varprojlim_{\alpha \in \Lambda} (Y_\alpha, \tau_\alpha)$ is said to be *reduced* if each $\pi_\alpha(Y)$ is dense in Y_α for each $\alpha \in \Lambda$.

Proposition 5.1 ([33]) *Let $(Y, \tau) = \varprojlim_{\alpha \in \Lambda} Y_\alpha$ be a reduced projective limit such that each Y_α has the approximation property. Then, Y has the approximation property.*

For the theory of projective and inductive limits, we refer to [30, 31, 33].

Let us now consider inductive limit of weighted spaces of holomorphic functions. Assume that $V = \{v_n\}$ is a countable decreasing family, i.e., $v_{n+1} \leq v_n$ for each n, of radial rapidly decreasing weights on E. Corresponding to V, inductive limit of weighted spaces is defined as $VH(E) = \bigcup_{n \geq 1} \mathcal{H}_{v_n}(E)$ endowed with the locally convex inductive topology $\tau_{\mathcal{I}}$. Since the closed unit ball B_{v_n} of each $\mathcal{H}_{v_n}(E)$ is τ_0 compact, $VH(E)$ is complete by Mujica's completeness theorem, namely,

Theorem 5.2 ([39]) *Let $(E, \tau) = \varinjlim_{n \in \mathbb{N}} E_n$ be an (LB)-space, and suppose that there exists a locally convex Hausdorff topology $\tilde{\tau} < \tau$ on E such that the closed unit ball B_n of each E_n is $\tilde{\tau}$-compact. Then,*

$$F = \{u \in E' : u|B_n \textit{ is } \tilde{\tau}\textit{-continuous for each } n \in \mathbb{N}\}$$

endowed with the topology of uniform convergence on the sets B_n, is a Fréchet space such that the evaluation mapping $J : E \to F'$ given by $J(x)(u) = u(x)$ for each $x \in E$ and $u \in F$, is a topological isomorphism from E onto F_i' (the inductive dual of F) and hence E must be complete.

The predual of $VH(E)$ defined as

$$VG(E) = \{\phi \in VH(E)' : \phi|B_{v_n} \text{ is } \tau_0\text{-continuous for each } n \in \mathbb{N}\ \}$$

is endowed with the topology of uniform convergence on the sets B_{v_n}. Also, $VG(E) = \varprojlim_{n \in \mathbb{N}} G_{v_n}(E)$ is a reduced projective limit, cf. [6]. Combining this fact with Propositions 2.9 and 5.1, we get

Theorem 5.3 *Let $V = \{v_n\}$ denote a countable decreasing family of radial rapidly decreasing weights on E. Then, E has the approximation property if and only if $VG(E)$ has the approximation property.*

Following [6], a family V of radial rapidly decreasing weights satisfies *condition* (A) if for each $m \in \mathbb{N}$, there exist $D > 0$, $R > 1$, and $n \in \mathbb{N}$, $n \geq m$ such that

$$\|P^j f(0)\|_n \leq \frac{D}{R^j}\|f\|_m$$

for each $j \in \mathbb{N}$ and $f \in \mathcal{H}_{v_m}(U)$.

For the final result of this section, we make use of the following result proved in [28]

Theorem 5.4 *If $V = \{v_n\}$ is a family of weights satisfying condition (A), then the sequence of spaces $\{\mathcal{Q}^m E)\}_{m=1}^{\infty}$ forms an $\mathcal{S}$-absolute decomposition for $VG(E)$ with respect to the topology of uniform convergence on B_{v_n}'s for each n.*

Finally, we have

Theorem 5.5 *Let $V = \{v_n\}$ denote a countable decreasing family of radial rapidly decreasing weights on E satisfying condition (A). Then, E has the compact approximation property if and only if $VG(E)$ has the compact approximation property.*

Proof It follows directly from Propositions 2.12, 2.15 and Theorem 5.4. □

Note 5.1 As Proposition 5.1 is not known to be true for the compact approximation property, Theorem 5.5 cannot be derived for the family V which does not satisfy condition (A). However, for $V = \{v\}$, the above result holds, though the singleton family of weights does not satisfy condition (A), cf. [28, Remark 4.4].

References

1. R.M. Aron, J.B. Prolla, Polynomial approximation of differentiable functions on Banach spaces. J. Reine Angew. Math. **313**, 195–216 (1980)
2. R.M. Aron, M. Schottenloher, Compact holomorphic mappings and the approximation property. J. Funct. Anal. **21**, 7–30 (1976)
3. R.M. Aron, C. Hervés, M. Valdivia, Weakly continuous mappings on Banach spaces. J. Funct. Anal. **52**, 189–204 (1983)
4. S. Banach, Theorie des Operations Lineaires, Warszawa (1932)
5. J.A. Barroso, *Introduction to Holomorphy*, vol. 106, North-Holland Mathematics Studies (North-Holland, Amsterdam, 1985)
6. M.J. Beltran, Linearization of weighted (LB)-spaces of entire functions on Banach spaces. Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM **106**(1), 275–286 (2012)
7. M.J. Beltran, Operators on weighted spaces of holomorphic functions. Ph.D. thesis, Universitat Politècnica de València, València (2014)
8. S. Berrio, G. Botelho, Ideal topologies and corresponding approximation properties. Ann. Acad. Sc. Fenn. Math. **41**, 265–285 (2016)
9. K.D. Bierstedt, Gewichtete Raume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt II. J. Reine Angew. Math. **260**, 133–146 (1973)
10. K.D. Bierstedt, J. Bonet, Biduality in Frechet and (LB)-spaces, in *Progress in Functional Analysis*, ed. by K.D. Bierstedt et al. North Holland Mathematics Studies, vol. 170 (North Holland, Amsterdam, 1992), pp. 113–133

11. K.D. Bierstedt, W.H. Summers, Biduals of weighted Banach spaces of analytic functions. J. Aust. Math. Soc. Ser. A **54**, 70–79 (1993)
12. K.D. Bierstedt, R.G. Miese, W.H. Summers, A projective description of weighted inductive limits. Trans. Am. Math. Soc. **272**, 107–160 (1982)
13. K.D. Bierstedt, J. Bonet, A. Galbis, Weighted spaces of holomorphic functions on balanced domains. Mich. Math. J. **40**, 271–297 (1993)
14. C. Boyd, S. Dineen, P. Rueda, Weakly uniformly continuous holomorphic functions and the approximation property. Indag. Math. (N.S.) **12**, 147–156 (2001)
15. E. Caliskan, Approximation of holomorphic mappings on infinite dimensional spaces. Rev. Mat. Complut. **17**, 411–434 (2004)
16. E. Caliskan, Bounded holomorphic mappings and the compact approximation property in Banach spaces. Port. Math. **61**, 25–33 (2004)
17. E. Caliskan, The bounded approximation property for the predual of the space of bounded holomorphic mappings. Studia Math. **177**, 225–233 (2006)
18. E. Caliskan, The bounded approximation property for spaces of holomorphic mappings on infinite dimensional spaces **279**, 705–715 (2006)
19. E. Caliskan, The bounded approximation property for weakly uniformly continuous type holomorphic mappings. Extracta Math. **22**, 157–177 (2007)
20. P.G. Casazza, Approximation properties, in *Handbook of the Geometry of Banach Spaces*, vol. 1, ed. by W.B. Johnson, J. Lindenstrauss (Elsevier, Amsterdam, 2001), pp. 271–316
21. C. Choi, J. Kim, Weak and quasi approximation properties in Banach spaces. J. Math. Anal. Appl. **316**, 722–735 (2006)
22. M. Delgado, C. Pineiro, An approximation property with respect to an operator ideal. Studia Math. **214**, 67–75 (2013)
23. S. Dineen, *Complex Analysis in Locally Convex Spaces*. North-Holland Mathematics Studies, vol. 57 (North-Holland, Amsterdam, 1981)
24. S. Dineen, *Complex Analysis on Infinite Dimensional Spaces* (Springer, London, 1999)
25. D. Garcia, M. Maestre, P. Rueda, Weighted spaces of holomorphic functions on Banach spaces. Studia Math. **138**(1), 1–24 (2000)
26. A. Grothendieck, Produits tensoriels topologiques et espaces nuclaires. Mem. Am. Math. Soc. **16** (1955)
27. M. Gupta, D. Baweja, Weighted spaces of holomorphic functions on Banach spaces and the approximation property. Extracta Math. (to appear)
28. M. Gupta, D. Baweja, The bounded approximation property for weighted (LB)-spaces of holomorphic mappings on Banach spaces, preprint
29. M. Gupta, D. Baweja, The bounded approximation property for the weighted spaces of holomorphic mappings on Banach spaces. Glas. Math. J. (to appear)
30. J. Horvath, *Topological Vector Spaces and Distributions* (Addison-Wesley, London, 1966)
31. H. Jarchow, *Locally Convex Spaces* (B.G. Teubner, Stuttgart, 1981)
32. E. Jorda, Weighted vector-valued holomorphic functions on Banach spaces, *Abstract and Applied Analysis*, vol. 2013 (Hindawi Publishing Corporation, 2013)
33. G. Köthe, Topological vector spaces. II, *Grundlehren der Mathematischen Wissenschaften*, vol. 237 (Springer, New York, 1979)
34. S. Lassalle, P. Turco, On p-compact mappings and the p-approximation property. J. Math. Anal. Appl. **389**, 1204–1221 (2012)
35. A. Lima, E. Oja, The weak approximation property. Math. Ann. **333**, 471–484 (2005)
36. A. Lima, V. Lima, E. Oja, Bounded approximation properties via integral and nuclear operators. Proc. Am. Math. Soc. **138**, 287–297 (2010)
37. J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces. I. Ergeb. Math. Grenzgeb., Bd., vol. 92 (Springer, Berlin, 1977)
38. P. Mazet, *Analytic Sets in Locally Convex Spaces*. North-Holland Mathematics Studies, vol. 89 (North-Holland Publishing Co, Amsterdam, 1984)
39. J. Mujica, A completeness criteria for inductive limits of Banach spaces, *Functional Analysis: Holomorphy and Approximation Theory II*. North-Holland Mathematics Studies, vol. 86 (North-Holland, Amsterdam, 1984), pp. 319–329

40. J. Mujica, *Complex Analysis in Banach Spaces*. North-Holland Mathematics Studies, vol. 120 (North-Holland, Amsterdam, 1986)
41. J. Mujica, Linearization of bounded holomorphic mappings on Banach spaces. Trans. Am. Math. Soc. **324**(2), 867–887 (1991)
42. J. Mujica, M. Valdivia, Holomorphic germs on Tsirlson's space. Proc. Am. Math. Soc. **123**, 1379–1384 (1995)
43. L. Nachbin, *Topology on Spaces of Holomorphic Mappings* (Springer, New York, 1969)
44. K.F. Ng, On a theorem of Dixmier. Math. Scand. **29**, 279–280 (1971)
45. E. Oja, The strong approximation property. J. Math. Anal. Appl. **338**(1), 407–415 (2008)
46. L.A. Rubel, A.L. Shields, The second duals of certain spaces of analytic functions. J. Aust. Math. Soc. **11**, 276–280 (1970)
47. P. Rueda, On the Banach Dieudonne theorem for spaces of holomorphic functions. Quaest. Math. **19**, 341–352 (1996)
48. R. Ryan, Applications of topological tensor products to infinite dimensional holomorphy. Ph.D. thesis, Trinity College, Dublin (1980)
49. M. Schottenloher, ϵ-product and continuation of analytic mappings. Analyse Funcionelle et applications (Hermann, Paris, 1975), pp. 261–270
50. G. Willis, The compact approximation property does not imply the approximation property. Studia Math. **103**, 99–108 (1992)

Bloch Mappings on Bounded Symmetric Domains

Tatsuhiro Honda

Abstract We introduce Bloch mappings on bounded symmetric domains which can be infinite dimensional and generalize Bonk's distortion theorem on $\mathbb{C}$ to locally biholomorphic Bloch mappings on finite dimensional bounded symmetric domains. As an application, we give a lower bound of the Bloch constant for these locally biholomorphic Bloch mappings. Finally, we show that there exist no isometric composition operators from the space $H^\infty(\mathbb{B}_X)$ of bounded and holomorphic functions on $\mathbb{B}_X$ into the α-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$ on $\mathbb{B}_X$.

Keywords Bloch mapping · Bounded symmetric domain · JB*-triple

1 Introduction

Let $\mathbb{U}$ be the unit disk in $\mathbb{C}$. The Bloch theorem states that a holomorphic function $f : \mathbb{U} \to \mathbb{C}$ with $f'(0) = 1$ maps a domain in $\mathbb{U}$ biholomorphically onto a disk with radius $r(f)$ greater than some positive absolute constant. The 'best possible' constant $\mathbf{B}$ for all such functions, that is, $\mathbf{B} = \inf\{r(f) : f$ is holomorphic on $\mathbb{U}$ and $f'(0) = 1\}$, is called the Bloch constant. The classical Bloch space $\mathcal{B}$ is the space of holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ satisfying $\|f\|_{Bloch} := \sup_{z\in\mathbb{U}}(1 - |z|^2)|f'(z)| < \infty$ endowed with the norm $\|f\|_{\mathcal{B}} := |f(0)| + \|f\|_{Bloch} < \infty$ so that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ becomes a Banach space.

The concept of a Bloch function has been extended to various complex domains in higher dimensions. Hahn [22] first introduced the notion of a $\mathbb{C}^n$-valued Bloch mapping on a finite dimensional bounded homogeneous domain, under the name 'of normal mapping of finite order.' Timoney [52] gave several equivalent definitions for $\mathbb{C}$-valued Bloch functions on a finite dimensional bounded homogeneous domain. Blasco et al. [5] extended to infinite dimensional Hilbert balls, where a Hilbert ball is the open unit ball of a Hilbert space and is a rank one bounded symmetric domain. Chu et al. [17] characterize Bloch functions on bounded symmetric domains, which

T. Honda (✉)
Hiroshima Institute of Technology, Hiroshima 731-5193, Japan
e-mail: thonda@cc.it-hiroshima.ac.jp

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_3

may be infinite dimensional, by extending several well-known equivalent conditions for Bloch functions on the open unit disk U in $\mathbb{C}$.

Bonk [6] proved the following distortion theorem:

$$\Re f'(z) \geq \frac{\sqrt{3} - |z|}{\left(1 - \frac{|z|}{\sqrt{3}}\right)^3} \text{ for } |z| \leq \frac{1}{\sqrt{3}}$$

which implies readily a result of Ahlfors [1] that the Bloch constant is greater than $\sqrt{3}/4$. Bonk's distortion theorem has been extended by Liu in [40, Theorem 7] to the class $H_{loc}(\mathbb{B}^n, \mathbb{C}^n)$ of $\mathbb{C}^n$-valued locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. For the class $H_{\text{loc}}(\mathbb{U}^n, \mathbb{C}^n)$ of locally biholomorphic mappings on the unit polydisk $\mathbb{U}^n$ in $\mathbb{C}^n$, the following distortion theorem has been shown by Wang and Liu [53, Theorem 3.2].

Ohno [45] investigated the weighted composition operators from the Hardy space H^∞ to the Bloch space on the unit disk in $\mathbb{C}$. Li and Stević [38, 39], Zhang and Chen [58] studied weighted composition operators from H^∞ to the α-Bloch space. Allen and Colonna [3] characterized the bounded weighted composition operators from H^∞ to the Bloch space of a bounded homogeneous domain and derived operator norm estimates. Colonna et al. [18] obtained sharper estimates on the operator norm of the multiplication operators from H^∞ to the Bloch space on a general bounded symmetric domain and determined such norm precisely in the case when the symbol of the operator fixes the origin as well as when the domain is the Euclidean ball or a bounded symmetric domain that has the unit disk as a factor, up to a biholomorphic transformation, and the symbol is not subjected to any restriction. They used this norm to show that for a large class of bounded symmetric domains D, there are no isometries among these multiplication operators.

In this chapter, we generalize the above results for Bloch mappings to any bounded symmetric domain in $\mathbb{C}^n$ realized as the unit ball $\mathbb{B}_X$ of an n-dimensional JB*-triple X. Kaup [35] showed that the bounded symmetric domains in complex Banach spaces are exactly the open unit balls of JB*-triples which are complex Banach spaces equipped with a Jordan triple structure. Note that a complex Banach space is a JB*-triple if, and only if, its open unit ball is homogeneous. All four types of classical Cartan domains are the open unit balls of JB*-triples, and the same holds for any finite product of these domains ([32], see also [33]). Therefore, the open unit balls of JB*-triples can be regarded as higher-dimensional generalizations of the open unit disk in the complex plane and a natural extension of the finite dimensional distortion theorems should be the ones on the open unit ball $\mathbb{B}_X$ of a finite dimensional JB*-triple X. Recently, Hamada and Kohr [31] gave a definition of α-Bloch mappings on $\mathbb{B}_X$ which is a generalization of α-Bloch functions on the unit disk in $\mathbb{C}$ by using the Bergman operator of the underlying JB*-triple (Definition 3.7). When $\alpha = 1$, it is equivalent to the definition of Bloch mappings on $\mathbb{B}^n$ by Liu [40] (see also [24]). By using the Jordan theory, we can generalize several results on α-Bloch functions on the unit disk in $\mathbb{C}$ to α-Bloch mappings on any bounded symmetric domain in $\mathbb{C}^n$.

2 Preliminaries

Let B_X be the unit ball of a complex Banach space X. Let Y be a complex Banach space. A holomorphic mapping $f : B_X \to Y$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B_X$. A holomorphic mapping $f : B_X \to Y$ is said to be biholomorphic if $f(B_X)$ is a domain in Y, f^{-1} exists, and holomorphic on $f(B_X)$. Let $L(X, Y)$ denote the set of continuous linear operators from X to Y. Let I_X be the identity in $L(X) = L(X, X)$.

Extending É. Cartan's [9] classification of finite dimensional bounded symmetric domains, it has been shown in [35] that every bounded symmetric domain, including the infinite dimensional ones, is biholomorphic to the open unit ball of a JB*-triple. A JB*-triple is a complex Banach space X equipped with a continuous Jordan triple product

$$(x, y, z) \in X \times X \times X \mapsto \{x, y, z\} \in X$$

satisfying

(J$_1$) $\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,
(J$_2$) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
(J$_3$) $x \square x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,
(J$_4$) $\|\{x, x, x\}\| = \|x\|^3$

where for $x, y \in X$, the *box operator* $x \square y : X \to X$ is defined by $x \square y(\cdot) = \{x, y, \cdot\}$ and (J$_2$) is called the *Jordan triple identity*.

Example 2.1 (i) A complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ is a JB*-triple in the triple product

$$\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x).$$

(ii) The unit polydisk U^n is the unit ball of the JB*-triple with the triple product

$$\{x, y, z\} = (x_i \overline{y_i} z_i)_{1 \le i \le n}, \quad x = (x_i),\ y = (y_i),\ z = (z_i) \in \mathbb{C}^n.$$

We refer to [14, 41, 49] for relevant details of JB*-triples and references. We recall some of them which will be needed in later.

An element $u \in X$ is called a tripotent if $\{u, u, u\} = u$. Two tripotents u and v are said to be orthogonal if $D(u, v) = 0$, where $D(u, v) = 2u \square v$. Orthogonality is a symmetric relation. A tripotent u is said to be maximal if the only tripotent which is orthogonal to u is 0. A tripotent u is said to be minimal if it cannot be written as a sum of two nonzero orthogonal tripotents. A frame is a maximal family of pair-wise orthogonal, minimal tripotents. The cardinality of all frames is the same and is called the rank r of X. A real subspace S of X is called a flat subspace of X if S is a real triple subsystem of X:

$$(x, y, z \in S) \Longrightarrow \{x, y, z\} \in S$$

and

$$\{x, y, z\} = \{y, x, z\} \quad \text{for } x, y, z \in S.$$

By [49, Proposition VI.3.2], the cardinality of the basis of all maximal flat subspaces is the same and equal to the rank r. A subspace I of X is called a triple ideal if $\{X, X, I\} + \{X, I, X\} \subset I$. A JB*-triple is simple if it has no nontrivial (norm) closed triple ideals.

Let B be the unit ball of a JB*-triple X. Then, for each $a \in B$, the Möbius transformation g_a defined by

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x \,\square\, a)^{-1}x, \tag{1}$$

is a biholomorphic mapping of B onto itself with $g_a(0) = a$, $g_a(-a) = 0$, and $g_{-a} = g_a^{-1}$.

Let $\dim X < \infty$. A point $u \in \overline{B}_X$ is said to be an extreme point of $\overline{B}_X$ if the only $x \in X$ satisfying $\|u + \lambda x\| \leq 1$ for all real numbers λ with $|\lambda| \leq 1$ is $x = 0$. Let $\mathcal{E}$ be the set of all extreme points of $\overline{B}_X$. By the Krein-Milman theorem (see e.g., [23, Chapter 4]), $\mathcal{E}$ is nonempty, since $\overline{B}_X$ is a compact subset of X. A subset Γ of $\overline{B}_X$ is called the Bergman-Shilov boundary of B_X if Γ is the smallest closed subset of $\overline{B}_X$ where every continuous function on $\overline{B}_X$ which is holomorphic on B_X attains its maximum absolute value.

Let $H^2(B_X)$ be the Bergman space of holomorphic functions on B_X which are square-integrable with respect to the Lebesgue measure on B_X. Let $k(z, \overline{w})$ be the Bergman kernel of B_X, that is, the reproducing kernel of the Hilbert space $H^2(B_X)$. The Bergman metric at $x \in B_X$ is defined by

$$h_x(u, v) = \partial_u \overline{\partial}_v \log k(x, \overline{x}).$$

For $x \in X$, $h_0(x, x)^{1/2}$ is called the Euclidean norm on X.

Let $(X, \|\cdot\|)$ be a JB*-triple, and let $H(B_X)$ denote the set of holomorphic mappings from B_X into $\mathbb{C}^n$. Let $\|\cdot\|_e$ denote the Euclidean norm on $\mathbb{C}^n$. For $A \in L(X, \mathbb{C}^n)$, let

$$\|A\|_{X,e} = \sup\{\|Az\|_e : \|z\| = 1\}.$$

and

$$\|A\|_e = \sup\{\|Az\|_e : \|z\|_e = 1\}.$$

We refer to [14, Theorem 3.2.3] for the proof of the following result which was due to several authors [32, 37, 41].

Proposition 2.2 *Let B_X be the unit ball of a JB*-triple X. Then, the Bergman-Shilov boundary Γ of B_X coincides with each of the following sets:*

(i) the set of maximal tripotents of X;

(ii) the set of extreme points of $\overline{B}_X$.

Further, if $\dim X < \infty$*, then these sets also coincide with the set of points of maximum Euclidean norm in* $\overline{B}_X$.

The last assertion above was due to Hamada et al. [27, Proposition 2.4].

Now, let X be a finite dimensional JB*-triple and we recall the constant $c(B_X)$ which was defined in [28]. Let h_0 be the Bergman metric on X at 0 and let

$$c(B_X) = \frac{1}{2} \sup_{x,y \in B_X} |h_0(x, y)|.$$

Let u be an arbitrary maximal tripotent in X. By Proposition 2.2, we have

$$c(B_X) = \frac{1}{2} h_0(u, u).$$

Since $\operatorname{tr} D(y, a) = h_0(y, a)$ by [41, Theorem 2.10], where $D(y, a) = 2\{y, a, \cdot\}$, we have

$$c(B_X) = \frac{1}{2} \operatorname{tr} D(u, u).$$

Let

$$X = V_0(u) \oplus V_1(u) \oplus V_2(u)$$

be the Peirce decomposition of X, where $V_j(u)$ is the eigenspace of $D(u, u)$ with the eigenvalue j for $j = 0, 1, 2$. Then, we have

$$c(B_X) = \frac{1}{2} (\dim V_1(u) + 2 \dim V_2(u)).$$

Since $V_0(u) = 0$ by [49, Proposition VI.2.4 (iii)], we have

$$c(B_X) = \frac{1}{2} (\dim X + \dim V_2(u)). \tag{2}$$

Moreover, u can be included in a maximal flat subspace S with basis of orthogonal tripotents $u_1, \ ldots, u_r$ such that $u = u_1 + \cdots + u_r$, where r is the rank of X. Let

$$X = \bigoplus_{0 \le i \le j \le r} V_{ij}(\mathbf{u})$$

be the Peirce decomposition with respect to $\mathbf{u} = (u_1, \ldots, u_r)$, where

$$V_{ij}(\mathbf{u}) = \{v \in X : D(u_l, u_l)v = (\delta_{li} + \delta_{lj})v, 1 \le l \le r\},$$

for $(i, j) \ne (0, 0)$ and $V_{00}(\mathbf{u}) = \{0\}$. Then by [49, p.504], we have

$$V_1(u) = \bigoplus_{1 \le j \le r} V_{0j}(\mathbf{u}), \quad V_2(u) = \bigoplus_{1 \le i \le j \le r} V_{ij}(\mathbf{u})$$

Since $e_i \in V_{ii}(\mathbf{u})$, dim $V_2(u) \ge r$. Therefore, from (2), we have

$$c(B_X) \ge \frac{\dim X + r}{2} \ge \frac{\dim X + 1}{2}.$$

Assume that X is simple. Then, $V_{ij}(\mathbf{u})$ $(1 \le i < j \le r)$ have the same dimension a, $V_{ii}(\mathbf{u})$ $(1 \le i \le r)$ have the same dimension 1, and $V_{0j}(\mathbf{u})$ $(1 \le j \le r)$ have the same dimension b by [49, Theorem VI.3.5]. Therefore, we have

$$\dim V_1(u) = br, \quad \dim V_2(u) = r + \frac{r(r-1)}{2}a.$$

and thus,

$$c(B_X) = \frac{1}{2} r g,$$

where $g = 2 + a(r-1) + b$ is the genus of X.

3 Bloch Mappings

The concept of a Bloch mapping on a finite dimensional bounded symmetric domain was first introduced by Hahn [22]. The following definition of Bloch mappings for dimension free bounded symmetric domains is the same as the one given in [40, 51] which is equivalent to Hahn's definition in finite dimensions.

For each $z_0 \in \mathbb{B}_X$, we define a family $F_f(z_0)$ of functions on $\mathbb{B}_X$ by

$$F_f(z_0) = \{f \circ g - (f \circ g)(z_0) : g \in \mathrm{Aut}(\mathbb{B}_X)\}.$$

We recall that a family $\mathcal{F} \subset H(U, \mathbb{C})$ is called *normal* if every sequence in $\mathcal{F}$ admits a subsequence which converges uniformly on compact subsets of U. A classical result states that $\mathcal{F}$ is normal if and only if it is uniformly bounded on compact sets in U (cf. [2, p. 216]). The following theorem is due to [17].

Theorem 3.1 *Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple X and let $f \in H(\mathbb{B}_X, \mathbb{C})$. The following conditions are equivalent:*

(1) f is a Bloch function.
(2) The radii of the schlicht disks in the range of f are bounded above.
(3) f is uniformly continuous as a function from the metric space $(\mathbb{B}_X, \rho)$ to the metric space $(\mathbb{C}$, Euclidean distance$)$.
(4) The family $F_f(z_0)$ is bounded on $\mathbb{B}_X(0, r)$ for $0 < r < 1$ and $z_0 \in \mathbb{B}_X$.
(5) $\|f\|_{\mathcal{B}(\mathbb{B}_X),s} < \infty$.

(6) *The family* $\{f \circ h : h \in H(U, \mathbb{B}_X)\}$ *consists of Bloch functions on* U *with uniformly bounded Bloch semi-norm.*
(7) *The family* $\{f \circ h - (f \circ h)(0) : h \in H(U, \mathbb{B}_X)\}$ *is normal.*

Theorem 3.2 *Let* B_X *be the unit ball of a JB*-triple* X. *If* f *is a Bloch mapping on* B_X, *then we have*

$$\|Df(z)\|_{X,e} \leq \frac{\|f\|_{\mathcal{B}}}{1-\|z\|^2}, \quad z \in B_X.$$

Proof Let $z \in B_X \setminus \{0\}$ be fixed and let $g_z \in \mathrm{Aut}(B_X)$ be the M̈obius transformation such that $g_z(0) = z$ and $g_z^{-1} = g_{-z}$.

By [14, Corollary 3.2.14] (see also [26, 37]),

$$\|g_{-z}(z)\| = \frac{1}{1-\|z\|^2}.$$

Therefore, we have

$$\|Df(z)\|_{X,e} \leq \|D(f \circ g_z)(0)\|_{X,e}\, \|Dg{-}z(z)\| \frac{\|f\|_{\mathcal{B}}}{1-\|z\|^2}.$$

□

Let $\beta(K)$ denote the set of Bloch mappings f with $\|f\|_{\mathcal{B}} \leq K$, where $1 \leq K \leq +\infty$.

Definition 3.3 Let B_X be the unit ball of an n-dimensional JB*-triple X. We define the *prenorm* $\|f\|_0$ of $f \in H(B_X)$ by

$$\|f\|_0 = \sup\left\{(1-\|z\|^2)^{c(B_X)/n} |\det Df(z)|^{1/n} : z \in B_X\right\}.$$

The following lemmas are obtained by the first author [24].

Lemma 3.4 *Let* B_X *be the unit ball of an* n*-dimensional JB*-triple* X.

(i) If f *is a Bloch mapping on* B_X, *then we have*

$$\|Df(z)\|_{X,e} \leq \frac{\|f\|_{\mathcal{B}}}{1-\|z\|^2}, \quad z \in B_X.$$

(ii) If f *is a Bloch mapping, then we have*

$$\|f\|_0 \leq \sup\left\{|\det Dg(0)|^{1/n} : g \in F_f\right\} < +\infty.$$

(iii) If $\|f\|_0 < +\infty$, *then*

$$|\det Df(z)| \leq \frac{\|f\|_0^n}{(1-\|z\|^2)^{c(B_X)}}, \quad z \in B_X.$$

(iv) If $\|f\|_0 = 1$ *and* $\det Df(0) = 1$*, then* $|\det Df(z)| = 1 + o(\|z\|)$.

When the target is the unit disk in $\mathbb{C}$, Chu et al. [17] obtained the following Schwarz Pick Lemma (cf. [5, Theorem 4.2], [30, Theorem 4.6]).

Lemma 3.5 *Let* $f \in H^\infty(\mathbb{B}_X)$ *be such that* $\|f\|_\infty \le 1$*. Then, we have*

$$\|Df(z)\|_{X,e} \le \frac{1 - |f(z)|^2}{1 - \|z\|_X^2}, \quad z \in \mathbb{B}_X.$$

For $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^* : \ l_x(x) = \|x\|, \ \|l_x\| = 1\}.$$

Then, $T(x) \neq \emptyset$ in view of the Hahn-Banach theorem. Let $H(U)$ denote the set of holomorphic functions on the unit disk U in $\mathbb{C}$.

Lemma 3.6 *Let* $u \in \partial B_X$ *be fixed and let*

$$f(z) = \left(\int_0^{l_u(z)} \psi(t)dt\right) u + z - l_u(z)u, \quad z \in B_X,$$

where $l_u \in T(u)$ *and* $\psi \in H(U)$*. Then,* $f \in H(B_X)$*,* $f(0) = 0$*, and* $\det Df(z) = \psi(l_u(z))$ *for* $z \in B_X$.

The following definition is given by Hamada and Kohr [31].

Definition 3.7 Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X, and let $\alpha > 0$. A mapping $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is called an α-Bloch mapping if

$$\|f\|_\alpha + \|f(0)\|_e < +\infty,$$

where $\|f\|_\alpha$ denotes the α-Bloch semi-norm of f defined by

$$\|f\|_\alpha = \sup_{z \in \mathbb{B}_X} \|Df(z) \circ B(z,z)^{\alpha/2}\|_{X,e}.$$

Let $\mathcal{B}^\alpha_{X,n}(\mathbb{B}_X)$ be the space of α-Bloch mappings $f : \mathbb{B}_X \to \mathbb{C}^n$. We note that the space $\mathcal{B}^\alpha_{X,n}(\mathbb{B}_X)$ is a complex Banach space with respect to the norm $\|\cdot\|_{\mathcal{B}^\alpha_n}$ given by

$$\|f\|_{\mathcal{B}^\alpha_n} = \|f\|_\alpha + \|f(0)\|_e, \quad f \in \mathcal{B}^\alpha_{X,n}(\mathbb{B}_X).$$

Remark 3.8 (i) α-Bloch mappings on $\mathbb{B}_X$ are also β-Bloch mappings on $\mathbb{B}_X$ for $\alpha \le \beta$.

Indeed, since

$$\|Df(z) \circ B(z,z)^{\beta/2}\|_{X,e} \le \|f\|_\alpha \sup_{z \in \mathbb{B}_X} \|B(z,z)^{(\beta-\alpha)/2}\|_{X,e} \le \|f\|_\alpha, \quad z \in \mathbb{B}_X,$$

the conclusion follows.

Since 1-Bloch mappings are equivalent to Bloch mappings ([24], cf. [51]), it follows that any Bloch mapping is also an α-Bloch mapping, for $\alpha \geq 1$.

(ii) Taking into account the Cauchy integral formula for vector-valued holomorphic mappings, the bounded mappings in $H(\mathbb{B}_X, \mathbb{C}^n)$ are Bloch mappings, so they are also α-Bloch mappings for $\alpha \geq 1$.

(iii) In view of Lemma 3.9 (i), it follows that α-Bloch mappings are bounded on $\mathbb{B}_X$ for $\alpha \in (0, 1)$.

Hamada and Kohr [31] proved the following generalization of [24, Lemma 2.8] to the case of α-Bloch mappings.

Lemma 3.9 *Let $\mathbb{B}_X$ be the unit ball of an n-dimensional JB*-triple X. If $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is an α-Bloch mapping, then we have*

$$\|Df(z)\|_{X,e} \leq \frac{\|f\|_\alpha}{(1 - \|z\|_X^2)^\alpha}, \quad z \in \mathbb{B}_X.$$

In the case that $\mathbb{B}_X$ is the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, Chen et al. [13] gave another definition of α-Bloch mappings on $\mathbb{B}^n$ in $\mathbb{C}^n$ as following.

Definition 3.10 Let $\mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$, and let $\alpha > 0$. Let $f \in H(\mathbb{B}^n, \mathbb{C}^n)$. We say that f is an α-Bloch mapping in the sense of Chen, Ponnusamy, and Wang if

$$\|f(0)\|_e + \sup_{z \in \mathbb{B}^n} (1 - \|z\|_e^2)^\alpha \|Df(z)\|_{X,e} < \infty.$$

Remark 3.11 (i) Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X, and let $\alpha > 0$. Let $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ be an α-Bloch mapping. Then, Lemma 3.9 (i) holds. Thus, if f is an α-Bloch mapping in the sense of Definition 3.7, then f is also an α-Bloch mapping in the sense of Definition 3.10.

(ii) Let $f = (f_1, \ldots, f_n) \in H(\mathbb{B}_X, \mathbb{C}^n)$. Then, f is an α-Bloch mapping if and only if each f_j is an α-Bloch function in the sense of Definition 3.7. Now, if $\alpha = 1$, then f_j is a Bloch function on $\mathbb{B}_X$ if and only if $\|D(f_j \circ g)(0)\|_{X,e}$ is uniformly bounded for $g \in \mathrm{Aut}(\mathbb{B}_X)$. In particular, if $\mathbb{B}_X = \mathbb{B}^n$, then f_j is a Bloch function if and only if

$$\sup_{z \in \mathbb{B}^n} (1 - \|z\|_e^2) \|Df_j(z)\|_{e,e} < +\infty, \quad j = 1, \ldots, n,$$

in view of [61]. Consequently, if $\alpha = 1$ and $\mathbb{B}_X = \mathbb{B}^n$, then $f \in H(\mathbb{B}^n, \mathbb{C}^n)$ is a Bloch mapping in the sense of Definition 3.7 if and only if f is a Bloch mapping in the sense of Definition 3.10.

For $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^* : \ l_x(x) = \|x\|_X, \ \|l_x\| = 1\},$$

where X^* is the dual space of X. Then, $T(x) \neq \emptyset$ in view of the Hahn-Banach theorem. Let $H(\mathbb{U})$ denote the set of holomorphic functions on the unit disk $\mathbb{U}$ in $\mathbb{C}$, that is $H(\mathbb{U}) = H(\mathbb{U}, \mathbb{C})$.

Let $f : \mathbb{B}_X \to \mathbb{C}$ be a holomorphic function, and let $\alpha > 0$. We say that f is an α-Bloch function on $\mathbb{B}_X$ if $f \in \mathcal{B}^\alpha_{X,1}(\mathbb{B}_X)$. We write $\mathcal{B}^\alpha(\mathbb{B}_X) = \mathcal{B}^\alpha_{X,1}(\mathbb{B}_X)$ and $\|f\|_{\mathcal{B}^\alpha} = \|f\|_{\mathcal{B}^\alpha_X}$ for $f \in \mathcal{B}^\alpha(\mathbb{B}_X)$. For $f \in H(\mathbb{B}_X, \mathbb{C})$, we set

$$Q^\alpha_f(z) = \|Df(z) \circ B(z,z)^{\alpha/2}\|_{X,e}$$

$$\text{i.e. } Q^\alpha_f(z) = \sup\left\{|Df(z) \circ B(z,z)^{\alpha/2}(x)| : x \in X, \|x\|_X = 1\right\}.$$

Then,

$$\|f\|_\alpha = \sup\{Q^\alpha_f(z) : z \in \mathbb{B}_X\} \text{ and } \|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha.$$

The following lemma is useful.

Lemma 3.12 *Let $\mathbb{B}_X$ be the unit ball of JB*-triples X. Let $f \in H(\mathbb{B}_Y, \mathbb{C})$.*
(i) *$Q^\alpha_f(z) \le Q^1_f(z)$ holds for $z \in \mathbb{B}_X$, $\alpha \ge 1$.*
(ii) *$Q^\alpha_f(0) = Q^1_f(0)$ holds for $\alpha \ge 1$.*
(iii) *Let g_{-a} be the Möbius transformation for $a \in \mathbb{B}_X$. Then, for $\alpha \ge 1$,*

$$Q^\alpha_{f \circ g_{-a}}(a) \le Q^\alpha_f(0).$$

Proof (i)

$$\begin{aligned} Q^\alpha_f(z) &= \|Df(z) \circ B(z,z)^{\alpha/2}\|_{X,e} \\ &\le \|Df(z) \circ B(z,z)^{\frac{1}{2}}\|_{X,e}\|B(z,z)^{\frac{\alpha-1}{2}}\|_{X,e} \\ &\le \|Df(z) \circ B(z,z)^{\frac{1}{2}}\|_{X,e} = Q^1_f(z). \end{aligned}$$

(ii) Since $B(0,0) = Id$, we have

$$Q^\alpha_f(0) = \|Df(0) \circ B(0,0)^{\alpha/2}\|_{X,e} = \|Df(0) \circ B(0,0)^{1/2}\|_{X,e} = Q^1_f(0).$$

(iii)

$$\begin{aligned} Q^\alpha_{f \circ g_{-a}}(a) &= \|D(f \circ g_{-a})(a) \circ B(a,a)^{\alpha/2}\|_{X,e} \\ &= \|Df(g_{-a}(a)) \circ Dg_{-a}(a) \circ B(a,a)^{\frac{\alpha}{2}}\|_{X,e} \\ &= \|Df(0) \circ B(a,a)^{-\frac{1}{2}} \circ B(a,a)^{\frac{\alpha}{2}}\|_{X,e} \\ &\le \|Df(0) \circ B(0,0)^{\frac{\alpha}{2}}\|_{X,e}\|B(a,a)^{\frac{\alpha-1}{2}}\| \\ &\le Q^\alpha_f(0). \end{aligned}$$

□

Next, we recall some basic facts on some subdomains of the unit disk U.

Definition 3.13 Let $\Omega \subset \mathbb{C}$ be a domain including the origin, and let f and g be holomorphic functions on Ω. We say that f is subordinate to g if there exists a holomorphic function $v : \Omega \to \Omega$ such that $v(0) = 0$ and $f = g \circ v$. We write $f \prec g$ to denote this subordination relation.

Next, for $a \in \mathbb{C}$ and $r > 0$, let

$$U(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$$

and for $r > 0$, let $\Delta(1, r)$ be a horodisk in U, that is,

$$\Delta(1, r) = \left\{ z \in U : \frac{|1 - z|^2}{1 - |z|^2} < r \right\} = U\left(\frac{1}{1 + r}, \frac{r}{1 + r} \right).$$

Then, $\partial\Delta(1, r)$ is a circle internally tangent to the unit circle at 1.

In the case $r > 1$, Wang [55, Lemma 1] obtained the following lemma.

Lemma 3.14 *Let $r > 1$. Assume that $h \in H(U)$, $h(0) = a \in \mathbb{R}$ and that there exists a positive number $s > 0$ such that $h(\Delta(1, r)) \subset \{w : \Re w < s\}$. Then*
(i) $h(z) \prec G_0(z) = b\frac{z+1}{z-1} + b + a$ on $\Delta(1, r)$, where $b = \frac{r(s-a)}{r-1} > 0$.
(ii) $\Re h(x) \geq G_0(x) = \frac{2bx}{x-1} + a$ for $0 < x < 1$ with equality holds for some x if and only if $h = G_0$.
(iii) $\Re h(-x) \leq G_0(-x) = \frac{2bx}{x+1} + a$ for $0 < x \leq \frac{r-1}{r+1}$ with equality holds for some x if and only if $h = G_0$.

The following lemma was proved in Wang and Liu [53, Lemma 2.2].

Lemma 3.15 *Let g be a holomorphic function on $U \cup \{1\}$. Assume that $g(U) \subset U \setminus \{0\}$ and $g(1) = 1$. Then, $g'(1) = \alpha > 0$ and*

$$|g(x)| \geq \exp\left\{ -2\alpha \frac{1 - x}{1 + x} \right\}, \quad \textit{for all } x \in (-1, 1).$$

4 Distortion Theorems

In this section, we give a distortion theorem for locally biholomorphic Bloch mappings on the unit ball of a finite dimensional JB*-triple. This theorem is a generalization of [40, Theorem 7], [55, Theorem 1], and [53, Theorem 3.2] to the unit ball of a finite dimensional JB*-triple. Let $H_{loc}(B_X)$ denote the set of locally biholomorphic mappings on B_X.

Theorem 4.1 *Let B_X be the unit ball of a finite dimensional JB*-triple X. Let $\alpha \in (0, 1]$ and let $m(\alpha)$ be the unique root of the equation*

$$e^{-c(B_X)x}(1+x)^{c(B_X)} = \alpha \tag{3}$$

in the interval $[0, +\infty)$. If $f \in H_{loc}(B_X)$, $\|f\|_0 = 1$, and $\det Df(0) = \alpha$, then

(i)

$$|\det Df(z)| \geq \frac{\alpha}{(1-\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)\|z\|}{1-\|z\|}\right\} \tag{4}$$

for $z \in B_X$.

(ii)

$$|\det Df(z)| \leq \frac{\alpha}{(1+\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{2c(B_X)\|z\|}{1+\|z\|}\right\} \tag{5}$$

for $\|z\| \leq \frac{m(\alpha)}{2+m(\alpha)}$.

Moreover, the estimates (4) and (5) are sharp.

Proof Let $c = c(B_X)$ and let

$$r(t) = e^{-ct}(1+t)^c, \quad t \in [0, +\infty).$$

Then, $r(t)$ is decreasing on $[0, +\infty)$, $r(0) = 1$, and $r(+\infty) = 0$. Therefore, there exists a unique $m(\alpha) \in [0, +\infty)$ such that

$$e^{-cm(\alpha)}(1+m(\alpha))^c = \alpha.$$

(i) Let $z \in B_X \setminus \{0\}$ be fixed, and let $s = \|z\|$. Consider the holomorphic function $|\det Df(s\cdot)| : \overline{B}_X \to \mathbb{C}$ which attains its maximum on $\overline{B_X}$ at a Bergman-Shilov boundary point $u \in \partial B_X$, which is a maximal tripotent in X by Proposition 2.2.

First, we consider the case $\alpha \in (0, 1)$ in which $m(\alpha) > 0$.

Let

$$g(\zeta) = (1-\zeta)^{2c} \det Df(\zeta u), \quad \zeta \in U.$$

Then, $g \in H(U)$, $g(\zeta) \neq 0$ on U, and $g(0) = \alpha$. Since $\|f\|_0 = 1$, by using Lemma 3.9 (iii), we have

$$|g(\zeta)| \leq \left(\frac{|1-\zeta|^2}{1-|\zeta|^2}\right)^c.$$

Let $h(\zeta) = \log g(\zeta)$, where the branch of the logarithm is chosen such that $h(0) = \log g(0) = \log \alpha$ is real. Then, we have

$$\Re h(\zeta) = \log |g(\zeta)| \leq c \log \frac{|1-\zeta|^2}{1-|\zeta|^2}, \quad \zeta \in U.$$

Therefore, we have

$$h(\Delta(1, 1 + m(\alpha))) \subset \{w : \Re w < c \log(1 + m(\alpha))\}.$$

In view of Lemma 3.14 (i), we obtain that $h \prec G_0$ on $\Delta(1, 1 + m(\alpha))$, where

$$G_0(\zeta) = b\frac{\zeta + 1}{\zeta - 1} + b + \log \alpha,$$

$$b = \frac{1 + m(\alpha)}{m(\alpha)}(c \log(1 + m(\alpha)) - \log \alpha) = c(1 + m(\alpha)).$$

In the last equality, we use the equality

$$e^{-cm(\alpha)}(1 + m(\alpha))^c = \alpha.$$

For any $x \in (0, 1)$, we obtain from Lemma 3.14 (ii) that

$$\log |g(x)| = \Re h(x) \geq c(1 + m(\alpha))\frac{2x}{x - 1} + \log \alpha.$$

This implies that

$$|g(x)| \geq \alpha \exp\left\{c(1 + m(\alpha))\frac{-2x}{1 - x}\right\}.$$

If we put $x = \|z\|$ in the above inequality, then we obtain the inequality (4) for $\alpha \in (0, 1)$.

Next, we consider the case $\alpha = 1$. Then $m(\alpha) = 0$. Let

$$g(\zeta) = \left(\frac{1 + \zeta}{2}\right)^{2c} \det Df\left(\frac{1 - \zeta}{2}u\right), \quad \zeta \in U.$$

Then, g is holomorphic on $U \cup \{1\}$ and $g(1) = 1$. Since $\|f\|_0 = 1$ and $\det Df(0) = 1$, by using Lemma 3.9 (iii) and (iv), we have $g'(1) = c$ and

$$\begin{aligned}
|g(\zeta)| &= \left|\frac{1 + \zeta}{2}\right|^{2c} \left|\det Df\left(\frac{1 - \zeta}{2}u\right)\right| \\
&\leq \left(\left|1 - \frac{1 - \zeta}{2}\right|^2 \frac{1}{1 - \left|\frac{1-\zeta}{2}\right|^2}\right)^c \\
&< 1
\end{aligned}$$

for $\zeta \in U$, since

$$\frac{1-\zeta}{2} \in U\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{z \in U : \frac{|1-z|^2}{1-|z|^2} < 1\right\}.$$

This implies that $g(U) \subset U \setminus \{0\}$. By Lemma 3.15, we obtain that

$$|g(x)| \geq \exp\left\{-2c\frac{1-x}{1+x}\right\}$$

for all $x \in (-1, 1)$. If we put $x = 1 - 2\|z\|$ in the above inequality, then we obtain the inequality (4) for $\alpha = 1$.

(ii) If $\alpha = 1$, then $m(\alpha) = 0$ and the inequality (5) follows immediately from (3).

Now let $\alpha \in (0, 1)$. Let $s = \frac{m(\alpha)}{2+m(\alpha)}$ and as before, the holomorphic map $|\det Df(s\cdot)|$ on $\overline{B}_X$ achieves its minimum on the set $\{z \in X : \|z\| \leq s\}$ at some Bergman-Shilov boundary point $u \in \partial\overline{B}_X$. We note that $-u$ is a maximal tripotent in X.

As in the proof of (i), define

$$g(\zeta) = (1-\zeta)^{2c} \det Df(-\zeta u) \qquad (\zeta \in U)$$

and define the mappings h and G_0 as in the proof of (i).

By the arguments in (i) and Lemma 3.14 (iii), we have

$$\Re h(-x) \leq G_0(-x) = 2c(1+m(\alpha))\frac{x}{x+1} + \log\alpha$$

for $0 < x \leq \frac{m(\alpha)}{2+m(\alpha)}$. For $\|z\| \leq s$, if we put $x = \|z\|$ in the above inequality, then we obtain the inequality (5).

Finally, we will show that the estimates (4) and (5) are sharp. Indeed, let $u \in \partial B_X$ be arbitrarily fixed and let

$$F(z) = \left(\int_0^{l_e(z)} \psi(t)dt\right) e + z - l_e(z)e,$$

where $l_u \in T(u)$ and

$$\psi(\zeta) = \frac{\alpha}{(1-\zeta)^{2c}} \exp\left\{(1+m(\alpha))\frac{-2c\zeta}{1-\zeta}\right\} \in H(U).$$

Then, $F \in H(B_X)$, $F(0) = 0$, and $\det DF(z) = \psi(l_u(z))$ by Lemma 3.6. Therefore, $\det DF(0) = \psi(0) = \alpha$. For any $z \in B_X$, let $\zeta = l_u(z)$. Since $e^{-cm(\alpha)}(1+m(\alpha))^c = \alpha$, we have

$$
\begin{aligned}
(1-\|z\|^2)^c|\det DF(z)| &\leq (1-|l_u(z)|)^c|\psi(l_u(z))| \\
&= \left(\frac{1-|\zeta|^2}{|1-\zeta|^2}\right)^c \alpha \left|\exp\left((1+m(\alpha))\frac{-2c\zeta}{1-\zeta}\right)\right| \\
&= \left(\frac{1-|\zeta|^2}{|1-\zeta|^2} b \exp\left(1-b\Re\left(1+\frac{2\zeta}{1-\zeta}\right)\right)\right)^c \\
&= (bt\exp(1-bt))^c \\
&\leq 1,
\end{aligned}
$$

where $b = 1 + m(\alpha)$ and

$$
t = \frac{1-|\zeta|^2}{|1-\zeta|^2} > 0.
$$

Note that in the last inequality, we used the inequality

$$
xe^{1-x} \leq 1 \quad \text{for } x > 0.
$$

Therefore, $\|F\|_0 \leq 1$. Also, let $z = \zeta u$. Then, $\|z\| = |\zeta|$, $l_u(z) = \zeta$, and the equality $(1-\|z\|^2)^c|\det DF(z)| = 1$ holds when $t = 1/b$. This implies that $\|F\|_0 = 1$. Since $\det DF(\pm\|z\|u) = \psi(\pm\|z\|)$ for all $z \in B_X$, F attains the equalities in (4) and (5). This completes the proof. □

Remark 4.2 (i) Let $\mathbb{B}^n$ be the Euclidean unit ball of $\mathbb{C}^n$ (that is, the Type I$(1,n)$ JB*-triple). Then, $c(\mathbb{B}^n) = (n+1)/2$. Therefore, Theorem 4.1 reduces to [40, Theorem 7], [55, Theorem 1].

(ii) Let U^n be the unit polydisk of $\mathbb{C}^n$. The Bergman kernel of U^n is as follows:

$$
k_{U^n}(z, \overline{w}) = \frac{1}{\pi^n}\prod_{j=1}^{n}\frac{1}{(1-z_j\overline{w}_j)^2}.
$$

Then, the Bergman metric at 0 is

$$
h_0(u, v) = 2\sum_{j=1}^{n} u_j\overline{v}_j.
$$

Thus, $c(U^n) = n$. Therefore, if $\alpha = 1$, then Theorem 4.1 reduces to [53, Theorem 3.2].

(iii) If $B_X = U$ is the unit disk in $\mathbb{C}$, then Theorem 4.1 reduces to Bonk et al. [7, Theorem 3].

5 Bloch Constant

In this section, we further assume that

$$\inf\{\|z\|_u : z \in \partial B_X\} \geq 1. \tag{6}$$

This assumption is not so strong, because the unit polydisk satisfies this condition and for any homogeneous unit ball B_X in $\mathbb{C}^n$, there exists a constant $c > 0$ such that cB_X satisfies the inequality (6). Under the above assumption, we give lower estimates for the radius of the largest univalent ball in the image of f centered at $f(0)$.

Let $\mathbb{B}^n(b, r)$ denote the Euclidean ball with center b and radius r. For $f \in H(B_X)$, a schlicht ball $\mathbb{B}^n(f(a), r)$ of f centered at $f(a)$ is that f maps an open subset G of B_X containing a biholomorphically onto this ball $\mathbb{B}^n(f(a), r)$.

For a point $a \in B_X$, let $r(a, f)$ be the largest Euclidean length of a schlicht ball of f centered at $f(a)$.

Definition 5.1 A point $z_0 \in B_X$ is called a critical point of $f \in H(B_X)$ if det $Df(z_0) = 0$. $f(z_0)$ is called a critical value of f.

The following lemma is a generalization of Liu [40, Lemma 2] to the unit ball of a finite dimensional JB*-triple. Since the proof of [40, Lemma 2] can be applied to our case, we omit it. Let $\mathbb{B}^n(b, r)$ denote the Euclidean ball with center b and radius r.

Lemma 5.2 *Let B_X be the unit ball of an n-dimensional JB*-triple X. Let $f \in H(B_X)$, G be an open subset of B_X, and $a \in G$. If f maps G biholomorphically onto the schlicht ball $\mathbb{B}^n(f(a), r(a, f))$, then either G and B_X have a common boundary point or there exists a critical value $f(z_0)$ on the boundary of the ball $\mathbb{B}^n(f(a), r(a, f))$ with the critical point z_0 on the boundary of G.*

The following lemma was proved in Hamada and Kohr [30].

Lemma 5.3 *Assume that the condition (6) is satisfied. Let $A \in L(\mathbb{C}^n)$. Then, the following inequalities hold:*

$$\|A\|_{X,e} \geq |\det A|^{1/n},$$

$$\|Aw\|_e \geq \frac{|\det A|}{\|A\|_{X,e}^{n-1}}, \quad w \in \partial B_X, \quad \text{if } \|A\|_{X,e} > 0. \tag{7}$$

For a locally biholomorphic Bloch mapping f, we obtain the following lower estimate for the radius of the largest ball in the image of f centered at $f(0)$. The following theorem is a generalization of [40, Theorem 8], [55, Theorem 2], and [53, Theorem 3.4] to the unit ball of a finite dimensional JB*-triple.

Theorem 5.4 *Let B_X be the unit ball of an n-dimensional JB*-triple X. Assume that the condition (6) is satisfied. If $f \in \beta(K) \cap H_{loc}(B_X)$, $\|f\|_0 = 1$, and* $\det Df(0) = \alpha \in (0, 1]$, *then*

$$r(0, f) \geq K^{1-n}\alpha \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)t}{1-t}\right\} dt$$
$$\geq \frac{\alpha K^{1-n}}{2c(B_X)(1+m(\alpha))}$$

where $m(\alpha)$ is the unique root of the equation

$$e^{-c(B_X)x}(1+x)^{c(B_X)} = \alpha$$

in the interval $[0, +\infty)$.

Proof Let $c = c(B_X)$. By Lemma 5.2, $r(0, f)$ is equal to the Euclidean distance from $f(0)$ to a boundary point of $f(B_X)$, since f is locally biholomorphic on B_X. Hence, there exists a line segment Γ of Euclidean length $r(0, f)$ from $f(0)$ to a point in $\partial f(B_X)$. Note that $r(0, f)$ is the largest nonnegative number r such that there exists a domain $V \subset B_X$ which is mapped biholomorphically onto $\mathbb{B}^n(f(0), r)$ by f. Let $\gamma = (f|_V)^{-1}(\Gamma)$. Then, γ is a smooth curve which is not relatively compact in B_X. By (7), we have

$$r(0, f) = \int_\Gamma \|dw\|_e = \int_\gamma \|Df(z)dz\|_e = \int_\gamma \left\|Df(z)\frac{dz}{\|dz\|}\right\|_e \|dz\|$$
$$\geq \int_\gamma \frac{|\det Df(z)|}{\|Df(z)\|_{X,e}^{n-1}} \|dz\|.$$

From Theorem 4.1 (i) and Lemma 3.9 (i), we have

$$\int_\gamma \frac{|\det Df(z)|}{\|Df(z)\|_{X,e}^{n-1}} \|dz\|$$
$$\geq K^{1-n}\alpha \int_\gamma \frac{(1-\|z\|^2)^{n-1}}{(1-\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)\|z\|}{1-\|z\|}\right\} \|dz\|$$
$$\geq K^{1-n}\alpha \int_\gamma \frac{(1-\|z\|^2)^{n-1}}{(1-\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)\|z\|}{1-\|z\|}\right\} d\|z\|,$$

since $d\|z\| \leq \|dz\|$ a.e. on γ by [34, Lemma 1.3]. Therefore, we have

$$r(0, f) \geq K^{1-n}\alpha \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)t}{1-t}\right\} dt.$$

Since $c(B_X) \geq (n+1)/2$, we also have

$$r(0, f) \geq K^{1-n}\alpha \int_0^1 \frac{1}{(1-t)^2} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)t}{1-t}\right\} dt$$
$$\geq \frac{\alpha K^{1-n}}{2c(B_X)(1+m(\alpha))}.$$

This completes the proof. □

Remark 5.5 (i) Let $\mathbb{B}^n$ be the Euclidean unit ball of $\mathbb{C}^n$ (that is, the Type I$(1, n)$ JB*-triple). Then, $c(\mathbb{B}^n) = (n+1)/2$. Therefore, Theorem 5.4 reduces to [40, Theorem 8], [55, Theorem 2].

(ii) Let U^n be the unit polydisk of $\mathbb{C}^n$. Then, $c(U^n) = n$. Therefore, if $\alpha = 1$, then Theorem 5.4 reduces to [53, Theorem 3.4].

(iii) When $n = 1$ and $B_X = U$, then Theorem 5.4 reduces to [7, Corollary 3].

6 Composition Operators

Let B_X be the unit ball of an n-dimensional JB*-triple X. Then, we obtain the following lemma.

Lemma 6.1 *For* $\alpha \geq 1$, $H^\infty(\mathbb{B}_X) \subset \mathcal{B}^\alpha(\mathbb{B}_X)$ *and the inclusion mapping* $i : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$ *is a linear operator satisfying*

$$\|f\|_\alpha \leq \|f\|_\infty.$$

Proof Let $f \in H^\infty(\mathbb{B}_X)$. We may assume $\|f\|_\infty = 1$. Since $Q_f^\alpha(z) \leq Q_f^1(z)$ for $z \in \mathbb{B}_X$ by Lemma 3.12, We have

$$\|f\|_\alpha \leq \|f\|_1 = \sup\{Q_f^1(z) : z \in \mathbb{B}_X\}.$$

Let g_a be the Möbius transformation for $a \in \mathbb{B}_X$. Then, we have $f \circ g_a \in H^\infty(\mathbb{B}_X)$ and $\|f \circ g_a\|_\infty \leq 1$. By Lemma 3.5, we have $Q_f^1(a) = \|Df(a) \circ B(a, a)^{1/2}\|_{X,e} = \|Df(a) \circ Dg_a(0)\|_{X,e} = \|D(f \circ g_a)(0)\|_{X,e} \leq 1 - |f \circ g_a(0)|^2 \leq \|f\|_\infty$. □

Let $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. By Lemma 6.1, the composition operator $C_\varphi : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$ with symbol φ, defined by

$$C_\varphi(f)(z) = f \circ \varphi(z) = f(\varphi(z)) \text{ for } f \in H^\infty(\mathbb{B}_X), z \in \mathbb{B}_X,$$

is well defined. Allen and Colonna [3, Corollary 5.6] proved the following theorem when $\mathbb{B}_X$ is a bounded homogeneous domain in $\mathbb{C}^n$ and $\alpha = 1$. Hamada [25] obtained the following theorem when $\alpha = 1$. The following theorem is a generalization to the α-Bloch space.

Theorem 6.2 *Let* $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$, $\alpha \geq 1$. *Then,* $C_\varphi : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$ *is bounded and*

$$1 \leq \|C_\varphi\| < 2$$

holds.

Proof Using the constant function $\mathbf{1} \in H^\infty(\mathbb{B}_X)$,

$$1 = \|\mathbf{1} \circ \varphi\|_{\mathcal{B}^\alpha} \leq \|C_\varphi\|.$$

On the other hand, we set $\theta^\alpha_\varphi(z) = \sup\{Q^\alpha_{f\circ\varphi}(z) : f \in H^\infty(\mathbb{B}_X), \|f\|_\infty \leq 1\}$ and $\theta^\alpha_\varphi = \sup_{z\in\mathbb{B}_X} \theta^\alpha_\varphi(z)$. Since $\theta^\alpha_\varphi(z) \leq 1$ for all $z \in \mathbb{B}_X$ by Lemmas 3.12 and 6.1, we have $\theta^\alpha_\varphi \leq 1$.

For $f \in H^\infty(\mathbb{B}_X)$ with $\|f\|_\infty \leq 1$, by the maximum principle for holomorphic functions, $|f(z)| < 1$ for all $z \in \mathbb{B}_X$. Moreover,

$$Q^\alpha_{f\circ\varphi}(z) \leq \theta^\alpha_\varphi(z) \leq \theta^\alpha_\varphi \leq 1.$$

So, we have

$$\|C_\varphi(f)\|_\alpha = \|f \circ \varphi\|_\alpha = \sup_{z\in\mathbb{B}_X} Q^\alpha_{f\circ\varphi}(z) \leq 1.$$

Therefore,

$$\|C_\varphi(f)\|_{\mathcal{B}^\alpha} = \|C_\varphi(f)\|_\alpha + |C_\varphi(f)(0)| < 2.$$

It follows from this that $\|C_\varphi\| < 2$. □

Allen and Colonna [3, Theorem 6.4] proved the following theorem when $\mathbb{B}_X$ is the unit disk in $\mathbb{C}$ and $\alpha = 1$. Hamada [25] obtained the following theorem when $\alpha = 1$. Using the Bloch norm introduced in Sect. 2, we can generalize to any bounded symmetric domain.

Theorem 6.3 *Let* $\mathbb{B}_X$ *be the unit ball of a finite dimensional JB*-triple* X. *Then, there exist no isometric composition operators from* $H^\infty(\mathbb{B}_X)$ *to* $\mathcal{B}^\alpha(\mathbb{B}_X)$ *for* $\alpha \geq 1$.

Proof Assume that C_φ is an isometry from $H^\infty(\mathbb{B}_X)$ to $\mathcal{B}^\alpha(\mathbb{B}_X)$. Let $a \in \partial\mathbb{B}_X$ be fixed and let $f(z) = l_a(z)$. Then, we have

$$|l_a(\varphi(0))| + \|l_a(\varphi)\|_\alpha = \|C_\varphi(f)\|_{\mathcal{B}^\alpha} = \|f\|_\infty = 1. \tag{8}$$

Let

$$f_+(z) = \frac{1 + f(z)}{2}, \quad f_-(z) = \frac{1 - f(z)}{2}.$$

Since $\|f_+\|_\infty = \|f_-\|_\infty = 1$, we have $\|C_\varphi(f_+)\|_{\mathcal{B}} = \|C_\varphi(f_-)\|_{\mathcal{B}} = 1$. That is,

$$|1 + l_a(\varphi(0))| + \|l_a(\varphi)\|_{\mathcal{B},s} = 2 = |1 - l_a(\varphi(0))| + \|l_a(\varphi)\|_{\mathcal{B},s}$$

By (8), we have

$$|1 + l_a(\varphi(0))| = 1 + |l_a(\varphi(0))| = |1 - l_a(\varphi(0))|.$$

Therefore, $l_a(\varphi(0)) = 0$. Since $a \in \partial\mathbb{B}_X$ is arbitrary, we deduce that $\varphi(0) = 0$.

Next, we have $\|C_\varphi(f^2)\|_{\mathcal{B}^\alpha} = \|f^2\|_\infty = 1$. On the other hand, by using the Schwarz Pick Lemma (Lemma 3.5), we have

$$\begin{aligned}
&|D(f \circ \varphi)^2(b) \circ B(b,b)^{\frac{\alpha}{2}}(x)| \\
&= |2f(\varphi(b))D(f \circ \varphi)(b) \circ D(g_b \circ g_{-b})(b) \circ B(b,b)^{\frac{\alpha}{2}}(x)| \\
&= |2f(\varphi(b))D(f \circ \varphi)(g_b(0)) \circ Dg_b(g_{-b}(b) \circ Dg_{-b}(b) \circ B(b,b)^{\frac{\alpha}{2}}(x)| \\
&= |2f(\varphi(b))D(f \circ \varphi \circ g_b)(0) \circ B(b,b)^{-\frac{1}{2}} \circ B(b,b)^{\frac{\alpha}{2}}(x)| \\
&\le 2|f(\varphi(b))| \, \|D(f \circ \varphi \circ g_b)(0)\|_{X,e} \, \|B(b,b)^{\frac{\alpha-1}{2}}(x)\| \\
&\le 2|f(\varphi(b))|(1 - |f(\varphi(b))|^2) \\
&\le 2 \sup_{x \in [0,1]} \max(x - x^3) = \frac{4}{3\sqrt{3}}
\end{aligned}$$

for $b \in \mathbb{B}_X$, $x \in X \setminus \{0\}$ with $\|x\| = 1$. Hence

$$\begin{aligned}
Q^\alpha_{C_\varphi(f^2)}(b) &= \sup\{|D(f \circ \varphi)^2(b) \circ B(b,b))^{\frac{\alpha}{2}}(x)|; x \in X \setminus \{0\}, \|x\| = 1\} \\
&\le \frac{4}{3\sqrt{3}} < 1.
\end{aligned}$$

This is a contradiction. $\square$

References

1. L.V. Ahlfors, An extension of Schwarz's lemma. Trans. Amer. Math. Soc. **43**, 359–364 (1938)
2. L.V. Ahlfors, *Complex Analysis* (McGraw-Hill, New York, 1966)
3. R.F. Allen, F. Colonna, Weighted composition operators from H^∞ to the Bloch space of a bounded homogeneous domain. Integral Equ. Oper. Theory **66**, 21–40 (2010)
4. J.M. Anderson, J.G. Clunie, Ch. Pommerenke, On Bloch functions and normal functions. J. Reine Angew. Math. **270**, 12–37 (1974)
5. O. Blasco, P. Galindo, A. Miralles, Bloch functions on the unit ball of an infinite dimensional Hilbert space. J. Funct. Anal. **267**, 1188–1204 (2014)
6. M. Bonk, On Bloch's constant. Proc. Amer. Math. Soc. **110**, 889–894 (1990)
7. M. Bonk, D. Minda, H. Yanagihara, Distortion theorems for locally univalent Bloch functions. J. Anal. Math. **69**, 73–95 (1996)
8. M. Bonk, D. Minda, H. Yanagihara, Distortion theorem for Bloch functions. Pacific. J. Math. **179**, 241–262 (1997)
9. É. Cartan, Sur les domaines bornés homogènes de l'espace den variables complexes Abh. Math. Sem. Univ. Hamburg **11**, 116–162 (1935)
10. H. Chen, P.M. Gauthier, On Bloch's constant. J. Anal. Math. **69**, 275–291 (1996)

11. H. Chen, P.M. Gauthier, Bloch constants in several variables. Trans. Am. Math. Soc. **353**, 1371–1386 (2001)
12. H. Chen, P.M. Gauthier, The Landau theorem and Bloch theorem for planar harmonic and pluriharmonic mappings. Proc. Amer. Math. Soc. **139**, 583–595 (2011)
13. S. Chen, S. Ponnusamy, X. Wang, Landau-Bloch constants for functions in α-Bloch spaces and Hardy spaces. Complex Anal. Oper. Theory. **6**, 1025–1036 (2012)
14. C.-H. Chu, Jordan structures in geometry and analysis, *Cambridge Tracts in Mathematics*, vol. 6 (Cambridge University Press, Cambridge, 2012)
15. C.-H. Chu, H. Hamada, T. Honda, G. Kohr, Distortion theorems for convex mappings on homogeneous balls. J. Math. Anal. Appl. **369**, 437–442 (2010)
16. C.-H. Chu, H. Hamada, T. Honda, G. Kohr, Distortion of locally biholomorphic Bloch mappings on bounded symmetric domains. J. Math. Anal. Appl. **441**, 830–843 (2016)
17. C.H. Chu, H. Hamada, T. Honda, G. Kohr, Bloch functions on bounded symmetric domains. J. Funct. Anal. **272**, 2412–2441 (2017)
18. F. Colonna, G.R. Easley, D. Singman, Norm of the multiplication operators from H^∞ to the Bloch space of a bounded symmetric domain. J. Math. Anal. Appl. **382**, 621–630 (2011)
19. C.H. FitzGerald, S. Gong, The Bloch theorem in several complex variables. J. Geom. Anal. **4**, 35–58 (1994)
20. I. Graham, D. Varolin, Bloch constants in one and several complex variables. Pac. J. Math. **174**, 347–357 (1996)
21. K.T. Hahn, Higher dimensional generalizations of the Bloch constant and their lower bounds. Trans. Amer. Math. Soc. **179**, 263–274 (1973)
22. K.T. Hahn, Holomorphic mappings of the hyperbolic space into the complex Euclidean space and the Bloch theorem. Canad. J. Math. **27**, 446–458 (1975)
23. D.J. Hallenbeck, T.H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory* (Pitman, Boston, 1984)
24. H. Hamada, A distortion theorem and the Bloch constant for Bloch mappings in $\mathbb{C}^n$, J. Anal. Math. (to appear)
25. H. Hamada, Weighted composition operators from H^∞ to the Bloch space of infinite dimensional bounded symmetric domains. Complex Anal. Oper. Theory (to appear)
26. H. Hamada, T. Honda, G. Kohr, Bohr's theorem for holomorphic mappings with values in homogeneous balls. Isr. J. Math. **173**, 177–187 (2009)
27. H. Hamada, T. Honda, G. Kohr, Linear invariance of locally biholomorphic mappings in the unit ball of a JB*-triple. J. Math. Anal. Appl. **385**, 326–339 (2012)
28. H. Hamada, T. Honda, G. Kohr, Trace-order and a distortion theorem for linearly invariant families on the unit ball of a finite dimensional JB*-triple. J. Math. Anal. Appl. **396**, 829–843 (2012)
29. H. Hamada, T. Honda, G. Kohr, Growth and distortion theorems for linearly invariant families on homogeneous unit balls in $\mathbb{C}^n$. J. Math. Anal. Appl. **407**, 398–412 (2013)
30. H. Hamada, G. Kohr, Pluriharmonic mappings in $\mathbb{C}^n$ and complex Banach spaces. J. Math. Anal. Appl. **426**, 635–658 (2015)
31. H. Hamada, G. Kohr, α-Bloch mappings on bounded symmetric domains in $\mathbb{C}^n$ (preprint)
32. L.A. Harris, Bounded symmetric homogeneous domains in infinite dimensional spaces, *Proceedings on Infinite Dimensional Holomorphy, Internat. Conference University of Kentucky, Lexington, KY,1973*, Lecture Notes in Mathematics (Springer, Berlin, 1974), pp. 13–40
33. L.K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains. Translations of Mathematical Monographs*, vol. 6, American Mathematical Society,Providence (1963)
34. T. Kato, Nonlinear semigroups and evolution equations. J. Math. Soc. Japan **19**, 508–520 (1967)
35. W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. Math. Z. **183**, 503–529 (1983)
36. W. Kaup, Hermitian Jordan, triple systems and the automorphisms of bounded symmetric domains. Math. Appl. **303**, 204–214 (1994)

37. W. Kaup, H. Upmeier, Jordan algebras and symmetric Siegel domains in complex Banach spaces. Math. Z. **157**, 179–200 (1977)
38. S. Li, S. Stević, Weighted composition operators from H^∞ to the Bloch space on the polydisc. Abstr. Appl. Anal. **2007**, Art. ID 48478, 13 pp (2007)
39. S. Li, S. Stević, Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball. Taiwan. J. Math. **12**, 1625–1639 (2008)
40. X.Y. Liu, Bloch functions of several complex variables. Pac. J. Math. **152**, 347–363 (1992)
41. O. Loos, *Bounded Symmetric Domains and Jordan Pairs* (University of California, Irvine, 1977)
42. K.M. Madigan, Composition operators on analytic Lipschitz spaces. Proc. Amer. Math. Soc. **119**, 465–473 (1993)
43. K.M. Madigan, A. Matheson, Compact composition operators on the Bloch space. Trans. Amer. Math. Soc. **347**, 2679–2687 (1995)
44. M.T. Malavé Ramírez, J. Ramos Ferenández, On a criterion for continuity and compactness of composition operators acting on α-Bloch spaces, C. R. Acad. Sci. Paris, Ser. I. **351**, 23–26 (2013)
45. S. Ohno, Weighted composition operators between H^∞ and the Bloch space. Taiwan. J. Math. **5**, 555–563 (2001)
46. S. Ohno, K. Stroethoff, R. Zhao, Weighted composition operators between Bloch-type spaces. Rocky Mt. J. Math. **33**, 191–215 (2003)
47. Ch. Pommerenke, On Bloch functions. J. London Math. Soc. **2**, 689–695 (1970)
48. R.C. Roan, Composition operators on a space of Lipschitz functions. Rocky Mt. J. Math. **10**, 371–379 (1980)
49. G. Roos, Jordan triple systems, pp. 425–534, in J. Faraut, S. Kaneyuki, A. Koranyi, Q.-K. Lu, G. Roos, Analysis and geometry on complex homogeneous domains, Progress in Mathematics, 185, *Birkhauser Boston* (Inc, Boston, MA, 2000)
50. J. Shi, L. Luo, Composition operators on the Bloch space. Acta Math Sinica, Engl. ser. **16**, 85–98 (2000)
51. R.M. Timoney, Bloch functions in several complex variables, I. Bull. Lond. Math. Soc. **12**, 241–267 (1980)
52. R.M. Timoney, Bloch functions in several complex variables, II. J. Reine Angew. Math. **319**, 1–22 (1980)
53. J. Wang, T. Liu, Bloch constant of holomorphic mappings on the unit polydisk of $\mathbb{C}^n$. Sci. China Ser. A. **51**, 652–659 (2008)
54. J. Wang, T. Liu, Distortion theorem for Bloch mappings on the unit ball $\mathcal{B}^n$. Acta Math. Sin. (Engl. Ser.) **25**, 1583–1590 (2009)
55. F.D. Wicker, Generalized Bloch mappings in complex Hilbert space. Can. J. Math. **29**, 299–306 (1977)
56. Z.M. Yan, S. Gong, Bloch constant of holomorphic mappings on bounded symmetric domains. Sci. China Ser. A. **36**, 285–299 (1993)
57. M.Z. Zhang, W. Xu, Composition operators on α-Bloch spaces on the unit ball. Acta Math Sinica, Engl. ser. **23**, 1991–2002 (2007)
58. M. Zhang, H. Chen, Weighted composition operators of H^∞ into α-Bloch spaces on the unit ball. Acta Math. Sin. (Engl. Ser.) **25**, 265–278 (2009)
59. Z. Zhou, J. Shi, Compactness of composition operators on the Bloch space in classical bounded symmetric domains. Michigan Math. J. **50**, 381–405 (2002)
60. K. Zhu, Bloch type spaces of analytic functions. Rocky Mt. J. Math. **23**, 1143–1177 (1993)
61. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball* (Springer, New York, 2005)

Certain Class of Meromorphically Multivalent Functions Defined by a Differential Operator

Ghazi S. Khammash and Praveen Agarwal

Abstract In this paper, we introduce the subclasses $T_p(\alpha, \delta, A, B, n)$ and T_p^* $(\alpha, \delta, A, B, n)$ of meromorphic multivalent functions in the punctured unit disk $U^* = \{z \in C : 0 < |z| < 1\}$ by using a differential operator $D_{\delta,p}^n f(z)$. We obtain coefficient estimates, distortion theorem, radius of convexity and closure theorems for the class $T_p^*(\alpha, \delta, A, B, n)$. The familiar concept of neighborhoods of analytic functions is also extended and applied to the functions considered here.

Keywords Meromorphic functions · p-valent · Neighborhoods · Integral operator

2010 Mathematics Subject Classifications 30C45

1 Introduction

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathrm{N} = \{1, 2, 3 \ldots\}), \tag{1}$$

which are analytic in the punctured unit disk $U^* = \{z \in C : 0 < |z| < 1\} = U/\{0\}$.

Also, let Ω_p denote the subclass of $\sum_p$ of meromorphic multivalent functions in U^*, which have the power series representation as:

G.S. Khammash
Mathematics Department, Al-Aqsa University, Gaza Strip, Palestine
e-mail: ghazikhamash@yahoo.com

P. Agarwal (✉)
Department of Mathematics, Anand International College of Engineering,
Jaipur 303012, India
e-mail: goyal.praveen2011@gmail.com

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_4

$$f(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} a_{k+p} z^{k+p} (a_{k+p} \geq 0). \tag{2}$$

A function $f(z) \in \sum_p$ is said to be p-valent meromorphically starlike function of order α, if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U^*), \tag{3}$$

for some $\alpha(0 \leq \alpha < p)$. We denoted the class of all meromorophic p-valent starlike functions of order α by $\sum_p(\alpha)$. Further, a function $f(z)$ in $\sum_p$ is said to be meromorophic p-valent convex of order α if and only if

$$\operatorname{Re}\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \qquad (z \in U^*), \tag{4}$$

for some $\alpha(0 \leq \alpha < p)$. We denote the class of all meromorophic p-valent convex functions of order α by $K_p(\alpha)$. The classes $\sum_p(\alpha)$ and $K_p(\alpha)$ and various other subclasses of $\sum_p$ have been studied rather extensively by Aouf et al. ([3, 5, 6]), Joshi and Srivastava [9], Kulkarni et al. [10], Mogra [13], Owa et al. [14], and others. For $\alpha = 0$, we obtain the class $\sum(p)$ and $K\,(p)$ of meromorophic p-valent starlike and convex functions with respect to the origin.

Denote by $\sum_p^*(\alpha)$ and $K_p^*(\alpha)$ the classes obtained by considering intersection, respectively, of the classes $\sum_p(\alpha)$ and $K_p(\alpha)$ with Ω_p, i.e.,

$$\begin{aligned} \sum\nolimits_p^* (\alpha) &= \sum\nolimits_p (\alpha) \cap \Omega_p \quad ; \quad (0 \leq \alpha < p) \\ K_p^* (\alpha) &= K_p (\alpha) \cap \Omega_p \quad ; \quad (0 \leq \alpha < p) \end{aligned} \tag{5}$$

The function $f\,(z)$ is said to be subordinate to $F\,(z)$, if there exists a function $w\,(z)$ analytic in U with $w\,(0) = 0$ and $|w\,(z)| < 1\,(z \in U)$, such that $f\,(z) = F\,(w\,(z))$. In such a case, we write $f\,(z) \prec F\,(z)$. In particular, if F is univalent, then $f\,(z) \prec F\,(z)$ if and only if $f\,(0) = F\,(0)$ and $f\,(U) \subset F\,(U)$.

For $f(z) \in \sum_p$ given by (1) and $g(z) \in \sum_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{k+p} z^{k+p} \, (p \in \mathrm{N} = \{1, 2, 3 \ldots\}), \tag{6}$$

the Hadamard product (or convolution) of f and g is denoted by $(f{*}g)\,(z)$ and defined by

$$(f * g)\,(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{k+p} b_{k+p} z^{k+p} \tag{7}$$

The extended linear derivative operator of Ruscheweyh type, $R_p^\gamma : \sum_p \to \sum_p$, is defined by the following convolution:

$$R_p^\gamma f(z) = \frac{1}{z^p(1-z)^{\gamma+1}} * f(z) \left(\gamma > -1;\ f \in \sum\nolimits_p\right). \tag{8}$$

In terms of binomial coefficient, (8) can be written as

$$R_p^\gamma f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \binom{\gamma+k+2p}{\gamma} a_{p+k} z^{p+k} \left(\gamma > -1;\ f \in \sum\nolimits_p\right). \tag{9}$$

In particular when $\lambda = n$ ($n \in N$), it is easily observed from (8) and (9) that

$$R_p^n f(z) = \frac{z^{-p}(z^{n+p} f(z))^{(n)}}{n!} \ (n \in N_0 = N \cup \{0\}), \tag{10}$$

so that (9) becomes

$$R_p^n f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \binom{n+k+2p}{n} a_{p+k} z^{p+k} \left(n \in N_0;\ f \in \sum\nolimits_p\right). \tag{11}$$

The definition (8) of linear operator R_p^λ is motivated essentially by familiar Ruscheweyh operator D^γ, which has been used widely on the space of analytic and univalent functions (see, for details, Rusheweyh [16], Raina and Srivastava [15], Yang [20]).

For the function $f(z) \in \sum_p$, Aouf [4] define the following differential operator:

$$\begin{aligned}
S_p^0 f(z) &= f(z) \\
S_p^1 f(z) &= \frac{1}{p} z f'(z) + \frac{2}{z^p} \\
&= \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(1 + \frac{k}{p}\right) a_{p+k} z^{p+k} = S_p f(z). \qquad (p \in N). \\
&\quad \vdots \\
S_p^2 f(z) &= S_p(D_p^1 f(z)). \\
S_p^n f(z) &= S_p(S_p^{n-1} f(z)) \\
&= \frac{1}{p} z \left(S_p^{n-1} f(z)\right)' + \frac{2}{z^p} (n, p \in N).
\end{aligned} \tag{12}$$

It can be easily seen that

$$S_p^n f(z) = \tfrac{1}{z^p} + \sum_{k=0}^{\infty} \left(1 + \frac{k}{p}\right)^n a_{p+k} z^{p+k} (n \in N_0 = N \cup \{0\}\,;\, p \in N). \tag{13}$$

With the aid of the deferential operator $S_p^n f(z)$ and the Ruscheweyh derivative $R_p^{\gamma} f(z)$, we define the following differential operator for the function $f(z) \in \Omega_p$

$$D_{\delta,p}^n f(z) = (1-\delta) S_p^n f(z) + \delta R_p^n f(z), \tag{14}$$

for $n \in N$ and $\delta \geq 0$.

Let $f(z)$ be given by (1), and then by making use of (11), (13) and (14) can be easily written as

$$D_{\delta,p}^n f(z) = \frac{1}{z^p} - \sum_{K=0}^{\infty} Q_K(n,\delta,p) a_{p+k} z^{p+k}, \tag{15}$$

where

$$Q_K(n,\delta,p) = (1-\delta)\left(1 + \frac{k}{p}\right)^n + \delta \binom{n+k+2p}{n}, \tag{16}$$

for $n \in N$ and $\delta \geq 0$.

With the aid of the differential operator $D_{\delta,p}^n f(z)$, we define the following subclasses of multivalent and meromorphic functions.

Definition 1 A function $f(z) \in \sum_p$ defined by (1) is said to be in the class $T_p(\alpha,\delta,A,B,n)$ if it satisfies the following subordination condition:

$$1 + \frac{\left(z(D_{\delta,p}^n f(z)\right)''}{\left(D_{\delta,p}^n f(z)\right)'} \prec -\frac{p + [pB + (A-B)(P-\alpha)]z}{1+Bz} (z \in U), \tag{17}$$

or, equivalently, if the following inequality holds true:

$$\left| \frac{1 + \frac{\left(z(D_{\delta,p}^n f(z)\right)''}{\left(D_{\delta,p}^n f(z)\right)'} + p}{B\left(1 + \frac{\left(z(D_{\delta,p}^n f(z)\right)''}{\left(D_{\delta,p}^n f(z)\right)'}\right) + [pB + (A-B)(P-\alpha)]} \right| < 1 (z \in U). \tag{18}$$

Also let $T_p^*(\alpha,\delta,A,B,n) = T_p(\alpha,\delta,A,B,n) \cap \Omega_p$.

$$(0 \leq \alpha < P; -1 \leq A < B \leq 1; 0 < B \leq 1; p \in N; n \in N_0; \delta \geq 0)$$

It may be noted for suitable choice of $\delta, A, B, n, p, \lambda,$ and α. The class T_p^* (α,δ,A,B,n) extends several classes of analytic and p-valent meromorphic functions such that Aouf and Shammaky [7], Srivastava et al. [18] and Uralegaddi and Ganigi [19].

2 Basic Properties of the Class $T^*_{\delta,p}(\alpha, A, B, n)$

We first determine a necessary and sufficient condition for a function $f(z) \in \Omega_p$ of the form (2) to be in the class $T^*_p(\alpha, \delta, A, B, n)$.

Theorem 1 *Let the function $f(z) \in \Omega_p$ defined by (2), then $f(z) \in T^*_p(\alpha, \delta, A, B, n)$ if and only if*

$$\sum_{k=0}^{\infty}(k+p)\,Q_K(n,\delta,p)\,[(k+p)\,(B+1)+p(A+1)+(B-A)\alpha]\,a_{k+p}$$
$$\leq p(B-A)(P-\alpha) \tag{19}$$

$$(0 \leq \alpha < P;\, A+B \geq 0;\, -1 \leq A < B \leq 1;\, 0 < B \leq 1;\, p \in N;\, n \in N_0;\, \lambda \geq 0;\, \delta \geq 0)$$

where $Q_K(n,\delta,p)$ is given by (16).

Proof Suppose that the function $f(z) \in \Omega_p$ defined by (2) be in the class $T^*_p(\alpha, \delta, A, B, n)$, then from (18) we have

$$\left|\frac{\left(z(D^n_{\lambda,\delta,p}f(z)\right)'' + (1+p)\left(D^n_{\lambda,\delta,p}f(z)\right)'}{B\left(\left(D^n_{\lambda,\delta,p}f(z)\right)' + z(D^n_{\lambda,\delta,p}f(z))''\right) + [pB+(A-B)(P-\alpha)]\left(D^n_{\lambda,\delta,p}f(z)\right)'}\right| =$$
$$\left|\frac{\left\{-\sum_{k=0}^{\infty}(k+p)\,(k+2p)\,Q_K(n,\lambda,\delta,p)a_{k+p}z^{k+2p}\right\}}{P(B-A)(P-\alpha)-\sum_{K=0}^{\infty}(k+p)\,Q_K(n,\lambda,\delta,p)[(k+p)\,B+(B-A)\alpha+Ap]a_{k+p}z^{k+2p}}\right|$$
$$< 1 (z \in U). \tag{20}$$

Since $|\text{Re}\{z\}| \leq |z|$ for any z, choosing z to be real and letting $z \to 1^-$ through real value, then (21) yields

$$\sum_{k=0}^{\infty}(k+p)\,(k+2p)\,Q_K(n,\lambda,\delta,p)a_{k+p}$$
$$\leq P(B-A)(P-\alpha)-\sum_{K=0}^{\infty}(k+p)\,Q_K(n,\lambda,\delta,p)[(k+p)\,B+(B-A)\alpha+Ap]a_{k+p}, \tag{21}$$

which leads us immediately to the coefficient inequality (19).

Next in order to prove the converse we assume that the inequality (19) holds true, then we observe that

$$\left|\frac{\left(z(D^n_{\delta,p}f(z)\right)'' + (1+p)\left(D^n_{\delta,p}f(z)\right)'}{B\left(\left(D^n_{\delta,p}f(z)\right)' + z(D^n_{\delta,p}f(z))''\right) + [pB + (A-B)(P-\alpha)]\left(D^n_{\delta,p}f(z)\right)'}\right| =$$

$$\leq \frac{\sum_{k=0}^{\infty}(k+p)\,(k+2p)\,Q_K(n,\delta,p)a_{k+p}}{P(B-A)(P-\alpha) - \sum_{K=0}^{\infty}(k+p)\,Q_K(n,\delta,p)[(k+p)\,B + (B-A)\alpha + Ap]a_{k+p}}$$

$$< 1(z \in U). \tag{22}$$

Hence by maximum modulus theorem, we have $f(z) \in T^*_p(\alpha, \delta, A, B, n)$. This completes the proof of Theorem.

Corollary 1 *Let the function $f(z) \in \Omega_p$ defined by (2), if $f(z) \in T^*_{\delta,p}(\alpha, A, B, n)$, then*

$$a_{k+p} \leq \frac{P(B-A)(P-\alpha)}{(k+p)\,Q_K(n,\delta,p)\,[(k+p)\,(B+1) + p(A+1) + (B-A)\alpha]} \quad (k \geq 0,\ p \in N). \tag{23}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} - \frac{P(B-A)(P-\alpha)}{(k+p)\,Q_K(n,\delta,p)\,[(k+p)\,(B+1) + p(A+1) + (B-A)\alpha]}z^{k+p}$$
$$(k \geq 0,\ p \in N). \tag{24}$$

Next we prove the following distortion and growth properties for the class $T^*_p(\alpha, \delta, A, B, n)$.

Theorem 2 *If a function $f(z) \in \Omega_p$ defined by (2) is in the class $T^*_p(\alpha, \delta, A, B, n)$, then*

$$\left[\frac{(p+m-1)!}{(p-1)!} - \frac{p!(B-A)(P-\alpha)}{(p-m)!Q_0(n,\delta,p)\,[p(A+B+2) + (B-A)\alpha]}r^{2p}\right]r^{-p-m}$$
$$\leq \left|f^m(z)\right| \leq \left[\frac{(p+m-1)!}{(p-1)!} + \frac{p!(B-A)(P-\alpha)}{(p-m)!Q_0(n,\delta,p)\,[p(A+B+2) + (B-A)\alpha]}r^{2p}\right]r^{-p-m}, \tag{25}$$

$$(0 < |z| = r < 1,\ 0 \leq m < p),$$

where the result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} - \frac{(B-A)(P-\alpha)}{Q_0(n,\delta,p)\,[p(A+B+2)+(B-A)\alpha]} z^p (p \in N), \tag{26}$$

and

$$Q_0(n,\delta,p) = 1 + \delta\left[\binom{n+2p}{n} - 1\right].$$

$$(0 \le \alpha < P;\, A+B \ge 0;\, -1 \le A < B \le 1;\, 0 < B \le 1;\, p \in N;\, n \in N_0;\, \delta \ge 0)$$

Proof For $f(z) \in T_p^*(\alpha, \delta, A, B, n)$, we find from Theorem 1 that

$$\begin{aligned} &p Q_0(n,\delta,p)\,[p(A+B+2)+(B-A)\alpha] \sum_{k=0}^{\infty} a_{k+p} \\ &\le \sum_{k=0}^{\infty} (k+p)\, Q_K(n,\delta,p)\,[(k+p)\,(B+1)+p(A+1)+(B-A)\alpha]\, a_{k+p} \\ &\le p(B-A)(P-\alpha), \end{aligned}$$

or

$$\sum_{k=0}^{\infty} a_{k+p} \le \frac{(B-A)(P-\alpha)}{Q_0(n,\delta,p)\,[p(A+B+2)+(B-A)\alpha]}, \tag{27}$$

Now by differentiating $f(z)$ in (2) m times, we have

$$f^m(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-p-m} - \sum_{k=0}^{\infty} \frac{(k+p)!}{(k+p-m)!} \left|a_{k+p}\right| z^{k+p-m},$$
$$(m \in N_0,\, P \in N, m < P) \tag{28}$$

Thus, for $0 \le |z| = r < 1$,

$$\begin{aligned} \left|f^m(z)\right| &= \left|(-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-p-m} - \sum_{k=0}^{\infty} \frac{(k+p)!}{(k+p-m)!} a_{k+p} z^{k+p-m}\right| \\ &\le \frac{(p+m-1)!}{(p-1)!} r^{-p-m} + \sum_{k=0}^{\infty} \frac{(k+p)!}{(k+p-m)!} a_{k+p} r^{k+p-m} \\ &\le \frac{(p+m-1)!}{(p-1)!} r^{-p-m} + \frac{p!}{(p-m)!} r^{p-m} \sum_{k=0}^{\infty} a_{k+p} \\ &\le \frac{(p+m-1)!}{(p-1)!} r^{-p-m} + \frac{p!}{(p-m)!} \frac{(B-A)(P-\alpha)}{Q_0(n,\delta,p)\,[p(A+B+2)+(B-A)\alpha]} r^{p-m}, \end{aligned}$$

similarly

$$\left|f^{m}(z)\right| \geq \frac{(p+m-1)!}{(p-1)!}r^{-p-m} - \frac{p!}{(p-m)!}r^{p-m}\sum_{k=0}^{\infty}a_{k+p}$$
$$\geq \frac{(p+m-1)!}{(p-1)!}r^{-p-m} - \frac{p!}{(p-m)!}\frac{(B-A)(P-\alpha)}{Q_0(n,\delta,p)\,[p(A+B+2)+(B-A)\alpha]}r^{p-m}.$$

The sharpness of each inequality in (25) satisfies the function $f(z)$ given by (26).

Next, we determine the radii of meromorphically p-valent starlikeness and convexity of order $\gamma (0 \leq \gamma < p)$ for functions in the class $T_p^*(\alpha, \delta, A, B, n)$.

Theorem 3 *If a function $f(z) \in \Omega_p$ defined by (2) is in the class $T_p^*(\alpha, \delta, A, B, n)$, then*

(i) $f(z)$ is meromorphically p-valent starlike of order $\gamma (0 \leq \gamma < p)$ in $|z| < r_1$, where

$$r_1 = \inf_{k\geq 0}\left\{Q_K(n,\delta,p)\frac{(k+p)\,(p-\gamma)\,[(k+p)\,(B+1)+p(A+1)+(B-A)\alpha]}{p(k+p+\gamma)(B-A)(P-\alpha)}\right\}^{\frac{1}{k+2p}}, \tag{29}$$

(ii) $f(z)$ is meromorphically p-valent convex of order $\gamma (0 \leq \gamma < p)$ in $|z| < r_2$, where

$$r_2 = \inf_{k\geq 0}\left\{Q_K(n,\delta,p)\frac{(p-\gamma)\,[(k+p)\,(B+1)+p(A+1)+(B-A)\alpha]}{(k+p+\gamma)(B-A)(P-\alpha)}\right\}^{\frac{1}{k+2p}}.$$
$$(0 \leq \alpha < P;\, A+B \geq 0;\, -1 \leq A < B \leq 1;\, 0 < B \leq 1;\, p \in N;\, n \in N_0;\, \delta \geq 0). \tag{30}$$

The result is sharp.

Proof (i) from (2), we easily get

$$\left|\frac{z\frac{f'(z)}{f(z)}+p}{z\frac{f'(z)}{f(z)}-p+2\gamma}\right| \leq \frac{\sum_{k=0}^{\infty}(k+2p)\,a_{k+p}\,|z|^{k+2p}}{2(\gamma+p)-\sum_{k=0}^{\infty}(k+2\gamma)a_{k+p}\,|z|^{k+2p}}.$$

Thus, we have the desired inequity:

$$\left|\frac{z\frac{f'(z)}{f(z)}+p}{z\frac{f'(z)}{f(z)}-p+2\gamma}\right| \leq 1 \qquad (0 \leq \gamma < p,\, p \in N), \tag{31}$$

if

$$\sum_{k=0}^{\infty}\frac{(k+p+\gamma)}{(p-\gamma)}a_{k+p}\,|z|^{k+2p} \leq 1. \tag{32}$$

Hence, by Theorem 1, (32) will be true if

$$\frac{(k+p+\gamma)}{(p-\gamma)}\,|z|^{k+2p} \leq \frac{(k+p)\,Q_K(n,\delta,p)\,[(k+p)\,(B+1)+p(A+1)+(B-A)\alpha]}{p(B-A)(P-\alpha)}$$

$$(k \geq 0, p \in N). \tag{33}$$

The inequality (33) leads us immediately to $|z| < r_1$, where r_1 is given by (29).

(ii) In order to prove the second assertion of the theorem, we find from (2) that

$$\left|\frac{1+z\frac{f''(z)}{f'(z)}+p}{1+z\frac{f''(z)}{f'(z)}-p+2\gamma}\right| \leq \frac{\sum_{k=0}^{\infty}(k+p)(k+2p)a_{k+p}\,|z|^{k+2p}}{2p(p-\gamma)-\sum_{k=0}^{\infty}(k+2\gamma)a_{k+p}\,|z|^{k+2p}}.$$

Thus, we have the desired inequity:

$$\left|\frac{1+z\frac{f''(z)}{f'(z)}+p}{1+z\frac{f''(z)}{f'(z)}-p+2\gamma}\right| \leq 1 \qquad (0 \leq \gamma < p,\ p \in N), \tag{34}$$

If

$$\sum_{k=0}^{\infty}\frac{(k+p)\,(k+p+\gamma)}{p(p-\gamma)}a_{k+p}\,|z|^{k+2p} \leq 1. \tag{35}$$

Hence, by Theorem 1, (35) will be true if

$$\frac{(k+p)\,(k+p+\gamma)}{p(p-\gamma)}\,|z|^{k+2p} \leq \frac{(k+p)\,Q_K(n,\delta,p)\,[(k+p)\,(B+1)+p(A+1)+(B-A)\alpha]}{p(B-A)(P-\alpha)}$$

$$(k \geq 0,\ p \in N), \tag{36}$$

The inequality (36) leads us immediately to $|z| < r_2$, where r_2 is given by (30). Each of these results is sharp for the function $f(z)$ given by (26).

Next, we prove closure theorems for the class $T_p^*(\alpha, \delta, A, B, n)$.

Theorem 4 *Let*

$$f_{-1} = \frac{1}{z^p} \tag{37}$$

and

$$f_{p+k}(z) = \frac{1}{z^p} - \frac{p(B-A)(P-\alpha)}{(p+k)Q_k(n,\delta,p)\,[(k+p)\,(B+1)+p(A+1)+(B-A)\alpha]}z^{p+k}$$

$$(k \geq 0;\ p \in N;\ n \in N_0). \tag{38}$$

Then $f(z)$ in the class $T_p^*(\alpha, \delta, A, B, n)$ if and only if it can expressed in the form

$$f(z) = \sum_{k=-1}^{\infty} \mu_{p+k} f_{P+K}(z), \tag{39}$$

where

$$\mu_{p+k} \geq 0 \, and \sum_{k=-1}^{\infty} \mu_{p+k} = 1.$$

Proof Let $f(z) = \sum_{k=-1}^{\infty} \mu_{p+k} f_{P+K}(z)$, where $\mu_{p+k} \geq 0$ and $\sum_{k=-1}^{\infty} \mu_{p+k} = 1$.

Then

$$f(z) = \sum_{k=-1}^{\infty} \mu_{p+k} f_{P+K}(z),$$

$$f(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} \mu_{p+k} \frac{p(B-A)(P-\alpha)}{(p+k)Q_k(n,\delta,p)\left[(k+p)(B+1)+p(A+1)+(B-A)\alpha\right]} z^{p+k}.$$

Then

$$\begin{aligned} &\sum_{k=0}^{\infty} \mu_{p+k} \frac{p(B-A)(P-\alpha)}{(p+k)Q_k(n,\delta,p)\left[(k+p)(B+1)+p(A+1)+(B-A)\alpha\right]} \\ &\times \frac{(p+k)Q_k(n,\delta,p)\left[(k+p)(B+1)+p(A+1)+(B-A)\alpha\right]}{p(B-A)(P-\alpha)} \\ &= \sum_{k=0}^{\infty} \mu_{p+k} = 1 - \mu_{p-1} \leq 1, \end{aligned}$$

which shows that $f(z) \in T_{\delta,p}^*(\alpha, A, B, n)$.

Conversely, let $f(z) \in T_{\delta,p}^*(\alpha, A, B, n)$, then

$$a_{k+p} \leq \frac{p(B-A)(P-\alpha)}{(p+k)Q_k(n,\delta,p)\left[(k+p)(B+1)+p(A+1)+(B-A)\alpha\right]}.$$

Set

$$\mu_{p+k} = \frac{p(B-A)(P-\alpha)}{(p+k)Q_k(n,\delta,p)\left[(k+p)(B+1)+p(A+1)+(B-A)\alpha\right]} a_{k+p},$$

and

$$\mu_{p-1} = 1 - \sum_{k=0}^{\infty} \mu_{p+k}.$$

It follows that $f(z) = \sum_{k=-1}^{\infty} \mu_{p+k} f_{P+K}(z)$. This completes the proof of Theorem.

Theorem 5 *The class $T_p^*(\alpha, \delta, A, B, n)$ is closed under convex linear combinations.*

Proof Let each of the functions

$$f_j(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} a_{k+p,j} z^{p+k} (a_{k+p,j} \geq 0;\ j = 1, 2) \tag{40}$$

be in the class $T_p^*(\alpha, \delta, A, B, n)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = (1-t) f_1(z) + t f_2(z) \in T_p^*(\alpha, \delta, A, B, n)(0 \leq t \leq 1), \tag{41}$$

is also in the class $T_p^*(\alpha, \delta, A, B, n)$, since

$$h(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} \left[(1-t) a_{k+p,1} + t a_{k+p,2}\right] z^{k+p} \quad (0 \leq t \leq 1). \tag{42}$$

With the aid of Theorem 1, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} (p+k) Q_k(n, \delta, p) \left[(k+p)(B+1) + p(A+1) + (B-A)\alpha\right] \left[(1-t) a_{k+p,1} + t a_{p+k,2}\right] \\
&= (1-t) \sum_{k=0}^{\infty} (p+k) Q_k(n, \delta, p) \quad \left[(k+p)(B+1) + p(A+1) + (B-A)\alpha\right] a_{p+k,1} \\
&\quad + t \sum_{k=0}^{\infty} (p+k) Q_k(n, \delta, p) \quad \left[(k+p)(B+1) + p(A+1) + (B-A)\alpha\right] a_{p+k,2} \\
&\leq (1-t) p(B-A)(P-\alpha) + t p(B-A)(P-\alpha) = p(B-A)(P-\alpha),
\end{aligned}$$

which shows that $h(z)) \in T_p^*(\alpha, \delta, A, B, n)$.

3 Neighborhoods and Partial Sums for the Class $T_p^*(\alpha, \delta, A, B, n)$

Following the earlier work (based upon the familiar concept of neighborhoods of analytic function) by Goodman [8] and Rusheweyh [17] and (more recently) by Altinatas et al. ([1, 2]) and Liu and Srivastava ([11, 12]), We begin by introducing here the δ-neighborhood of a function $f(z) \in \Omega_p$ of the form (2) by means of the definition below:

$$N_\delta(f) = \Bigg\{ g \in \Omega_p : g(z) = z^{-p} - \sum_{k=0}^{\infty} b_{k+p} z^{k+p} \left(b_{k+p} \geq 0\right) \text{ and }$$
$$\sum_{k=0}^{\infty} Q_K(n, \delta, p) \frac{(k+p)\left[(B-A)\alpha + k(1+B) + p(1+|A|)\right]}{p(B-A)(P-\alpha)} \left|a_{k+p} - b_{k+p}\right| \leq \delta \Bigg\}, \tag{43}$$

$$(0 \leq \alpha < P; -1 \leq A < B \leq 1; 0 < B \leq 1; p \in N; n \in N_0; \delta \geq 0)$$

Theorem 6 *Let $\delta > 0$. If the function $f(z) \in \Omega_p$ defined by (2) satisfies the following condition:*

$$\frac{f(z) + \varepsilon z^{-p}}{1+\varepsilon} \in T^*_{\delta,p}(\alpha, A, B, n), \tag{44}$$

for any complex number ε such that $|\varepsilon| < \delta$, then $N_\delta(f) \subset T^*_p(\alpha, \delta, A, B, n)$.

Proof We see from (18) that $g(z) \in T^*_p(\alpha, \delta, A, B, n)$ if and only if for any complex number σ, $|\sigma| = 1$, we have

$$\frac{1 + \frac{\left(z(D^n_{\delta,p} f(z))\right)''}{\left(D^n_{\delta,p} f(z)\right)'} + p}{B\left(1 + \frac{\left(z(D^n_{\delta,p} f(z))\right)''}{\left(D^n_{\delta,p} f(z)\right)'}\right) + [pB + (A-B)(P-\alpha)]} \neq \sigma,$$

which is equivalent to

$$\frac{g(z) * h(z)}{z^{-p}} \neq 0 \qquad (z \in U), \tag{45}$$

where, for convenience,

$$h(z) = z^{-p} - \sum_{k=0}^{\infty} c_{k+p} z^{k+p},$$

$$c_{k+p} = (k+p)\, Q_K(n, \delta, p) \frac{[\sigma(B-A)\alpha - (k+p)(1-\sigma B) - p(1-\sigma A)]}{\sigma P(B-A)(P-\alpha)}. \tag{46}$$

It follows from (46) that

$$\begin{aligned}\left|c_{k+p}\right| &\leq (k+p)\, Q_K(n, \delta, p) \frac{[(B-A)\alpha + k(1+B) + p(1+|A|)]}{P(B-A)(P-\alpha)} \\ &= (k+p)\, Q_K(n, \delta, p) \frac{[(B-A)\alpha + k(1+B) + p(1+|A|)]}{p(B-A)(P-\alpha)}\end{aligned}$$

If $f(z) \in \Omega_p$ defined by (2) satisfies (44), then (45) yields

$$\left| \frac{f(z)^{*} h(z)}{z^{-p}} \right| \geq \delta. \tag{47}$$

Now, we suppose that

$$\varphi(z) = z^{-p} - \sum_{k=0}^{\infty} d_{k+p} z^{k+p} \in N_{\delta}(f) \left(d_{k+p} \geq 0 \right). \tag{48}$$

We easily seen that

$$\left| \frac{[\varphi(z) - f(z)] * h(z)}{z^{-p}} \right| = \left| \sum_{k=0}^{\infty} \left(d_{k+p} - a_{k+p} \right) c_{k+p} z^{k+2p} \right|$$

$$\leq |z| \sum_{k=0}^{\infty} (k+p)\, Q_K(n, \delta, p) \frac{[(B-A)\alpha + k(1+B) + p\,(1+|A|)]}{p(B-A)(P-\alpha)} \left| \left(d_{k+p} - a_{k+p} \right) \right| < \delta.$$

$(z \in U; \delta > 0)$

Thus for any number σ such that $|\sigma| = 1$, we have

$$\frac{\varphi(z) * h(z)}{z^{-p}} \neq 0 \qquad (z \in U),$$

which implies that $\varphi(z) \in T_p^{*}(\alpha, \delta, A, B, n)$. This completes the proof of the theorem.

Theorem 7 *Let the function $f(z) \in \Omega_p$ defined by (2) and define the partial sum $s_1(z)$ and $s_m(z)$ as follows:*

$$s_1(z) = z^{-p},$$

and

$$s_m(z) = z^{-p} - \sum_{k=0}^{m} a_{k+p} z^{k+p} \qquad (m = 0, 1, 2, \ldots), \tag{49}$$

Suppose also that

$$\sum_{k=0}^{\infty} d_{k+p} a_{k+p} \leq 1 \left(d_{k+p} = (k+p)\, Q_K(n, \delta, p) \frac{[(B-A)\alpha + k(1+B) + p\,(1+|A|)]}{p(B-A)(P-\alpha)} \right), \tag{50}$$

then we have that

(i) $f(z) \in T_p^{*}(\alpha, \delta, A, B, n)$

(ii)
$$\operatorname{Re}\left\{ \frac{f(z)}{s_m(z)} \right\} > 1 - \frac{1}{d_{m+p+1}} \qquad (m = 0, 1, 2, \ldots), \tag{51}$$

and

(iii)
$$\mathrm{Re}\left\{\frac{s_m(z)}{f(z)}\right\} > \frac{d_{m+p+1}}{d_{m+p+1}+1} \qquad (m = 0, 1, 2, \ldots), \tag{52}$$

The estimation in (51) and (52) is sharp.

Proof (i) It is not difficult to see that $z^{-p} \in T_p^*(\alpha, \delta, A, B, n)$. According to Theorem 6 and hypothesis (50), we have $N_1(z^{-p}) \subset T_p^*(\alpha, \delta, A, B, n)$, which follows that $f(z) \in T_p(\alpha, \delta, A, B, n)$.

(ii) Under the hypothesis in part (ii) of the theorem, we can see from (50) that

$$\sum_{k=0}^{m} a_{k+p} + d_{m+p+1} \sum_{k=m+1}^{\infty} a_{k+p} \leq \sum_{k=0}^{\infty} d_{k+p} a_{k+p} < 1. \tag{53}$$

by using hypothesis (50) again.

Upon setting

$$g_1(z) = d_{m+p+1}\left\{\frac{f(z)}{s_m(z)} - \left(1 - \frac{1}{d_{m+p+1}}\right)\right\} = 1 - \frac{d_{m+p+1}\displaystyle\sum_{k=m+1}^{\infty} a_{k+p} z^{k+2p}}{1 - \displaystyle\sum_{k=0}^{m} a_{k+p} z^{k+2p}} \quad , \tag{54}$$

and applying (53), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{m+p+1}\displaystyle\sum_{k=m+1}^{\infty} a_{k+p}}{2 - 2\displaystyle\sum_{k=0}^{m} a_{k+p} - d_{m+p+1}\displaystyle\sum_{k=m+1}^{\infty} a_{k+p}} \leq 1 \quad (z \in U,\ m \geq 0), \tag{55}$$

which readily yields the assertion (51) of Theorem 7. If we take

$$f(z) = z^{-p} - \frac{z^{m+p+1}}{d_{m+p+1}} \quad , \tag{56}$$

then

$$\frac{f(z)}{s_m(z)} = 1 - \frac{z^{m+2p-1}}{d_{m+p+1}} \to 1 - \frac{1}{d_{m+p+1}} \text{as } z \to 1,$$

which shows that the bound in (51) is best possible.

(iii) Similarly, if we put

$$g_2(z) = \left(1 + d_{m+p+1}\right)\left(\frac{s_m(z)}{f(z)} - \frac{d_{m+p+1}}{1 + d_{m+p+1}}\right) = 1 + \frac{\left(1 + d_{m+p+1}\right)\sum_{k=m+1}^{\infty} a_{k+p} z^{k+2p}}{1 - \sum_{k=0}^{\infty} a_{k+p} z^{k+2p}}, \tag{57}$$

we obtain the assertion (52) of Theorem 7. If we take

$$f(z) = z^{-p} - \frac{z^{m+p+1}}{d_{m+p+1}}, \tag{58}$$

then

$$\frac{s_m(z)}{f(z)} = \frac{d_{m+p+1}}{d_{m+p+1} - z^{m+2p-1}} \to \frac{d_{m+p+1}}{d_{m+p+1} - 1} \text{as } z \to 1,$$

which shows that the bound in (52) is best possible.

References

1. O. Altintas, On a subclass of certain starlike functions with negative coefficients. Math. Japon. **36**(3), 489–495 (1991)
2. O. Altintas, O. Ozkan, H.M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients. Appl. Math. Lett. **13**(3), 63–67 (2000)
3. M.K. Aouf, New criteria for multivalent meromorphic starlike functions of order alpha. Proc. Jpn. Acad. Ser. A Math. Sci. **69**, 66–70 (1993)
4. M.K. Aouf, A class of meromorphic multivalent functions with positive coefficient. Taiwan. J. Math. **12**, 2517–2533 (2008)
5. M.K. Aouf, H.M. Hossen, New criteria for meromorphic p-valent starlike functions. Tsukuba J. Math. **17**, 481–486 (1993)
6. M.K. Aouf, H.M. Srivastava, A new criteria for meromorphic p-valent convex functions of order alpha. Math. Sci. Res. Hot-line **1**(8), 7–12 (1997)
7. M.K. Aouf, A.E. Shammaky, A certain subclass of meromorphic p-valent convex functions with negative coefficients. J. Approx. Theory Appl. **1**(2), 123–143 (2005)
8. A.W. Goodman, Univalent functions and non analytic curves. Proc. Am. Math. Soc. **8**, 598–601 (1957)
9. S.B. Joshi, H.M. Srivastava, A certain family of meromorphically multivalent functions. Comput. Math. Appl. **38**, 201–211 (1999)
10. S.R. Kulkarni, U.H. Naik, H.M. Srivastava, A certain class of meromorphically p-valent quasi-convex functions. Pan Am. Math. J. **8**(1), 57–64 (1998)
11. J.L. Liu, H.M. Srivastava, A linear operator and associated families of mermorphically multivalent function. J. Math. Anal. Appl. **259**, 566–581 (2001)
12. J.L. Liu, H.M. Srivastava, Subclasses of meromorphically multivalent functions associated with a certain linear operator. Math. Comput. Model. **39**, 35–44 (2004)
13. M.L. Mogra, Meromorphic multivalent functions with positive coefficients I and II. Math. Japon. **35**, 1–11 and 1089–1098 (1990)
14. S. Owa, H.E. Darwish, M.K. Aouf, Meromorphic multivalent functions with positive and fixed second coefficients. Math. Japon. **46**, 231–236 (1997)

15. R.K. Raina, H.M. Srivastava, Inclusion and neighborhoods properties of some analytic and multivalent functions. J. Inequal. Pure Appl. Math. **7**(1) Article 5:1–6 (electronic) (2006)
16. S. Rusheweyh, New criteria for univalent functions. Proc. Am. Math. Soc. **49**, 109–115 (1975)
17. S. Ruscheweyh, Neighborhoods of univalent functions. Proc. Am. Math. Soc. **81**(4), 521–527 (1981)
18. H.M. Srivastava, H.M. Hossen, M.K. Aouf, A certain subclass of meromorphically functions with negative coefficients. Math. J. Ibaraki Univ. **30**, 33–51 (1998)
19. B.A. Uralegaddi, M.D. Ganigi, Meromorphic convex functions with negative coefficients. J. Math. Res. Expo. **1**, 21–26 (1987)
20. D. Yang, On a class of meromorphic starlike multivalent functions. Bull. Inst. Math. Acad. Sin. **24**, 151–157 (1996)

Bivariate Symmetric Discrete Orthogonal Polynomials

Y. Guemo Tefo, Iván Area and M. Foupouagnigni

Abstract In this paper, we analyze second-order linear partial difference equations having bivariate symmetric orthogonal polynomial solutions. We present conditions to have admissible, potentially self-adjoint partial difference equations of hypergeometric type having orthogonal polynomial solutions. For these solutions, we give explicitly the matrix coefficients of the three-term recurrence relations they satisfy. Finally, conditions in order to have symmetric orthogonal polynomial solutions are presented.

1 Introduction

The theory of univariate orthogonal polynomials has been deeply developed because of the strong relations with other areas of mathematics and of course because this type of special functions appear in with several applications in physics and engineering. Univariate orthogonal polynomials can be presented from, e.g., the differential

Y. Guemo Tefo
Faculty of Science, Department of Mathematics, University of Yaounde I,
Yaoundé, Cameroon
e-mail: guemoyves2000@yahoo.fr

I. Area (✉)
Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo,
Universidade de Vigo, Campus As Lagoas s/n, 32004 Ourense, Spain
e-mail: area@uvigo.es

M. Foupouagnigni
Department of Mathematics, Higher Teachers' Training College,
University of Yaounde I, Yaoundé, Cameroon
e-mail: mfoupouagnigni@aims-cameroon.org

M. Foupouagnigni
African Institute for Mathematical Sciences (AIMS), P.O. Box 608,
Limbe Crystal Gardens, South West Region, Cameroon

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_5

equations they satisfy, giving rise to many important differential equations of mathematical physics. As a consequence, univariate orthogonal polynomials appear in a natural way in the study of wave mechanics, heat conduction, electromagnetic theory, quantum mechanics or mathematical statistics, to cite some applications.

Moreover, the univariate symmetric orthogonal polynomials appear in many interesting applications. The well-known (univariate) Gegenbauer polynomials appear, e.g., in the resolution of the Gibbs phenomenon [8, 9] or in tissue segmentation of human brain MRI through preprocessing [1]. Also, as for the univariate discrete situation, the symmetric Kravchuk polynomials appear in the Fourier–Kravchuk transform used in Optics [5] or in the approximation of harmonic oscillator wave functions [6].

In this context, if $\{p_n(x)\}_{n\in\mathbf{N}}$ is a sequence of univariate polynomials, the Favard theorem [21] links the orthogonality with the three-term recurrence relation, which in the monic form can be expressed as

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad \gamma_n \neq 0.$$

In the univariate symmetric situations (both continuous and discrete), the symmetry of the polynomials implies that $\beta_n = 0$ for all $n = 0, 1, 2, \dots$. Moreover, in the classical case [16], it is possible to provide explicit expressions for the coefficients β_n and γ_n that appear in the above three-term recurrence relation satisfied by $\{p_n(x)\}_{n\in\mathbf{N}}$ from the coefficients in the second-order differential equation satisfied by the polynomials.

This idea of expressing the coefficients of the three-term recurrence relation in terms of the coefficients of the second-order differential equation satisfied by the polynomials has been extended to the bivariate continuous, discrete and their q-analogues cases [2–4, 18, 19], where now the coefficients of the three-term recurrence relations satisfied by bivariate orthogonal polynomials have been explicitly given in terms of the coefficients of the partial differential, difference or q-difference equation satisfied by the bivariate polynomials. In doing so, graded lexicographical order and the matrix vector representation have been used, first introduced by Kowalski [10, 11] and afterward considered by Xu [22, 23].

In the bivariate continuous case, the following partial differential equation was considered by Lyskova [13–15]

$$\begin{aligned}(a_1x^2 + b_1x + c_1)\frac{\partial^2}{\partial x^2}u(x, y) + (a_2y^2 + b_2y + c_2)\frac{\partial^2}{\partial x^2}u(x, y)\\ + 2(a_3xy + b_3x + c_3y + d_3)\frac{\partial^2}{\partial x\partial y}u(x, y)\\ + (e_1x + f_1)\frac{\partial}{\partial x}u(x, y) + (e_2y + f_2)\frac{\partial}{\partial y}u(x, y) + \lambda u(x, y) = 0,\end{aligned}$$

where a_i, b_i, c_i, d_i, e_i, f_i and λ are real numbers, which has the property that the partial derivatives of any solution also satisfy an equation of the same type (Lyskova class or hypergeometric equation). The above partial differential equation has been discretized in [2, 18, 24] by introducing uniquely two partial difference operators for the crossed second-order partial derivative $\dfrac{\partial^2}{\partial x \partial y}$ giving rise to

$$\begin{aligned}\sigma_{11}(x, y)\Delta_1\nabla_1 u(x, y) + \sigma_{12}(x, y)\Delta_1\nabla_2 u(x, y) + \sigma_{21}(x, y)\Delta_2\nabla_1 u(x, y)\\ + \sigma_{22}(x, y)\Delta_2\nabla_2 u(x, y) + \tau_1(x, y)\Delta_1 u(x, y) + \tau_2(x, y)\Delta_2 u(x, y) + \lambda u(x, y) = 0\,,\end{aligned} \tag{1}$$

where σ_{ij} and τ_i are polynomials of at most total degree two and one, respectively. In [24], the equation has been analyzed under the hypothesis of being admissible and potentially self-adjoint. Moreover, in [2, 18] the hypergeometric condition has been added, giving rise to Rodrigues formula for the bivariate orthogonal polynomial solutions.

In this paper, we consider the following second-order linear partial difference equation

$$\begin{aligned}\sigma_{11}(\mathbf{x})\Delta_1\nabla_1 u(\mathbf{x}) + \sigma_{22}(\mathbf{x})\Delta_2\nabla_2 u(\mathbf{x}) + \sigma_{12a}(\mathbf{x})\Delta_1\nabla_2 u(\mathbf{x})\\ + \sigma_{12b}(\mathbf{x})\Delta_2\nabla_1 u(\mathbf{x}) + \sigma_{12c}(\mathbf{x})\nabla_1\nabla_2 u(\mathbf{x}) + \sigma_{12d}(\mathbf{x})\Delta_1\Delta_2 u(\mathbf{x})\\ + \tau_1(\mathbf{x})\Delta_1 u(\mathbf{x}) + \tau_2(\mathbf{x})\Delta_2 u(\mathbf{x}) + \lambda u(\mathbf{x}) = 0,\end{aligned} \tag{2}$$

where σ_{ii}, $i = 1, 2$ and σ_{12k}, $k = a, b, c, d$, are polynomials of at most total degree two, and τ_i are polynomials of total degree one, λ is the spectral parameter, and we have used $(\mathbf{x}) = (x, y)$.

The paper is organized as follows. First, we obtain conditions in order that (2) be an equation of hypergeometric type. Conditions for (2) being an admissible equation are obtained in Sect. 3. Moreover, in Sect. 4 conditions for (2) being admissible and potentially self-adjoint are derived. In these conditions, in Sect. 5 we obtain explicit expressions for the matrix coefficients appearing in the three-term recurrence relations satisfied by the orthogonal polynomial solutions of (2). Finally, in Sect. 6 conditions in order to have symmetric orthogonal polynomial solutions are presented.

2 The Hypergeometric Class of the Linear Second-Order Partial Difference Equation

An important class of differential and difference equation has been analyzed, e.g., in [17]. A differential equation is said to be of hypergeometric class if all the derivatives of a solution of the equation are solution of an equation of the same type. In the bivariate case, this concept has been first analyzed by Lyskova [13–15].

Definition 1 We say that Eq. (2) is of hypergeometric type if all the difference derivatives $u^{(r,s)}(\mathbf{x}) = \Delta_1^r \Delta_2^s u(\mathbf{x})$ of any solution $u(\mathbf{x})$ of the Eq. (2) are solution of an equation of the same type.

Let us consider the second-order linear partial difference equation (2) where the polynomial coefficients are given by

$$\begin{aligned}
\sigma_{11}(\mathbf{x}) &= a_{11}x^2 + b_{11}y^2 + c_{11}xy + d_{11}x + e_{11}y + f_{11},\\
\sigma_{22}(\mathbf{x}) &= a_{22}x^2 + b_{22}y^2 + c_{22}xy + d_{22}x + e_{22}y + f_{22},\\
\sigma_{12a}(\mathbf{x}) &= a_{12a}x^2 + b_{12a}y^2 + c_{12a}xy + d_{12a}x + e_{12a}y + f_{12a},\\
\sigma_{12b}(\mathbf{x}) &= a_{12b}x^2 + b_{12b}y^2 + c_{12b}xy + d_{12b}x + e_{12b}y + f_{12b},\\
\sigma_{12c}(\mathbf{x}) &= a_{12c}x^2 + b_{12c}y^2 + c_{12c}xy + d_{12c}x + e_{12c}y + f_{12c},\\
\sigma_{12d}(\mathbf{x}) &= a_{12d}x^2 + b_{12d}y^2 + c_{12d}xy + d_{12d}x + e_{12d}y + f_{12d},\\
\tau_1(\mathbf{x}) &= \tau_{11}x + \tau_{12}y + \tau_{13},\\
\tau_2(\mathbf{x}) &= \tau_{21}x + \tau_{22}y + \tau_{23}.
\end{aligned}$$

Note that Eq. (2) reduces to the Eq. (1) considered in [2, 18, 24] in the particular case $\sigma_{12c} = \sigma_{12d} = 0$.

Next, we obtain conditions for (2) to be of hypergeometric type. Let us apply the Δ_1 operator to (2) and denote $\Delta_1 u(\mathbf{x}) = u^{(1,0)}(\mathbf{x})$. If we analyze each summand, we have

$$\Delta_1\left[\lambda u(\mathbf{x})\right] = \lambda u^{(1,0)}(\mathbf{x}) \tag{3}$$

$$\Delta_1\left[\tau_2(\mathbf{x})\Delta_2 u(\mathbf{x})\right] = \tau_2(\mathbf{x})\Delta_2 u^{(1,0)}(\mathbf{x}) + \Delta_1\tau_2(\mathbf{x})\Delta_2 u(x+1,y) \tag{4}$$

$$\Delta_1\left[\tau_1(\mathbf{x})\Delta_1 u(\mathbf{x})\right] = \Delta_1\tau_1(\mathbf{x})u^{(1,0)}(\mathbf{x}) + \tau_1(x+1,y)\Delta_1 u^{(1,0)}(\mathbf{x}) \tag{5}$$

$$\Delta_1\left[\sigma_{11}(\mathbf{x})\Delta_1\nabla_1 u(\mathbf{x})\right] = \sigma_{11}(\mathbf{x})\Delta_1\nabla_1 u^{(1,0)}(\mathbf{x}) + \Delta_1\sigma_{11}(\mathbf{x})\Delta_1 u^{(1,0)}(\mathbf{x}) \tag{6}$$

$$\Delta_1\left[\sigma_{22}(\mathbf{x})\Delta_2\nabla_2 u(\mathbf{x})\right] = \sigma_{22}(\mathbf{x})\Delta_2\nabla_2 u^{(1,0)}(\mathbf{x}) + \Delta_1\sigma_{22}(\mathbf{x})\Delta_2\nabla_2 u(x+1,y) \tag{7}$$

$$\Delta_1\left[\sigma_{12a}(\mathbf{x})\Delta_1\nabla_2 u(\mathbf{x})\right] = \Delta_1\sigma_{12a}(\mathbf{x})\nabla_2 u^{(1,0)}(\mathbf{x}) + \sigma_{12a}(x+1,y)\Delta_1\nabla_2 u^{(1,0)}(\mathbf{x}) \tag{8}$$

$$= \Delta_1\sigma_{12a}(\mathbf{x})\Delta_2 u^{(1,0)}(\mathbf{x}) - \Delta_1\sigma_{12a}(\mathbf{x})\Delta_2\nabla_2 u^{(1,0)}(\mathbf{x}) + \sigma_{12a}(x+1,y)\Delta_1\nabla_2 u^{(1,0)}(\mathbf{x})$$

$$\Delta_1\left[\sigma_{12b}(\mathbf{x})\Delta_2\nabla_1 u(\mathbf{x}\right] = \sigma_{12b}(\mathbf{x})\Delta_2\nabla_1 u^{(1,0)}(\mathbf{x}) + \Delta_1\sigma_{12b}(\mathbf{x})\Delta_2 u^{(1,0)}(\mathbf{x}) \tag{9}$$

$$\Delta_1\left[\sigma_{12d}(\mathbf{x})\Delta_1\Delta_2 u(\mathbf{x})\right] = \Delta_1\sigma_{12d}(\mathbf{x})\Delta_2 u^{(1,0)}(\mathbf{x}) + \sigma_{12d}(x+1,y)\Delta_1\Delta_2 u^{(1,0)}(\mathbf{x}) \tag{10}$$

$$\Delta_1\left[\sigma_{12c}(\mathbf{x})\nabla_1\nabla_2 u(\mathbf{x})\right] = \sigma_{12c}(\mathbf{x})\nabla_1\nabla_2 u^{(1,0)}(\mathbf{x}) + \Delta_1\sigma_{12c}(\mathbf{x})\nabla_2 u^{(1,0)}(\mathbf{x}) \tag{11}$$

$$= \sigma_{12c}(\mathbf{x})\nabla_1\nabla_2 u^{(1,0)}(\mathbf{x}) + \Delta_1\sigma_{12c}(\mathbf{x})\Delta_2 u^{(1,0)}(\mathbf{x}) - \Delta_1\sigma_{12c}(\mathbf{x})\Delta_2\nabla_2 u^{(1,0)}(\mathbf{x})$$

In order to $u^{(1,0)}(\mathbf{x})$ be solution of an equation of the same type, from Eq. (4) we have that $\Delta_1\tau_2(\mathbf{x}) = 0$, or equivalently $\tau_2(\mathbf{x})$ does not depend on x. Symmetrically, if we apply Δ_2 to (2) and since $u^{(0,1)}(\mathbf{x})$ must be solution of an equation of the same type, we obtain $\Delta_2\tau_1(\mathbf{x}) = 0$, or equivalently $\tau_1(\mathbf{x})$ does not depend on y. Using these properties and (5), we have that $\tau_1(\mathbf{x})$ must be a polynomial of degree 1 in x and $\tau_2(\mathbf{x})$ must be a polynomial of degree 1 in y in order to have an equation with constant eigenvalue. Moreover, from (6) we have that $\Delta_1\sigma_{11}(\mathbf{x})$ must be a polynomial of first degree in x, so it does not contain term in xy. Symmetrically, $\Delta_2\sigma_{22}(\mathbf{x})$ must be a polynomial of first degree in y, so it does not contain term in xy. From (7), $\Delta_1\sigma_{22} = 0$ and symmetrically $\Delta_2\sigma_{11} = 0$ which imply that σ_{11} does not depend on y and σ_{22} does not depend on x, respectively. If we collect the terms multiplying $\Delta_2 u^{(1,0)}(\mathbf{x})$, we obtain that $\Delta_1(\tau_2(\mathbf{x}) + \sigma_{12a}(\mathbf{x}) + \sigma_{12b}(\mathbf{x}) + \sigma_{12c}(\mathbf{x}) + \sigma_{12d}(\mathbf{x}))$ must be a polynomial of degree 1 in y, or equivalently $\Delta_1^2(\sigma_{12a}(\mathbf{x}) + \sigma_{12b}(\mathbf{x}) + \sigma_{12c}(\mathbf{x}) + \sigma_{12d}(\mathbf{x})) = 0$. Symmetrically, $\Delta_2^2(\sigma_{12a}(\mathbf{x}) + \sigma_{12b}(\mathbf{x}) + \sigma_{12c}(\mathbf{x}) + \sigma_{12d}(\mathbf{x})) = 0$. If we collect the terms in $\Delta_2\nabla_2 u^{(1,0)}(\mathbf{x})$, we observe that $\sigma_{22}(\mathbf{x}) - \Delta_1(\sigma_{12a}(\mathbf{x}) + \sigma_{12c}(\mathbf{x}))$ must not depend on x which implies that $\Delta_1^2(\sigma_{12a}(\mathbf{x}) + \sigma_{12c}(\mathbf{x})) = 0$. Symmetrically, $\Delta_2^2(\sigma_{12b}(\mathbf{x}) + \sigma_{12c}(\mathbf{x})) = 0$.

Lemma 1 *Equation (2) belongs to the hypergeometric class if and only if*

$$\begin{cases} \sigma_{11}(\boldsymbol{x}) = a_{11}x^2 + d_{11}x + f_{11}, \\ \sigma_{22}(\boldsymbol{x}) = b_{22}y^2 + e_{22}y + f_{22}, \\ \sigma_{12a}(\boldsymbol{x}) = a_{12a}x^2 + b_{12a}y^2 + c_{12a}xy + d_{12a}x + e_{12a}y + f_{12a}, \\ \sigma_{12b}(\boldsymbol{x}) = a_{12b}x^2 + b_{12b}y^2 + c_{12b}xy + d_{12b}x + e_{12b}y + f_{12b}, \\ \sigma_{12c}(\boldsymbol{x}) = -a_{12a}x^2 - b_{12b}y^2 + c_{12c}xy + d_{12c}x + e_{12c}y + f_{12c}, \\ \sigma_{12d}(\boldsymbol{x}) = -a_{12b}x^2 - b_{12a}y^2 + c_{12d}xy + d_{12d}x + e_{12d}y + f_{12d}, \\ \tau_1(\boldsymbol{x}) = \tau_{11}x + \tau_{13}, \\ \tau_2(\boldsymbol{x}) = \tau_{22}y + \tau_{23}. \end{cases} \tag{12}$$

Theorem 1 *Let us assume that (2) is of hypergeometric type. If $u(\boldsymbol{x})$ is solution of (2), then $u_\alpha(\boldsymbol{x}) = \Delta_1^r\Delta_2^s u(\boldsymbol{x})$ is a solution of the following equation belonging to the hypergeometric class:*

$$\begin{aligned} \sigma_{11}^{(r,s)}(\boldsymbol{x})\Delta_1\nabla_1 u_\alpha(\boldsymbol{x}) + \sigma_{22}^{(r,s)}(\boldsymbol{x})\Delta_2\nabla_2 u_\alpha(\boldsymbol{x}) + \sigma_{12a}^{(r,s)}(\boldsymbol{x})\Delta_1\nabla_2 u_\alpha(\boldsymbol{x}) \\ + \sigma_{12b}^{(r,s)}(\boldsymbol{x})\Delta_2\nabla_1 u_\alpha(\boldsymbol{x}) + \sigma_{12c}^{(r,s)}(\boldsymbol{x})\nabla_1\nabla_2 u_\alpha(\boldsymbol{x}) + \sigma_{12d}^{(r,s)}(\boldsymbol{x})\Delta_1\Delta_2 u_\alpha(\boldsymbol{x}) \\ + \tau_1^{(r,s)}(\boldsymbol{x})\Delta_1 u_\alpha(\boldsymbol{x}) + \tau_2^{(r,s)}(\boldsymbol{x})\Delta_2 u_\alpha(\boldsymbol{x}) + \mu^{(r,s)} u_\alpha(\boldsymbol{x}) = 0 \end{aligned} \tag{13}$$

where

$$\begin{aligned} \sigma_{11}^{(r,s)}(\boldsymbol{x}) &= \sigma_{11} - s\Delta_2(\sigma_{12b} + \sigma_{12c}) - \frac{s(s-1)}{2}\Delta_2^2\sigma_{12b}, \\ \sigma_{22}^{(r,s)}(\boldsymbol{x}) &= \sigma_{22} - r\Delta_1(\sigma_{12a} + \sigma_{12c}) - \frac{r(r-1)}{2}\Delta_1^2\sigma_{12a}, \\ \sigma_{12a}^{(r,s)}(\boldsymbol{x}) &= \sigma_{12a} + r\Delta_1(\sigma_{12a}) + \frac{r(r-1)}{2}\Delta_1^2\sigma_{12a}, \\ \sigma_{12b}^{(r,s)}(\boldsymbol{x}) &= \sigma_{12b} + s\Delta_2(\sigma_{12b}) + \frac{s(s-1)}{2}\Delta_2^2\sigma_{12b}, \\ \sigma_{12c}^{(r,s)}(\boldsymbol{x}) &= \sigma_{12c}, \end{aligned}$$

$$\sigma_{12d}^{(r,s)}(\boldsymbol{x}) = \sigma_{12d} + r\Delta_1\sigma_{12d} + s\Delta_2\sigma_{12d} + \frac{r(r-1)}{2}\Delta_1^2\sigma_{12d} + rs\Delta_1\Delta_2\sigma_{12d} + \frac{s(s-1)}{2}\Delta_2^2\sigma_{12d},$$

$$\tau_1^{(r,s)}(\boldsymbol{x}) = \tau_1 + r\Delta_1(\tau_1+\sigma_{11}) + s\Delta_2(\sigma_{12a}+\sigma_{12b}+\sigma_{12c}+\sigma_{12d}) + \frac{r(r-1)}{2}\Delta_1^2\sigma_{11} + rs\Delta_1\Delta_2(\sigma_{12a}+\sigma_{12d}) + \frac{s(s-1)}{2}\Delta_2^2(\sigma_{12b}+\sigma_{12d}),$$

$$\tau_2^{(r,s)}(\boldsymbol{x}) = \tau_2 + s\Delta_2(\tau_2+\sigma_{22}) + r\Delta_1(\sigma_{12a}+\sigma_{12b}+\sigma_{12c}+\sigma_{12d}) + \frac{s(s-1)}{2}\Delta_2^2\sigma_{22} + rs\Delta_1\Delta_2(\sigma_{12b}+\sigma_{12d}) + \frac{r(r-1)}{2}\Delta_1^2(\sigma_{12a}+\sigma_{12d}),$$

$$\mu^{(r,s)}(\boldsymbol{x}) = \lambda + r\Delta_1\tau_1 + s\Delta_2\tau_2 + \frac{r(r-1)}{2}\Delta_1^2\sigma_{11} + \frac{s(s-1)}{2}\Delta_2^2\sigma_{22} + rs\Delta_1\Delta_2(\sigma_{12a}+\sigma_{12b}+\sigma_{12c}+\sigma_{12d}).$$

Proof The result can be obtained by applying repeatedly the forward difference operators Δ_1 and Δ_2 to the initial Eq. (2).

3 Admissible Equations

The idea of admissibility was first introduced by Krall and Sheffer [12] in the case of second-order linear partial differential equations.

Definition 2 The second-order linear partial difference equation (2) is said to be admissible if there exists a sequence $\{\lambda_n\}$, $(n = 0, 1, 2\dots)$ such that for $\lambda = \lambda_n$ there are precisely $n+1$ linearly independent solutions in the form of polynomials of total degree n and there are no nontrivial solutions in the set of polynomials whose total degree is less than n.

Theorem 2 *The second-order linear partial difference equation* (2) *with polynomial coefficients given in* (12) *is admissible if and only if*

$$\begin{cases} \tau_{22} = \tau_{11}, \\ b_{22} = a_{11}, \\ c_{12d} = 2a_{11} - c_{12a} - c_{12b} - c_{12c}, \\ \lambda_n = -n((n-1)a_{11} + \tau_{11}). \end{cases}$$

Proof The proof can be done in a similar way as in the continuous case [15, 20].

As a consequence, we have that the polynomial coefficients of the second-order partial difference equation (2) have the form

$$
\begin{cases}
\sigma_{11}(\mathbf{x}) = a_{11}x^2 + d_{11}x + f_{11}, \\
\sigma_{22}(\mathbf{x}) = a_{11}y^2 + e_{22}y + f_{22}, \\
\sigma_{12a}(\mathbf{x}) = a_{12a}x^2 + b_{12a}y^2 + c_{12a}xy + d_{12a}x + e_{12a}y + f_{12a}, \\
\sigma_{12b}(\mathbf{x}) = a_{12b}x^2 + b_{12b}y^2 + c_{12b}xy + d_{12b}x + e_{12b}y + f_{12b}, \\
\sigma_{12c}(\mathbf{x}) = -a_{12a}x^2 - b_{12b}y^2 + c_{12c}xy + d_{12c}x + e_{12c}y + f_{12c}, \\
\sigma_{12d}(\mathbf{x}) = -a_{12b}x^2 - b_{12a}y^2 + (2a_{11} - c_{12a} - c_{12b} - c_{12c})xy \\
\qquad\qquad + d_{12d}x + e_{12d}y + f_{12d}, \\
\tau_1(\mathbf{x}) = \tau_{11}x + \tau_{13}, \\
\tau_2(\mathbf{x}) = \tau_{11}y + \tau_{23}, \\
\lambda_n = -n((n-1)a_{11} + \tau_{11}).
\end{cases}
\tag{14}
$$

4 Potentially Self-adjoint Difference Operators

The operator $\mathscr{D}$ defined by

$$
\begin{aligned}
\mathscr{D}u(\mathbf{x}) = {} & \sigma_{11}(\mathbf{x})\Delta_1\nabla_1 u(\mathbf{x}) + \sigma_{22}(\mathbf{x})\Delta_2\nabla_2 u(\mathbf{x}) + \sigma_{12a}(\mathbf{x})\Delta_1\nabla_2 u(\mathbf{x}) \\
& + \sigma_{12b}(\mathbf{x})\Delta_2\nabla_1 u(\mathbf{x}) + \sigma_{12c}(\mathbf{x})\nabla_1\nabla_2 u(\mathbf{x}) + \sigma_{12d}(\mathbf{x})\Delta_1\Delta_2 u(\mathbf{x}) \\
& + \tau_1(\mathbf{x})\Delta_1 u(\mathbf{x}) + \tau_2(\mathbf{x})\Delta_2 u(\mathbf{x}),
\end{aligned}
\tag{15}
$$

allows us to write the second-order linear partial difference equation as

$$
\mathscr{D}u(\mathbf{x}) + \lambda u(\mathbf{x}) = 0.
$$

The adjoint operator $\mathscr{D}^\dagger$ of $\mathscr{D}$ is defined by

$$
\begin{aligned}
\mathscr{D}^\dagger u = {} & \Delta_1\nabla_1(\sigma_{11}u) + \Delta_1\nabla_2(\sigma_{12b}u) + \Delta_2\nabla_1(\sigma_{12a}u) \\
& + \Delta_1\Delta_2(\sigma_{12c}u) + \nabla_1\nabla_2(\sigma_{12d}u) + \Delta_2\nabla_2(\sigma_{22}u) - \nabla_1(\tau_1 u) - \nabla_2(\tau_2 u).
\end{aligned}
$$

Definition 3 An operator $\mathscr{A}$ is self-adjoint if $\mathscr{A}^\dagger = \mathscr{A}$.

Definition 4 The operator $\mathscr{D}$ is potentially self-adjoint in a domain G if there exists in this domain a positive real function $\rho(\mathbf{x}) = \rho(x, y)$ such that the operator $\rho(\mathbf{x})\mathscr{D}$ is self-adjoint in the domain G.

If we multiply $\mathscr{D}$ defined in (15) through a positive function $\rho(\mathbf{x})$ in a certain domain G, we obtain that the operator is potentially self-adjoint provided that the following conditions are satisfied

$$\begin{cases} \rho(x-1,y-1)\sigma_{12d}(x-1,y-1) = \rho(x,y)\sigma_{12c}(x,y), \\ \rho(x-1,y)\sigma_{12a}(x-1,y) = \rho(x,y-1)\sigma_{12b}(x,y-1), \\ \rho(x-1,y)\varpi_3(x-1,y) = \rho(x,y)\varpi_1(x,y) \\ \rho(x,y-1)\varpi_4(x,y-1) = \rho(x,y)\varpi_2(x,y), \end{cases} \tag{16}$$

where

$$\begin{cases} \varpi_1(x,y) = \sigma_{11}(x,y) + \sigma_{12b}(x,y) - \sigma_{12c}(x,y), \\ \varpi_2(x,y) = \sigma_{22}(x,y) + \sigma_{12a}(x,y) - \sigma_{12c}(x,y), \\ \varpi_3(x,y) = \sigma_{11}(x,y) + \sigma_{12a}(x,y) - \sigma_{12d}(x,y) + \tau_1(x,y), \\ \varpi_4(x,y) = \sigma_{22}(x,y) + \sigma_{12b}(x,y) - \sigma_{12d}(x,y) + \tau_2(x,y). \end{cases} \tag{17}$$

We shall refer to the system of Eq. (16) as Pearson type system, in analogy to what happens in the univariate case [17].

The above relations for the function ρ can be written as

$$\begin{aligned} &\Delta_1(\varpi_1(\mathbf{x})\rho(\mathbf{x})) = \nabla_1(\rho(\mathbf{x})\,(\sigma_{11}(\mathbf{x}) + \sigma_{12a}(\mathbf{x}) - \sigma_{12d}(\mathbf{x}) + \tau_1(\mathbf{x}))), \\ &\Delta_2(\varpi_2(\mathbf{x})\rho(\mathbf{x})) = \nabla_2(\rho(\mathbf{x})\,(\sigma_{22}(\mathbf{x}) + \sigma_{12b}(\mathbf{x}) - \sigma_{12d}(\mathbf{x}) + \tau_2(\mathbf{x}))), \\ &\rho(x,y+1)\sigma_{12d}(x,y+1) = \rho(x+1,y)\sigma_{12b}(x+1,y), \\ &\rho(x,y)\sigma_{12d}(x,y) = \rho(x+1,y+1)\sigma_{12c}(x+1,y+1), \end{aligned}$$

and for determining the unknown function $\rho(\mathbf{x})$, we get the system

$$\begin{cases} \Delta_1\left(\rho(\mathbf{x})\sigma_{11}(\mathbf{x})\right) + \Delta_2\left(\rho(\mathbf{x})\sigma_{12a}(\mathbf{x})\right) + \nabla_2\left(\rho(\mathbf{x})\sigma_{12d}(\mathbf{x})\right) = \rho(\mathbf{x})\tau_1(\mathbf{x}), \\ \Delta_2\left(\rho(\mathbf{x})\sigma_{22}(\mathbf{x})\right) + \Delta_1\left(\rho(\mathbf{x})\sigma_{12b}(\mathbf{x})\right) + \nabla_1\left(\rho(\mathbf{x})\sigma_{12d}(\mathbf{x})\right) = \rho(\mathbf{x})\tau_2(\mathbf{x}), \\ \Delta_1\nabla_2\left(\sigma_{12b}(\mathbf{x})\rho(\mathbf{x})\right) = \Delta_2\nabla_1\left(\sigma_{12a}(\mathbf{x})\rho(\mathbf{x})\right), \\ \Delta_1\Delta_2\left(\sigma_{12c}(\mathbf{x})\rho(\mathbf{x})\right) = \nabla_1\nabla_2\left(\sigma_{12d}(\mathbf{x})\rho(\mathbf{x})\right). \end{cases} \tag{18}$$

The above system can be written in matrix form as

$$\mathbb{U}\begin{pmatrix} \frac{\Delta_1(\rho(\mathbf{x}))}{\rho(\mathbf{x})} \\ \frac{\Delta_2(\rho(\mathbf{x}))}{\rho(\mathbf{x})} \\ \frac{\nabla_1(\rho(\mathbf{x}))}{\rho(\mathbf{x})} \\ \frac{\nabla_2(\rho(\mathbf{x}))}{\rho(\mathbf{x})} \end{pmatrix} = \begin{pmatrix} \theta(x,y) \\ \xi(x,y) \\ \varphi(x,y) \\ \psi(x,y) \end{pmatrix}, \tag{19}$$

where

$$\mathbb{U} = \begin{pmatrix} \sigma_{11}(x,y) & \sigma_{12a}(x,y) & 0 & \sigma_{12d}(x,y) \\ \sigma_{12b}(x,y) & \sigma_{22}(x,y) & \sigma_{12d}(x,y) & 0 \\ \sigma_{12b}(x+1,y) & -\sigma_{12a}(x,y+1) & 0 & 0 \\ \sigma_{12c}(x+1,y) & 0 & 0 & \sigma_{12d}(x,y-1) \end{pmatrix},$$

the functions

$$\begin{cases} \theta(x, y) = \tau_1(\mathbf{x}) - \mathscr{G}_1(\mathbf{x})\Delta_1(\sigma_{11}(\mathbf{x})) - \mathscr{G}_2(\mathbf{x})\Delta_2(\sigma_{12a}(\mathbf{x})) - \mathscr{G}_3(\mathbf{x})\nabla_2(\sigma_{12d}(\mathbf{x})), \\ \xi(x, y) = \tau_2(\mathbf{x}) - \mathscr{G}_1(\mathbf{x})\Delta_1(\sigma_{12b}(\mathbf{x})) - \mathscr{G}_2(\mathbf{x})\Delta_2(\sigma_{22}(\mathbf{x})) - \mathscr{G}_4(\mathbf{x})\nabla_1(\sigma_{12d}(\mathbf{x})), \\ \varphi(x, y) = \sigma_{12a}(x, y+1) - \sigma_{12d}(x+1, y), \\ \psi(x, y) = \sigma_{12d}(x, y-1) - \sigma_{12c}(x+1, y), \end{cases} \tag{20}$$

and we have denoted

$$\begin{cases} \mathscr{G}_1(\mathbf{x}) = \dfrac{\varpi_3(x, y)}{\varpi_1(x+1, y)}, \\ \mathscr{G}_2(\mathbf{x}) = \dfrac{\varpi_4(x, y)}{\varpi_2(x, y+1)}, \\ \mathscr{G}_3(\mathbf{x}) = \dfrac{\varpi_2(x, y)}{\varpi_4(x, y-1)}, \\ \mathscr{G}_4(\mathbf{x}) = \dfrac{\varpi_1(x, y)}{\varpi_3(x-1, y)}. \end{cases} \tag{21}$$

5 The Three-Term Recurrence Relations

For any $\mathbf{x} = (x, y) \in \mathbb{R}^2$, we shall denote by $\mathbf{x}^n$ ($n \in \mathbb{N}_0$) the column vector of the monomials $x^{n-k}y^k$, whose elements are arranged in graded lexicographical order (see [7, p. 32]):

$$\mathbf{x}^n = (x^{n-k}y^k), \quad 0 \le k \le n, \quad n \in \mathbb{N}_0 . \tag{22}$$

Let $\{P^n_{n-k,k}(x, y)\}$ be a sequence of polynomials in the space Π^2_n of all polynomials of total degree at most n in two variables, $\mathbf{x} = (x, y)$, with real coefficients. These polynomials can be expressed as finite sums of terms of the form $ax^{n-k}y^k$, where $a \in \mathbb{R}$.

Let $\mathbb{P}_n$ denote the (column) polynomial vector

$$\mathbb{P}_n = (P^n_{n,0}(x, y), P^n_{n-1,1}(x, y), \ldots, P^n_{1,n-1}(x, y), P^n_{0,n}(x, y))^{\mathrm{T}}. \tag{23}$$

In these conditions, each polynomial vector $\mathbb{P}_n$ can be written in terms of the basis (22) as:

$$\mathbb{P}_n = G_{n,n}\mathbf{x}^n + G_{n,n-1}\mathbf{x}^{n-1} + \cdots + G_{n,0}\,\mathbf{x}^0, \tag{24}$$

where $G_{n,j}$ are matrices of size $(n+1) \times (j+1)$ and $G_{n,n}$ is a nonsingular square matrix of size $(n+1) \times (n+1)$.

Definition 5 A polynomial vector $\mathbb{P}_n$ is said to be monic if its leading matrix coefficient $G_{n,n}$ is the identity matrix (of size $(n+1) \times (n+1)$); i.e.:

$$\mathbb{P}_n = \mathbf{x}^n + G_{n,n-1}\mathbf{x}^{n-1} + G_{n,n-2}\mathbf{x}^{n-2} + \cdots + G_{n,0}\,\mathbf{x}^0\,. \tag{25}$$

For a monic polynomial $\mathbb{P}_n$, each of its polynomial entries $P^n_{n-k,k}(x, y)$ can be written as

$$P^n_{n-k,k}(x, y) = x^{n-k} y^k + \text{terms of lower total degree}\,. \tag{26}$$

In what follows we shall consider monic polynomials denoted by $\mathbb{P}_n$.

The following theorem [7] provides conditions for $\{\mathbb{P}_n\}_{n\geq 0}$ be an orthogonal polynomial sequence.

Theorem 3 *Let $\mathscr{L}$ be a positive definite moment linear functional acting on the space Π^2_n of all polynomials of total degree at most n in two variables, and $\{\mathbb{P}_n\}_{n\geq 0}$ be an orthogonal family with respect to $\mathscr{L}$. Then, for $n \geq 0$, there exist unique matrices $A_{n,j}$ of size $(n+1) \times (n+2)$, $B_{n,j}$ of size $(n+1) \times (n+1)$, and $C_{n,j}$ of size $(n+1) \times n$, such that*

$$x_j \mathbb{P}_n = A_{n,j}\mathbb{P}_{n+1} + B_{n,j}\mathbb{P}_n + C_{n,j}\mathbb{P}_{n-1}, \quad j = 1, 2, \tag{27}$$

with the initial conditions $\mathbb{P}_{-1} = 0$ and $\mathbb{P}_0 = 1$, where we have used the notations $x_1 = x$ and $x_2 = y$.

Next, we shall obtain explicit expressions for the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ appearing in the three-term recurrence relations (27), in terms of the polynomial coefficients of (2). These matrices allow us to compute the monic orthogonal polynomial solutions of (2), in case they exist.

In doing so, we shall repeatedly use the matrices $L_{n,j}$ of size $(n+1) \times (n+2)$

$$L_{n,1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad L_{n,2} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}. \tag{28}$$

It is easy to check the following important properties

$$\begin{cases} x\,\mathbf{x}^n = L_{n,1}\mathbf{x}^{n+1}, \quad y\,\mathbf{x}^n = L_{n,2}\mathbf{x}^{n+1}, \\ x^2\,\mathbf{x}^n = L_{n,1}L_{n+1,1}\mathbf{x}^{n+2}, \quad y^2\,\mathbf{x}^n = L_{n,2}L_{n+1,2}\mathbf{x}^{n+2}, \\ L_{n,2}L_{n+1,1} = L_{n,1}L_{n+1,2}. \end{cases} \tag{29}$$

Moreover for $j = 1, 2$,

$$L_{n,j}\, L_{n,j}^{\mathrm{T}} = I_{n+1}, \tag{30}$$

where I_{n+1} denotes the identity matrix of size $n+1$.

In order to obtain explicit expressions for the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ appearing in the three-term recurrence relations (27), in terms of the polynomial coefficients of (2) we shall need the action of the forward and backward difference operators Δ and ∇ on $\mathbf{x}^n$ given by

$$\Delta_j \mathbf{x}^n = \sum_{k=1}^{n} \mathbb{E}_{n,j}^{k}\, \mathbf{x}^{n-k}, \quad \nabla_j \mathbf{x}^n = \sum_{k=1}^{n} (-1)^{k+1} \mathbb{E}_{n,j}^{k}\, \mathbf{x}^{n-k},$$

where if we denote the entries of the matrices $\mathbb{E}_{n,j}^{r} = (e_{p,q,j}^{r})$ of size $(n+1) \times (n-r+1)$, we have that

$$e_{p,q,1}^{r}(n) = \begin{cases} \binom{n-p}{r}, & p = q, \\ 0, & p \neq q, \end{cases} \qquad e_{p,q,2}^{r}(n) = \begin{cases} \binom{p}{r}, & p = q + r, \\ 0, & p \neq q + r. \end{cases}$$

If we substitute the expansion (25) in (2), by equating the coefficients in $\mathbf{x}^n$, $\mathbf{x}^{n-1}$, and $\mathbf{x}^{n-2}$ we obtain the following explicit expressions for the matrices $G_{n,n-1}$ and $G_{n,n-2}$ in (25):

$$G_{n,n-1} = \mathbb{S}_n \mathbb{F}_{n-1}^{-1}(\lambda_n), \tag{31}$$
$$G_{n,n-2} = (\mathbb{T}_n + G_{n,n-1}\mathbb{S}_{n-1})\mathbb{F}_{n-2}^{-1}(\lambda_n) \tag{32}$$

where

$$\mathbb{F}_n(\lambda_l) = (\lambda_n - \lambda_l)\mathbb{I}_{n+1}.$$

The matrix $\mathbb{S}_n$ of size $(n+1) \times n$ is given in terms of the polynomial coefficients of the Eq. (2) as

$$\mathbb{S}_n = \begin{pmatrix} s_{1,1} & s_{1,2} & 0 & \cdots & \cdots & 0 \\ s_{2,1} & s_{2,2} & s_{2,3} & \ddots & & \vdots \\ s_{3,1} & s_{3,2} & \ddots & \ddots & \ddots & \vdots \\ 0 & s_{4,2} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & s_{n-1,n} \\ \vdots & & \ddots & \ddots & \ddots & s_{n,n} \\ 0 & \cdots & \cdots & 0 & s_{n+1,n-1} & s_{n+1,n} \end{pmatrix} \quad (n \geq 1), \tag{33}$$

where

$$
\begin{aligned}
s_{i,i+1} &= \big((1-i)(b_{12b}+b_{12c})+e_{11}\big)u(n-i+2), \qquad i=1,\ldots,n-1,\\
s_{i,i} &= (n-i+1)\left((n-i)\left(d_{11}+\frac{\tau_{11}}{2}+(i-1)(a_{11}-c_{12b}-c_{12c})\right)\right.\\
&\quad +(i-1)(e_{12a}+e_{12b}+e_{12c}+e_{12d})\\
&\quad \left.+\tau_{13}-(b_{12a}+b_{12c})u(i)\right), \qquad i=1,\ldots,n,\\
s_{i+1,i} &= i(n-i)(d_{12a}+d_{12b}+d_{12c}+d_{12d})\\
&\quad +u(i+1)\left(e_{22}+\frac{\tau_{11}}{2}+(n-i)(a_{11}-c_{12a}-c_{12c})\right)\\
&\quad -iu(n-i+1)(a_{12b}+a_{12c})+i\tau_{23}, \qquad i=1,\ldots,n,\\
s_{i+2,i} &= \Big(d_{22}-(n-i-1)(a_{12a}+a_{12c})\Big)u(i+2), \qquad i=1,\ldots,n-1,
\end{aligned}
$$

where $u(n)=n(n-3)+2$.

Moreover, the matrix $\mathbb{T}_n$ of size $(n+1)\times(n-1)$ is given in terms of the polynomial coefficients of the Eq. (2) as

$$
\mathbb{T}_n = \begin{pmatrix}
t_{1,1} & 0 & \cdots\cdots & & 0\\
t_{2,1} & t_{2,2} & \ddots & & \vdots\\
t_{3,1} & t_{3,2} & \ddots & \ddots & \vdots\\
0 & t_{4,2} & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & t_{n-1,n-1}\\
\vdots & & \ddots & \ddots & t_{n,n-1}\\
0 & \cdots & \cdots & 0 & t_{n+1,n-1}
\end{pmatrix} \qquad (n\geq 2), \tag{34}
$$

where, for $1\leq i\leq n-1$,

$$
\begin{aligned}
t_{i,i} =& -\frac{1}{12}\Big((i-n-1)(i-n)\big(a_{11}(i-n+1)(3i+n-6)\\
&+2(3(i-1)(-e_{12a}+e_{12b}+e_{12c}-e_{12d}\\
&+(i-2)(b_{12a}+b_{12b}))-6f_{11}+\tau_{11}(i-n+1)-3\tau_{13})\big)\Big),\\
t_{i+1,i} =& -\frac{1}{2}i(i-n)\Big((i-n+1)\big((i-1)(c_{12a}+c_{12b}-a_{11})-d_{12a}+d_{12b}+d_{12c}\\
&-d_{12d}\big)+(1-i)(e_{12a}-e_{12b}+e_{12c}-e_{12d})+2(f_{12a}+f_{12b}+f_{12c}+f_{12d})\Big),\\
t_{i+2,i} =& \frac{1}{12}i\Big((i^2-1)\big(a_{11}(-3i+4n-6)+2\tau_{11}\big)\\
&+6(i+1)\big((d_{12a}-d_{12b}+d_{12c}-d_{12d}\\
&-(i-n+2)(a_{12a}+a_{12b}))(i-n+1)+2f_{22}+\tau_{23})\Big).
\end{aligned}
$$

As in the case analyzed in [2], we have the following result providing the explicit expressions of the matrices appearing in the three-term recurrence relations satisfied by the monic polynomials:

Theorem 4 *The explicit expressions of the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ $(j = 1, 2)$ appearing in (27) in terms of the values of the leading coefficients $G_{n,n-1}$ and $G_{n,n-2}$, explicitly given in (31) and (32), respectively, are*

$$\begin{cases} A_{n,j} = L_{n,j}, \quad n \geq 0, \\ B_{0,j} = -L_{0,j} G_{1,0}, \quad B_{n,j} = G_{n,n-1} L_{n-1,j} - L_{n,j} G_{n+1,n}, \quad n \geq 1, \\ C_{1,j} = -(L_{1,j} G_{2,0} + B_{1,j} G_{1,0}), \\ C_{n,j} = G_{n,n-2} L_{n-2,j} - L_{n,j} G_{n+1,n-1} - B_{n,j} G_{n,n-1}, \quad n \geq 2\,, \end{cases} \tag{35}$$

where the matrices $L_{n,j}$ have been introduced in (28).

Since [7],

$$\operatorname{rank}(L_{n,j}) = n + 1 = \operatorname{rank}(C_{n+1,j}), \quad j = 1, 2, \quad n \geq 0, \tag{36}$$

the columns of the joint matrices

$$L_n = \left(L_{n,1}^T\,, L_{n,2}^T\right)^T \quad \text{and} \quad C_n = \left(C_{n,1}^T\,, C_{n,2}^T\right)^T\,,$$

of size $(2n + 2) \times (n + 2)$ and $(2n + 2) \times n$, respectively, are linearly independent, which implies that

$$\operatorname{rank}(L_n) = n + 2, \quad \operatorname{rank}(C_n) = n. \tag{37}$$

Therefore, the matrix L_n has full rank. As a consequence, there exists a unique matrix $D_n^\dagger$ of size $(n + 2) \times (2n + 2)$,

$$D_n^\dagger = \left(D_{n,1} | D_{n,2}\right) = \left(L_n^T L_n\right)^{-1} L_n^T, \tag{38}$$

such that

$$D_n^\dagger L_n = I_{n+2}.$$

where I_{n+2} denotes the identity matrix of size $n + 2$.

If we now consider the left inverse $D_n^\dagger$ of the joint matrix L_n

$$D_n^\dagger = \begin{pmatrix} 1 & & & 0 & & \\ & 1/2 & \bigcirc & 1/2 & & \bigcirc \\ & & \ddots & & \ddots & \\ \bigcirc & & 1/2 & \bigcirc & & 1/2 \\ & & & 0 & & 1 \end{pmatrix},$$

it is possible to give a recursive formula for the monic orthogonal polynomials

$$\mathbb{P}_{n+1} = D_n^{\dagger}\left[\begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n\right]\mathbb{P}_n - D_n^{\dagger} C_n \mathbb{P}_{n-1}, \quad n \geq 0, \tag{39}$$

with the initial conditions $\mathbb{P}_{-1} = 0, \mathbb{P}_0 = 1$, where $\otimes$ denotes the Kronecker product and

$$B_n = \left(B_{n,1}^T, B_{n,2}^T\right)^T, \quad C_n = \left(C_{n,1}^T, C_{n,2}^T\right)^T, \tag{40}$$

are matrices of size $(2n+2) \times (n+1)$ and $(2n+2) \times n$, respectively. We have therefore obtained another presentation of [7, (3.2.10)]. This idea has already been presented in [18].

6 Bivariate Symmetric Orthogonal Polynomials

In this section, we give conditions for an admissible second-order partial difference equation of hypergeometric type to have symmetric orthogonal polynomial solutions.

Definition 6 A polynomial $P_{n-k,k}^n(x, y)$ of degree n is said to be symmetric if $P_{n-k,k}^n(-x, -y) = (-1)^n P_{n-k,k}^n(x, y)$.

As a consequence of this definition, the coefficients of all the monomials of $P_{n-k,k}^n(x, y)$ of degree $n-(2\ell+1), \ell = 0, 1, \ldots, \lfloor\frac{n-1}{2}\rfloor$ are zero, where $\lfloor\cdot\rfloor$ denotes the floor function.

Theorem 5 *Let us assume that* (2) *is admissible and of hypergeometric type. The Eq.* (2) *has a symmetric sequence of polynomial solutions if and only if*

$$\begin{cases} \tau_{13} = \tau_{23} = 0, \\ d_{12a} + d_{12b} + d_{12c} + d_{12d} = 0, \\ e_{12a} + e_{12b} + e_{12c} + e_{12d} = 0, \\ c_{12c} = a_{11} - c_{12a}, \\ d_{12b} = -d_{12a}, \\ a_{12b} = a_{12a}, \\ c_{12b} = c_{12a}, \\ b_{12b} = b_{12a}, \\ d_{11} = e_{22} = -\dfrac{\tau_{11}}{2}. \end{cases} \tag{41}$$

Proof Suppose that the admissible second-order difference equation (2) has a symmetric sequence of polynomials solutions. Let us consider the monic symmetric sequence of polynomials solutions written in vector form as in (25).

If we substitute each vector polynomials $\mathbb{P}_n$ $(n = 0, 1, 2, 3)$ in (2) with coefficients (14), by equating the coefficients in $\mathbf{x}^{n-1}$ to zero, we obtain (41).

For the converse, let $\mathbb{P}_n = \sum_{i=0}^{n} G_{n,i}\mathbf{x}^i$ be a polynomial vector as in (24). We have

$$\Delta_r \nabla_t \mathbb{P}_n = \sum_{i=2}^{n}\sum_{k=0}^{i-2}\sum_{l=0}^{k}(-1)^{i-k} G_{n,i}\mathbb{E}_{i,t}^{i-k-1}\mathbb{E}_{k+1,r}^{k+1-l}\mathbf{x}^l$$

which is a polynomial of degree $n-2$. Then, for $0 \le p \le n-2$ the coefficient of $\mathbf{x}^p$ in $\Delta_r \nabla_t \mathbb{P}_n$ is

$$\sum_{i=p+2}^{n}\sum_{k=p}^{i-2}(-1)^{i-k} G_{n,i}\mathbb{E}_{i,t}^{i-k-1}\mathbb{E}_{k+1,r}^{k+1-p}.$$

For $1 \le p \le n-1$, the coefficient of $\mathbf{x}^p$ in $x_j \Delta_r \nabla_t \mathbb{P}_n$ is

$$\sum_{i=p+1}^{n}\sum_{k=p-1}^{i-2}(-1)^{i-k} G_{n,i}\mathbb{E}_{i,t}^{i-k-1}\mathbb{E}_{k+1,r}^{k+2-p} L_{p-1,j}.$$

For $2 \le p \le n$, the coefficient of $\mathbf{x}^p$ in $x_{j_1} x_{j_2} \Delta_r \nabla_t \mathbb{P}_n$ is

$$\sum_{i=p}^{n}\sum_{k=p-2}^{i-2}(-1)^{i-k} G_{n,i}\mathbb{E}_{i,t}^{i-k-1}\mathbb{E}_{k+1,r}^{k+3-p} L_{p-2,j_2} L_{p-1,j_1}.$$

We also have

$$\begin{aligned}
\nabla_1 \nabla_2 \mathbb{P}_n &= \sum_{i=2}^{n}\sum_{k=0}^{i-2}\sum_{l=0}^{k}(-1)^{i-l} G_{n,i}\mathbb{E}_{i,2}^{i-k-1}\mathbb{E}_{k+1,1}^{k+1-l}\mathbf{x}^l,\\
\Delta_1 \Delta_2 \mathbb{P}_n &= \sum_{i=2}^{n}\sum_{k=0}^{i-2}\sum_{l=0}^{k} G_{n,i}\mathbb{E}_{i,2}^{i-k-1}\mathbb{E}_{k+1,1}^{k+1-l}\mathbf{x}^l,\\
\Delta_1 \mathbb{P}_n &= \sum_{i=1}^{n}\sum_{k=0}^{i-1} G_{n,i}\mathbb{E}_{i,1}^{i-k}\mathbf{x}^k,\\
\Delta_2 \mathbb{P}_n &= \sum_{i=1}^{n}\sum_{k=0}^{i-1} G_{n,i}\mathbb{E}_{i,2}^{i-k}\mathbf{x}^k.
\end{aligned}$$

As a consequence, if we substitute $\mathbb{P}_n$ in (2) with coefficients (14) and conditions (41), the coefficients of $\mathbf{x}^p$ are

$$\begin{cases} \sum_{i=2}^{n}\sum_{k=0}^{i-2} G_{n,i}A^1_{i,k,0} + \lambda_n G_{n,0}, & p=0, \\ \sum_{i=3}^{n}\sum_{k=1}^{i-2} G_{n,i}A^1_{i,k,1} + \sum_{i=2}^{n}\sum_{k=0}^{i-2} G_{n,i}A^2_{i,k,1} + \sum_{i=1}^{n} G_{n,i}A^3_{i,1} + \lambda_n G_{n,1}, & p=1, \\ \sum_{i=p+2}^{n}\sum_{k=p}^{i-2} G_{n,i}A^1_{i,k,p} + \sum_{i=p+1}^{n}\sum_{k=p-1}^{i-2} G_{n,i}A^2_{i,k,p} + \sum_{i=p}^{n} G_{n,i}A^3_{i,p} & \\ \qquad + \sum_{i=p}^{n}\sum_{k=p-2}^{i-2} G_{n,i}A^4_{i,k,p} + \lambda_n G_{n,p}, & 2 \le p \le n-2, \\ G_{n,n}A^2_{n,n-2,n-1} + \sum_{i=n-1}^{n} G_{n,i}A^3_{i,n-1} + \sum_{i=n-1}^{n}\sum_{k=n-3}^{i-2} G_{n,i}A^4_{i,k,n-1} & \\ \qquad + \lambda_n G_{n,n-1}, & p=n-1, \\ G_{n,n}A^3_{n,n} + G_{n,n}A^4_{n,n-2,n} + \lambda_n G_{n,n}, & p=n. \end{cases}$$

where the matrices $A^k_{i,j}$ and $A^k_{n,m,r}$ can be computed recursively. Hence, we can write the coefficient of $\mathbf{x}^p$ as

$$G_{n,p}B_{p,p} + G_{n,p+1}B_{p,p+1} + \sum_{i=p+2}^{n} G_{n,i}B_{p,i}. \tag{42}$$

Notice the difference between these matrices and those appearing in the three-term recurrence relation. Especially for the case $2 \le p \le n-2$, we have

$$\begin{aligned} B_{p,p} &= A^3_{p,p} + A^4_{p,p-2,p} + \lambda_p I_{p+1}, \\ B_{p,p+1} &= A^2_{p-1,p-1,p} + A^3_{p+1,p} + A^4_{p+1,p-2,p} + A^4_{p+1,p-1,p}, \\ B_{p,i} &= \sum_{k=p}^{i-2} A^1_{i,k,p} + \sum_{k=p-1}^{i-2} A^2_{i,k,p} + A^3_{i,p} + \sum_{k=p-2}^{i-2} A^4_{i,k,p}. \end{aligned}$$

Therefore, (42) can be written as

$$\begin{pmatrix} B^T_{p,p} & B^T_{p,p+1} & B^T_{p,p+2} & \cdots & B^T_{p,n} \end{pmatrix} \begin{pmatrix} G^T_{n,p} \\ G^T_{n,p+1} \\ G^T_{n,p+2} \\ \vdots \\ G^T_{n,n} \end{pmatrix}.$$

By equating the coefficients of $1, \mathbf{x}, \mathbf{x}^2, \ldots, \mathbf{x}^n$ in this order in the equation

$$\mathscr{D}\mathbb{P}_n + \lambda_n \mathbb{P}_n = 0,$$

we find the linear homogeneous system of equations

$$\Omega_n G = \theta,$$

where θ is the zero matrix of size $\binom{n+2}{2} \times (n+1)$,

$$\Omega_n = \begin{pmatrix} B_{0,0}^T & B_{0,1}^T & B_{0,2}^T & \cdots & B_{0,n}^T \\ 0 & B_{1,1}^T & B_{1,2}^T & \cdots & B_{1,n}^T \\ \vdots & & \ddots & & \vdots \\ \vdots & & & B_{n-1,n-1}^T & B_{n-1,n}^T \\ 0 & \cdots & \cdots & 0 & B_{n,n}^T \end{pmatrix}$$

and

$$G = \begin{pmatrix} G_{n,0}^T \\ G_{n,1}^T \\ \vdots \\ \vdots \\ G_{n,n}^T \end{pmatrix}.$$

Now $B_{n,n}^T$ is the zero matrix, so it is always possible to choose a nonsingular matrix $G_{n,n}$ satisfying last $n+1$ equations. $B_{n-1,n}$ is the zero matrix, and $B_{n-1,n-1}$ is diagonal matrix with the entries $\lambda_n - \lambda_{n-1}$ in the main diagonal. We then obtain that $G_{n,n-1}$ is the zero matrix. By doing so we find that the matrices $G_{n,n-(2j+1)}$ in (24), $j = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ are identically zero and the proof is complete.

As a consequence, we have that the polynomial coefficients of the admissible second-order linear partial difference equation of hypergeometric type (2) have the form

$$\begin{cases} \sigma_{11}(\mathbf{x}) = a_{11}x^2 - \frac{\tau_{11}}{2}x + f_{11}, \\ \sigma_{22}(\mathbf{x}) = a_{11}y^2 - \frac{\tau_{11}}{2}y + f_{22}, \\ \sigma_{12a}(\mathbf{x}) = c_{12a}xy + d_{12a}x + e_{12a}y + f_{12a}, \\ \sigma_{12b}(\mathbf{x}) = c_{12a}xy - d_{12a}x + e_{12b}y + f_{12b}, \\ \sigma_{12c}(\mathbf{x}) = (a_{11} - c_{12a})xy + d_{12c}x + e_{12c}y + f_{12c}, \\ \sigma_{12d}(\mathbf{x}) = (a_{11} - c_{12a})xy + d_{12d}x + e_{12d}y + f_{12d}, \\ \tau_1(\mathbf{x}) = \tau_{11}x, \\ \tau_2(\mathbf{x}) = \tau_{11}y, \\ \lambda_n = -n((n-1)a_{11} + \tau_{11}), \end{cases} \tag{43}$$

where

$$
\begin{aligned}
e_{12a} &= \frac{1}{12}\left(-a_{11}+12f_{12c}-2\left(3\left(f_{11}+f_{22}\right)+\tau_{11}\right)\right)-\frac{1}{2}c_{12a}+d_{12a}+f_{12a},\\
e_{12b} &= \frac{1}{12}\left(6\left(-2\left(d_{12a}+f_{12a}+f_{12c}\right)+c_{12a}+f_{11}+f_{22}\right)+a_{11}+2\tau_{11}\right),\\
d_{12c} &= \frac{1}{6}\left(-6d_{12a}+2a_{11}+6f_{22}+\tau_{11}\right),\\
e_{12c} &= \frac{1}{4}\left(-2c_{12a}+a_{11}+2f_{11}-2f_{22}\right)+d_{12a}+f_{12a}+f_{12c},\\
d_{12d} &= d_{12a}-\frac{a_{11}}{3}-f_{22}-\frac{1}{6}\tau_{11},\\
e_{12d} &= \frac{1}{4}\left(-2\left(2\left(d_{12a}+f_{12a}+f_{12c}\right)-c_{12a}+f_{11}-f_{22}\right)-a_{11}\right).
\end{aligned}
$$

Moreover, under these assumptions we have that the matrices $\mathbb{S}_n$ defined in (33) are identically zero, which implies that the matrices $G_{n,n-1}$ defined in (31) are also identically zero. Therefore, the matrices $B_{n,j}$ explicitly given in (35) of the three-term recurrence relations (27) satisfied by $\mathbb{P}_n$ are also zero, giving the symmetry condition. Furthermore, in this symmetric situation, the coefficients of the matrices $\mathbb{T}_n$ defined in (34) are given by

$$
\begin{cases}
t_{i,i} & = -\frac{1}{12}(i-n-1)(i-n)\left(a_{11}(i-n+2)(3i+n-5)+12(i-2)f_{11}\right.\\
& \quad \left.+2\tau_{11}(2i-n)\right),\\
t_{i+1,i} & = \frac{1}{6}i(i-n)\left(6\left(2\left((i-2)\left(f_{12a}+f_{12c}\right)+(2i-n)d_{12a}\right)\right.\right.\\
& \quad \left.-(i-1)(i-n+2)c_{12a}\right)+a_{11}(i(3i-3n-1)+5n-6)\\
& \quad \left.+6f_{22}(n-2i)+\tau_{11}(n-2i)\right),\\
t_{i+2,i} & = -\frac{1}{12}i(i+1)\left((i-2)a_{11}(3i-4n+5)-12f_{22}(i-n+2)+2\tau_{11}(n-2i)\right).
\end{cases}
$$

Acknowledgements Y. Guemo Tefo and I. Area thank the hospitality of the African Institute for Mathematical Sciences (AIMS-Cameroon), where a significant part of this research was performed during their visits in November 2014, and May and June 2015. The work of I. Area has been partially supported by the Agencia Estatal de Innovacioón (AEI) of Spain under grant MTM2016-75140-P, co-financed by the European Community fund FEDER, and Xunta de Galicia, grants GRC 2015-004 and R 2016/022.

References

1. R. Archibald, K. Chen, A. Gelb, R. Renaut, Improving tissue segmentation of human brain MRI through preprocessing by the Gegenbauer reconstruction method. NeuroImage **20**(1), 489–502 (2003)

2. I. Area, E. Godoy, J. Rodal, On a class of bivariate second-order linear partial difference equations and their monic orthogonal polynomial solutions. J. Math. Anal. Appl. **389**(1), 165–178 (2012)
3. I. Area, E. Godoy, A. Ronveaux, A. Zarzo, Bivariate second-order linear partial differential equations and orthogonal polynomial solutions. J. Math. Anal. Appl. **387**(2), 1188–1208 (2012)
4. I. Area, N.M. Atakishiyev, E. Godoy, J. Rodal, Linear partial q-difference equations on q-linear lattices and their bivariate q-orthogonal polynomial solutions. Appl. Math. Comput. **223**, 520–536 (2013)
5. N.M. Atakishiyev, K.B. Wolf, Fractional Fourier-Kravchuk transform. J. Opt. Soc. Am. A **14**(7), 1467–1477 (1997)
6. N.M. Atakishiyev, L.E. Vicent, K.B. Wolf, Continuous vs. discrete fractional Fourier transforms. J. Comput. Appl. Math. **107**(1), 73–95 (1999)
7. C.F. Dunkl, Y. Xu, Orthogonal polynomials of several variables, *Encyclopedia of Mathematics and Its Applications*, vol. 81 (Cambridge University Press, Cambridge, 2001)
8. D. Gottlieb, C.W. Shu, On the Gibbs phenomenon and its resolution. SIAM Rev. **39**(4), 644–668 (1997)
9. S. Gottlieb, J.H. Jung, S. Kim, A review of David Gottlieb's work on the resolution of the Gibbs phenomenon. Commun. Comput. Phys. **9**(3), 497–519 (2011)
10. M.A. Kowalski, Orthogonality and recursion formulas for polynomials in n variables. SIAM J. Math. Anal. **13**(2), 316–323 (1982)
11. M.A. Kowalski, The recursion formulas for orthogonal polynomials in n variables. SIAM J. Math. Anal. **13**(2), 309–315 (1982)
12. H.L. Krall, I.M. Sheffer, Orthogonal polynomials in two variables. Ann. Mat. Pura Appl. **4**(76), 325–376 (1967)
13. A. Lyskova, Orthogonal polynomials in several variables. Sov. Math. Dokl. **43**(1), 264–268 (1991)
14. A.S. Lyskova, Polynomial solutions of a certain class of ordinary differential equations. Uspekhi Mat. Nauk **51**(3), 207–208 (1996)
15. A.S. Lyskova, On some properties of orthogonal polynomials of several variables. Uspekhi Mat. Nauk **52**(4(316)), 207–208 (1997)
16. A.F. Nikiforov, V.B. Uvarov, *Special Functions of Mathematical Physics. A Unified Introduction with Applications* (Birkhäuser Verlag, Basel, 1988)
17. A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer Series in Computational Physics (Springer, Berlin, 1991)
18. J. Rodal, I. Area, E. Godoy, Linear partial difference equations of hypergeometric type: orthogonal polynomial solutions in two discrete variables. J. Comput. Appl. Math. **200**(2), 722–748 (2007)
19. J. Rodal, I. Area, E. Godoy, Structure relations for monic orthogonal polynomials in two discrete variables. J. Math. Anal. Appl. **340**(2), 825–844 (2008)
20. P.K. Suetin, Orthogonal polynomials in two variables, *Analytical Methods and Special Functions*, vol. 3 (Gordon and Breach Science Publishers, Amsterdam, 1999)
21. G. Szegő, *Orthogonal Polynomials*, 4th edn. (American Mathematical Society, Providence, 1975) (American Mathematical Society, Colloquium Publications, vol. XXIII)
22. Y. Xu, On multivariate orthogonal polynomials. SIAM J. Math. Anal. **24**(3), 783–794 (1993)
23. Y. Xu, Multivariate orthogonal polynomials and operator theory. Trans. Am. Math. Soc. **343**(1), 193–202 (1994)
24. Y. Xu, Second-order difference equations and discrete orthogonal polynomials of two variables. Int. Math. Res. Not. **8**, 449–475 (2005)

New and Extended Applications of the Natural and Sumudu Transforms: Fractional Diffusion and Stokes Fluid Flow Realms

Fethi Bin Muhammed Belgacem, Rathinavel Silambarasan, Hammouch Zakia and Toufik Mekkaoui

Abstract The Natural transform is used to solve fractional differential equations for various values of fractional degrees α, and various boundary conditions. Fractional diffusion problems solutions are analyzed, followed by Stokes–Ekman boundary thickness problem. Furthermore, the Sumudu transform is applied for fluid flow problems, such as Stokes, Rayleigh, and Blasius, toward obtaining their solutions and corresponding boundary layer thickness.

Keywords Natural transform · Sumudu transform · Fractional diffusion · Fluid dynamics

2010 AMS Math. Subject Classification Primary 42A38, 42B10 · Secondary 58D30

1 Introduction

To obtain the solutions for engineering problems such as in magneto-hydro-dynamics or fluid dynamics whether through ordinary, partial or fractional differential equations, integral transform methods are often sought to the rescue. The advent is that

F.B.M. Belgacem (✉)
Faculty of Basic Education, Department of Mathematics, PAAET, Al-ardhiya, Kuwait
e-mail: fbmbelgacem@gmail.com

R. Silambarasan
IT Networking, Department of Information Technology, SITE, VIT University, Vellore, Tamilnadu, India
e-mail: silambu_vel@yahoo.co.in

H. Zakia · T. Mekkaoui
E3MI Group, Faculty of Sciences and Techniques of Errachidia, Department of Mathematics, Moulay Ismail University, 52000 Meknes, Morocco
e-mail: hammouch.zakia@gmail.com

T. Mekkaoui
e-mail: toufik_mekkaoui@yahoo.fr

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_6

they convert the differential problems to simplifiable algebraic problems in a possibly new domain with proxy units, the solution of which are then often inverted back to yield the sought solution. Fourier and Laplace transforms are the traditional integral transform icons in this regard [24, 45]. Based on the type of kernel used, various integral transforms and problem-solving techniques have risen to include Hankel, Mellin, Hilbert Jacobi, Gegenbauer, Radon, Wavelet and Curvelet transforms, and Z [43]. For instance, orthogonal polynomial kernels led to Legendre, Laguerre, and Hermite transforms [43].

For the function $f(t)$ defined in the set $A = \{f(t)|\exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}$, if $t \in (-1)^j \times [0, \infty)\}$, Natural transform is given by,

$$\begin{aligned}\mathbb{N}[f(t)] = R(s,u) &= \int_0^\infty e^{-st} f(ut)dt = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)dt \\ &= \frac{1}{s}\int_0^\infty e^{-t} f\left(\frac{ut}{s}\right)dt\ ;\ Re(s) > 0,\ u \in (-\tau_1, \tau_2).\end{aligned} \tag{1}$$

In (1), when $u \equiv 1$ gives Laplace transform and $s \equiv 1$ gives Sumudu transform, hence second and third integral equations define the respective Natural-Laplace dual (NLD) and Natural-Sumudu dual (NSD).

The above mentioned Natural transform combines the features of Laplace and Sumudu transforms and hence converges to both transforms upon variable substitutions in kernel. In this work, some Natural transform properties are reviewed and applied to fractional order diffusion equation in semi-infinite medium for its solution, and then for different values of α, the solution is analyzed. Followed by same, Natural transform is applied for Stokes–Ekman problem to obtain its layer thickness. Table comprising all new Natural transforms for certain functions is given. In the second half of this work, Sumudu transform applied for Stokes, Rayleigh, and Blasius problems to obtain their solutions and hence their layer thickness.

2 Natural Transform Properties

Properties and table of elementary functions and N transform are given with solutions of fluid flow over a plane wall is solved by the Natural transform (N-transform) in [1]. Assuming both initial and boundary conditions were null, Maxwell's simultaneous equations were solved by Natural transform in [2]. Extensive properties including multiple shifting, dual nature to Laplace and Sumudu transforms, and all other required properties of Natural transform with list of tables were studied in [3]. A more generalized Laplace, Sumudu, and Natural transforms definitions are given in [3], (Eqs (1.4-5), and (2.12-13) in [3]). Natural and its inverse transforms were derived from Fourier integral in [3]. Bromwich contour integral and Heaviside's expansions theorems for the inverse Natural transform were derived in [3]

(Theorems 5.3 and 5.4, [3]). The same reference contains multiple shifting results related to products and divisions which were derived in terms of s as well as u in [3].

Transverse electromagnetic planar waves propagating in lossy medium (TEMP) are solved for electric field using Natural transform [4]. Maxwells equations were extended for n dimensions and studied using Natural transform in [5]. The relations of integral transforms and $J_0(2\sqrt{vt})$ are studied in [6]. Fractional ODEs solved using Natural transform in [7]. Natural transform in distribution space $\mathcal{D}'$ and its dual space $\mathcal{D}$ is studied in [8]. Natural transform in distribution space and Boehmians is studied in [9]. Integral equations solved using Natural transform in [10]. Decomposition method with Natural transform employeed for solving Schrödinger equations in [11]. In [12] Q-theory of Natural transform discussed. Natural transform of basic functions calculated by Adomain decomposition method (ADM) in [13]. Laplace, Fourier, Sumudu, and Mellin transforms were back tracked from Natural transform in [14]. In [15], fractional PDEs are solved by Natural transform. Fluid PDEs are solved by Natural transform in [16]. Natural transform and the Homotopy Perturbation method (HPM) were hybridly joined to solve fractional PDEs in [17]. Fractional Natural transform, properties, and applications were studied in [18]. Some more applications of Natural transforms in solving PDEs were given in [19, 20].

Theorem 2.1 *If $f(t+NT) = -f(t)$, then*

$$\mathbb{N}[f(t)] = -\frac{1}{u(1+e^{-\frac{sT}{u}})}\int_0^T e^{-\frac{st}{u}} f(t)dt. \tag{2}$$

Proof The proof is straightforward, rewriting the second integral of (1) in the interval $[0, T]$ and $[T, \infty)$ so that $[0, \infty) = [0, T] \cup [T, \infty)$ and applying $f(t+NT) = -f(t)$, simplifying gives (2). □

Theorem 2.2

$$\mathbb{N}\left[\begin{cases} 0\,;\ t < \frac{b}{a} \\ f(at-b)\,;\ t > \frac{b}{a},\ a,\ b > 0 \end{cases}\right] = \frac{1}{a}e^{-\frac{sb}{au}} R\left(\frac{s}{a}, u\right). \tag{3}$$

Proof Applying second integral of (1) to the left-hand side of (3) and after simplifying completes the result. □

3 Natural Transform Applications to Fractional Order Diffusion Equation in Semi-Infinite Medium and Stokes–Ekman Layer Problem

Example 3.1 (*Fractional diffusion problem*) The fractional order diffusion equation in semi infinite medium $z > 0$, where initial temperature is zero in the whole medium and temperature at the boundary is $X_0 f(t)$ given in [42]. The problem is described by the following equations.

$$\frac{\partial^{\alpha} x(z,t)}{\partial t^{\alpha}} = \kappa \frac{\partial^2 x(z,t)}{\partial z^2} \; ; \; z \in (0, \infty) \; , \; t > 0. \tag{4}$$

The initial and boundary conditions are, respectively, given by,

$$x(z,0) = 0 \; ; \; z > 0. \tag{5}$$

$$x(0,t) = X_0 f(t) \; ; \; t > 0 \text{ and } x(z,t) = 0 \text{ as } z \to \infty. \tag{6}$$

Letting $\mathbb{N}[x(z,t)] = R(z,s,u)$ and $\mathbb{N}[f(t)] = R(s,u)$, Natural transform of (4) after initial condition (5) and boundary condition (6) is given by,

$$\frac{d^2 R(z,s,u)}{dz^2} - \left(\frac{s^{\alpha}}{\kappa u^{\alpha}}\right) R(z,s,u) = 0 \; , \; z > 0. \tag{7}$$

and

$$R(0,s,u) = X_0 R(s,u) \; ; \; R(z,s,u) \to 0 \text{ as } z \to \infty. \tag{8}$$

Now the solution of (7) along with (8) is given by,

$$R(z,s,u) = X_0 R(s,u) \exp\left(-z\sqrt{\frac{s^{\alpha}}{\kappa u^{\alpha}}}\right). \tag{9}$$

Inverse Natural transform of (9) from the application of convolution theorem of Natural transform [3] gives the solution of (4).

$$x(z,t) = X_0 \int_0^t f(t-\tau) g(z,\tau) d\tau = X_0 f(t) g(z,t). \tag{10}$$

where

$$g(z,t) = \mathbb{N}^{-1}\left[\exp\left(-z\sqrt{\frac{s^{\alpha}}{\kappa u^{\alpha}}}\right)\right].. \tag{11}$$

Now the solution $x(z,t)$ of (4) with boundary condition $f(t) = 1$ in (6) for different values of α in (4) is given in the Table 1.

Next the solution $x(z,t)$ of (4) with boundary condition $f(t) = t$ in (6) for different values of α in (4) is given in the Table 2.

Hence, the general solution of (4) is given by (10) and (11). Finally, for different values of α in (11), $g(z,t)$ is given in Table 3.

Example 3.2 (*Stokes–Ekman problem*) When both fluid and disk rotate with uniform angular velocity Ω about $z-$ axis, unsteady boundary layer flow in a semi-infinite body of viscous fluid bounded by an infinite horizontal disk at $z = 0$ is given by the

Table 1 Solutions of (4) with boundary condition $f(t) = 1$ in (6) for different valus of α in (4)

S. No	α in (3.1)	$x(z, t)$
1	–2	$X_0 J_0\left(\frac{2\sqrt{zt}}{\kappa^{1/4}}\right)$
2	–1	$X_0\left[{}_0F_2\left(;\frac{1}{2},1;\frac{z^2t}{4\kappa}\right) - 2z\sqrt{\frac{t}{\pi\kappa}}\,{}_0F_2\left(;\frac{3}{2},\frac{3}{2};\frac{z^2t}{4\kappa}\right)\right]$
3	1	$X_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$
4	2	$X_0\,\mathrm{H}\left(t-\frac{z}{\sqrt{\kappa}}\right) = \begin{cases} 0\,;\ t < \frac{z}{\sqrt{\kappa}} \\ \text{undefined}\,;\ t = \frac{z}{\sqrt{\kappa}} \\ X_0\,;\ t > \frac{z}{\sqrt{\kappa}} \end{cases}$

Table 2 Solutions of (4) with boundary condition $f(t) = t$ in (6) for different valus of α in (4)

S. No	α in (3.1)	$x(z, t)$
1	–2	$X_0\kappa^{\frac{1}{4}}\sqrt{\frac{t}{z}}\,J_1\left(\frac{2\sqrt{zt}}{\kappa^{1/4}}\right)$
2	–1	$X_0\left[{}_0F_2\left(;\frac{1}{2},2;\frac{z^2t}{4\kappa}\right)t - \frac{4zt^{\frac{3}{2}}}{3\sqrt{\pi\kappa}}\,{}_0F_2\left(;\frac{3}{2},\frac{5}{2};\frac{z^2t}{4\kappa}\right)\right]$
3	1	$X_0\left[\left(\frac{z^2+2\kappa t}{2\kappa}\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) - z\sqrt{\frac{t}{\pi\kappa}}e^{-\frac{z^2}{4\kappa t}}\right]$
4	2	$X_0\,\mathrm{H}\left(t-\frac{z}{\sqrt{\kappa}}\right)\left(t-\frac{z}{\sqrt{\kappa}}\right) = \begin{cases} 0\,;\ t < \frac{z}{\sqrt{\kappa}} \\ \text{undefined}\,;\ t = \frac{z}{\sqrt{\kappa}} \\ X_0\left(t-\frac{z}{\sqrt{\kappa}}\right)\,;\ t > \frac{z}{\sqrt{\kappa}} \end{cases}$

Table 3 Solutions $g(z, t)$ in (11) for different valus of α in (11)

S. No	α in (3.8)	$x(z, t)$
1	–2	$\delta(t) - \frac{1}{\kappa^{\frac{1}{4}}}\sqrt{\frac{2}{t}}\,J_1\left(\frac{2\sqrt{zt}}{\kappa^{\frac{1}{4}}}\right)$
2	1	$\frac{z}{2\sqrt{\pi\kappa t^2}}e^{-\frac{z^2}{4\kappa t}}$
3	2	$\delta\left(t-\frac{z}{\sqrt{\kappa}}\right) = \begin{cases} undefined\,;\ t = \frac{z}{\sqrt{\kappa}} \\ 0\,;\ otherwise \end{cases}$

following equations [42].

$$\frac{\partial q(z,t)}{\partial t} + 2\Omega i q(z,t) = \nu \frac{\partial^2 q(z,t}{\partial z^2} \; ; \; z > 0 \; , \; t > 0. \tag{12}$$

$$q(z,t) = ae^{i\omega t} + be^{-i\omega t} \text{ on } z = 0 \; , \; t > 0/ \tag{13}$$

$$q(z,t) = 0 \; ; \; z \to \infty \; , \; t > 0. \tag{14}$$

$$q(z,t) = 0 \; ; \; t \leq 0 \, \forall \, z > 0. \tag{15}$$

Here, q is the complex velocity field, ω is frequency of oscillation of disk, and a and b are complex constants [42].

Let $\mathbb{N}[q(z,t)] = R(z,s,u)$, Natural transform of (12) leads to,

$$\frac{d^2 R(z,s,u)}{dz^2} - \left(\frac{s + 2\Omega u i}{u\nu}\right) R(z,s,u) = 0. \tag{16}$$

Solution of (16) after initial and boundary conditions (8) and (7), respectively, yields,

$$R(z,s,u) = \frac{a}{s - i\omega u} \exp\left(-z\sqrt{\frac{s + 2\Omega u i}{u\nu}}\right) + \frac{b}{s + i\omega u} \exp\left(-z\sqrt{\frac{s + 2\Omega u i}{u\nu}}\right). \tag{17}$$

Inverse Natural transform of (17) gives the solution $q(z,t)$ of (12).

$$\begin{aligned} q(z,t) =& \frac{ae^{i\omega t}}{2}\left[\exp\left(-z\left(\frac{(2\Omega+\omega)i}{\nu}\right)\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{t(2\Omega+\omega)i}\right)\right] \\ &+ \frac{ae^{i\omega t}}{2}\left[\exp\left(z\left(\frac{(2\Omega+\omega)i}{\nu}\right)\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{t(2\Omega+\omega)i}\right)\right] \\ &+ \frac{be^{-i\omega t}}{2}\left[\exp\left(-z\left(\frac{(2\Omega+\omega)i}{\nu}\right)\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{t(2\Omega+\omega)i}\right)\right] \\ &+ \frac{be^{-i\omega t}}{2}\left[\exp\left(z\left(\frac{(2\Omega+\omega)i}{\nu}\right)\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{t(2\Omega+\omega)i}\right)\right] \end{aligned} \tag{18}$$

Upon $\omega = 0$ in (18) gives the Ekman layer thickness of order $\sqrt{\frac{\nu}{2\Omega}}$.

For the functions in [41], table constituting list of exponential functions and their Natural transforms is given which will be useful for future study.

4 Sumudu Transform Literature Review

Over the past decade, a new theoretical framework has been developed to model anomalous diffusion. The new framework is based around the physics of continuous time random walks and the mathematics of fractional calculus. One can ask what would be a differential having as its exponent a fraction. Although this seems removed from Geometry . . . it appears that one day these paradoxes will yield useful consequences. Gottfried Leibniz Fractional Diffusion.

When $s \equiv 1$ in (1), Natural transform converges to Sumudu transform, namely

$$\mathbb{S}[f(t)] = \int_0^\infty e^{-t} f(ut)dt = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)dt \ ; \ u \in (-\tau_1, \tau_2). \tag{19}$$

Sumudu transform is shown to dual of Laplace transform and used to solve production equation [21], followed by multiple shifting, convolutions, and table of Sumudu integral transforms given in [22]. Properties of shifting and derivative of functions, Taylor's theorems, Sumudu transform applications and its relations to number theory and matrices given in [23]. From the Fourier integral, Sumudu transform is derived and applied for TEMP waves in [24]. Inverse Sumudu transform is applied and obtained the solution for Bessel's differential equations and shown its relation to Laplace transform through Bessel function in [25]. In [26, 27, 29, 31, 32, 34] Sumudu transform applications used for fractional differential equations. In [28] Laplace transform defined for trigonometric functions and then new infinite series of trigonometric functions along with tables, examples discussed. Fractional Schnakenberg solved numerically in [30]. Fractional order systems in electrical circuits were studied in [33]. Boundary problems of double diffusiveness are studied in [35]. Sumudu transform applied for continuous everywhere and nowhere differentiable functions for smoothening the fracture in [36]. Sumudu computation in series format was derived through symbolic C++ pseudocode, using the Sumudu definition without any additional decomposition schemes, in [37]. From the bimodular elliptic functions, Sumudu transform of $\tan(x)$ and $\sec(x)$ is derived as continued fractions in [38]. In [39] different Sumudu transform definition, its properties for trigonometric functions including table of new infinite series expansions of trigonometric functions were studied. Magnetic field solution of TEMP waves, numerical results and Maple graphical study were given in [40].

5 Sumudu Transform Applications to Stokes, Rayleigh and Blasius Problems

Example 5.1 (*Stokes problem*) Flow in unsteady boundary layer induced in semi-infinite viscous fluid is bounded by an infinite horizontal disk at $z = 0$ due to oscillations of disk in its own plane with given frequency ω. Corresponding PDE is given by [42],

$$\frac{\partial x(z,t)}{\partial t} = \nu \frac{\partial x(z,t)}{\partial z^2} \ ; \ z > 0 \ , \ t > 0. \tag{20}$$

Initial conditions are

$$x(z,t) = 0 \ ; \ z \to \infty \ , \ t > 0. \tag{21}$$

$$x(z,0) = 0 \ ; \ t \leq 0 \ , \ \forall z > 0. \tag{22}$$

and the boundary condition is

$$x(z,t) = X_0 e^{i\omega t} \ ; \ z = 0 \ , \ t > 0. \tag{23}$$

Here, $x(z,t)$ is velocity of fluid and ν kinematic viscosity of fluid.

Let $\mathbb{S}[x(z,t)] = G(z,u)$. Sumudu transform of (20) and after initial and boundary conditions yields,

$$G(z,u) = \frac{X_0}{1 - i\omega u} \exp\left(-z\sqrt{\frac{1}{u\nu}}\right). \tag{24}$$

Sumudu inverting (24) gives velocity in unsteady boundary layer.

$$x(z,t) = \frac{X_0 e^{i\omega t}}{2}\left[e^{-z\sqrt{\frac{i\omega}{\nu}}}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - \sqrt{i\omega t}\right) + e^{z\sqrt{\frac{i\omega}{\nu}}}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + \sqrt{i\omega t}\right)\right]. \tag{25}$$

From which the thickness of Stokes boundary layer is $\sqrt{\frac{\nu}{\omega}}$.

Example 5.2 (*Rayleigh problem*) When the frequency $\omega = 0$ in Stokes problem, motion is generated from rest with constant velocity X_0 in fluid [42]. Therefore, from (24).

$$G(z,u) = X_0 \exp\left(-z\sqrt{\frac{1}{u\nu}}\right). \tag{26}$$

Sumudu inverting (26) gives the velocity $x(z,t)$.

$$x(z,t) = X_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}}\right). \tag{27}$$

Thickness of Rayleigh boundary layer is $\sqrt{\nu t}$.

Example 5.3 (*Blasius problem*) Unsteady boundary layer flow in semi-infinite body of viscous fluid is enclosed by infinite horizontal disk at $z = 0$ [42].

When the boundary condition is t in Stokes problem Example 5.1 leads to the Blasius problem. Therefore, Sumudu transformed (20)–(23) with $x(z, t) = X_0 t$ yields,

$$G(z, u) = X_0 u \exp\left(-z\sqrt{\frac{1}{u\nu}}\right). \tag{28}$$

Inverse Sumudu transform of (28) using Maple gives the velocity profile of Blasius problem.

$$x(z, t) = \frac{X_0}{2}\left[\left(\frac{z^2 + 2\nu v}{\nu}\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}}\right) - 2z\sqrt{\frac{t}{\pi\nu}}e^{-\frac{z^2}{4\nu t}}\right]. \tag{29}$$

6 Conclusion

With respect to fractional diffusion problem, following observations were found. When the boundary condition is constant.

- For $\alpha = -2$, velocity profile $x(z, t)$ is in terms of Bessel's function.
- For $\alpha = -1$, velocity profile $x(z, t)$ is in terms of hypergeometric function.
- For $\alpha = 1$, velocity profile $x(z, t)$ is in terms of complementary error function [42].
- For $\alpha = 2$, velocity profile $x(z, t)$ is in terms of Heaviside's function.
- When the boundary condition is t, velocity profiles for different α are given in Table 2.
- For $\alpha > -2$ and $\alpha > 2$, velocity profile $x(z, t)$ does not exists.
- Therefore, for constant and t boundary conditions, velocity $x(z, t)$ is defined for $\alpha \in [-2, 2]$.

Sumudu reciprocity property of [24] is shown in the realm of Stokes fluid flow problem and its descendent variations, and obtained solutions and layer thickness are in exact concordance with results in the literature [42].

For future studies, we relegate the still open problems regarding finding the velocity $x(z, t)$ when α takes fractional values in the interval $[-2, 2]$. Lists of Natural transforms of elementary functions given in Table 4 will be useful for further study. Moreover, we declare that we remain open to our readers comments, communications, and suggestions.

Table 4 Natural transform of elementary functions

S. No	$f(t)$	$\mathbb{N}[f(t)]$
1	$e^{-\alpha t}$	$\dfrac{1}{s+\alpha u}$
2	$te^{-\alpha t}$	$\dfrac{u}{(s+\alpha u)^2}$
3	$t^{v-1}e^{-\alpha t}$	$\dfrac{u^{v-1}\Gamma(v)}{(s+\alpha u)^v}$
4	$\dfrac{e^{-\alpha t}-e^{-\beta t}}{t}$	$\dfrac{1}{u}\log\left(\dfrac{s+\beta u}{s+\alpha u}\right)$
5	$\dfrac{(1-e^{-\alpha t})^2}{t^2}$	$\left(\dfrac{s+2\alpha u}{u^2}\right)\log\left(\dfrac{s+2\alpha u}{u}\right)$ $+\dfrac{s}{u^2}\log\left(\dfrac{s}{u}\right)-2\left(\dfrac{s+\alpha u}{u^2}\right)$ $\times\log\left(\dfrac{s+\alpha u}{u}\right)$
6	$\dfrac{!}{t}-\dfrac{(t+2)(1-e^{-t})}{2t^2}$	$\left(\dfrac{2s+u}{2u^2}\right)\log\left(\dfrac{s+u}{s}\right)-\dfrac{1}{u}$
7	$\dfrac{1}{1-e^{-t}}$	$\dfrac{1}{2u}\psi\left(\dfrac{s+u}{2u}\right)-\dfrac{1}{2u}\psi\left(\dfrac{s}{2u}\right)$
8	$\left(1-e^{-\frac{t}{\alpha}}\right)^{v-1}$	$\dfrac{\alpha}{u}B\left(\dfrac{\alpha s}{u},v\right)$
9	$\dfrac{t^n}{\left(1-e^{-\frac{t}{\alpha}}\right)}$	$\dfrac{(-\alpha)^{n+1}}{u}\psi^{(n)}\left(\dfrac{\alpha s}{u}\right)$
10	$\dfrac{t^{v-1}}{\left(1-e^{-\frac{t}{\alpha}}\right)}$	$\dfrac{\alpha^v\Gamma(v)}{u}\zeta\left(v,\dfrac{\alpha s}{u}\right)$
11	$\dfrac{1}{t(1-e^{-t})}-\dfrac{1}{t^2}-\dfrac{1}{2t}$	$\dfrac{1}{u}\left[\dfrac{s}{u}\left(1-\log\left(\dfrac{s}{u}\right)\right)+\log\Gamma\left(\dfrac{s}{u}\right)\right]$ $+\dfrac{1}{u}\left[\dfrac{1}{2}\log\left(\dfrac{s}{2u\pi}\right)\right]$
12	$\dfrac{1-e^{-\alpha t}}{1-e^{-t}}$	$\dfrac{1}{u}\left[\psi\left(\dfrac{s+\alpha u}{u}\right)-\psi\left(\dfrac{s}{u}\right)\right]$
13	$\dfrac{1-e^{-\alpha t}}{t(1+e^{-t})}$	$\dfrac{1}{u}\left[\dfrac{\Gamma\left(\frac{s}{2u}\right)\Gamma\left(\frac{2(s+\alpha u+u)}{4u}\right)}{\Gamma\left(\frac{2(s+u)}{4u}\right)\Gamma\left(\frac{2(s+\alpha u)}{4u}\right)}\right]$
14	$\dfrac{(1-e^{-t})^{v-1}}{(1-ze^{-t})^{\mu}}$	$\dfrac{1}{u}B\left(\dfrac{s}{u},v\right){}_2F_1\left(\mu,\dfrac{s}{u};\dfrac{s+uv}{u};z\right)$

(continued)

Table 4 (continued)

S. No	$f(t)$	$\mathbb{N}[f(t)]$
15	$\frac{(1-e^{-\alpha t})(1-e^{-\beta t})}{1-e^{-t}}$	$\frac{1}{u}\left[\psi\left(\frac{s+\alpha u}{u}\right)+\psi\left(\frac{s+\beta u}{u}\right)\right]$ $-\frac{1}{u}\left[\psi\left(\frac{s+(\alpha+\beta)u}{u}\right)-\psi\left(\frac{s}{u}\right)\right]$
16	$\frac{(1-e^{-\alpha t})(1-e^{-\beta t})}{t(1-e^{-t})}$	$\frac{1}{u}\left[\frac{\Gamma\left(\frac{s}{u}\right)\Gamma\left(\frac{s+(\alpha+\beta)u}{u}\right)}{\Gamma\left(\frac{s+\alpha u}{u}\right)\Gamma\left(\frac{s+\beta u}{u}\right)}\right]$
17	$\frac{(1-e^{-\alpha t})(1-e^{-\beta t})(1-e^{-\gamma t})}{t(1-e^{-t})}$	$\frac{1}{u}\left[\frac{\Gamma\left(\frac{s}{u}\right)\Gamma\left(\frac{s+(\alpha+\beta)u}{u}\right)}{\Gamma\left(\frac{s+\alpha u}{u}\right)\Gamma\left(\frac{s+\beta u}{u}\right)}\right]$ $\times\frac{1}{u}\left[\frac{\Gamma\left(\frac{s+(\beta+\gamma)u}{u}\right)\Gamma\left(\frac{s+(\gamma+\alpha)u}{u}\right)}{\Gamma\left(\frac{s+\gamma u}{u}\right)\Gamma\left(\frac{s+(\alpha+\beta+\gamma)u}{u}\right)}\right]$
18	$\frac{[\alpha+\sqrt{1-e^{-t}}]^{-v}+[\alpha-\sqrt{1-e^{-t}}]^{-v}}{\sqrt{1-e^{-t}}}$	$\frac{2^{(s+u)/u}e^{(s-uv)/ui\pi}\Gamma(s/u)}{u\Gamma(v)}$ $\times(\alpha^2-1)^{s/2u-v/2}Q_{s/u-1}^{v-s/u}(\alpha)$
19	$\begin{cases}0\,;\ 0<t<\beta\\ \frac{\left[e^{-\beta}\sqrt{1-e^{-2t}}-e^{-t}\sqrt{1-e^{-2\beta}}\right]^{v}}{\sqrt{1-e^{-2t}}}\,;\ t>\beta\end{cases}$	$\frac{\sqrt{\pi}\Gamma(s/u)\Gamma(v+1)e^{-\frac{\beta((s+uv)/u)}{2}}}{u2^{s/2u+v/2}\Gamma(s/2u+v/2+1/2)}$ $\times P_{(-s/2u+v/2)}^{(-s/2u-v/2)}(\sqrt{1-e^{-2\beta}})$
20	$e^{(\mu-1)t}(1-e^{-t})^{\mu^{-1/2}}$ $\times[(1-e^{-t})\sin(\theta)-i(1-e^{-t})\cos(\theta)]^{\mu^{-1/2}}$	$\frac{2^{\mu^{-1}}\Gamma(\mu+1/2)\Gamma(s/u-\mu+1)\sin^{\mu}(\theta)}{u\sqrt{\pi}\Gamma(s/u+\mu+1)}$ $\times e^{(s/u+1/2)i\theta+\mu/2-1/4i\pi}$ $\times[\pi P_{v}^{\mu}(\cos(\theta))+2iQ_{v}^{\mu}(\cos(\theta))]$
21	$\begin{cases}0\,;\ 0<t<\beta\\ e^{-\frac{t^2}{4\alpha}}\,;\ t>\beta\end{cases}$	$\frac{\sqrt{\alpha u}e^{as^2/u^2}}{u}\mathrm{Erfc}\left[\frac{s\sqrt{\alpha}}{u}+\frac{\beta}{2\alpha}\right]$
22	$te^{-\frac{t^2}{4\alpha}}$	$\frac{2\alpha}{u}-\frac{2\sqrt{\pi}s\alpha^{3/2}e^{\frac{as^2}{u^2}}}{u^2}\mathrm{Erfc}\left[\frac{s\sqrt{\alpha}}{u}\right]$
23	$\frac{e^{-\frac{t^2}{4\alpha}}}{\sqrt{t}}$	$\frac{\sqrt{s\alpha}}{u^{3/2}}e^{\frac{\alpha s^2}{2u^2}}\mathrm{K}_{\frac{1}{4}}\left(\frac{\alpha s^2}{2u^2}\right)$
24	$t^{v-1}e^{-\frac{t^2}{8\alpha}}$	$\frac{\Gamma(v)2^{v}\alpha^{v/2}}{u}e^{\frac{\alpha s^2}{u^2}}\mathrm{D}_{-v}\left(\frac{2\sqrt{\alpha}s}{u}\right)$
25	$e^{-\frac{t}{4\alpha}}$	$\frac{\sqrt{\alpha}}{u\sqrt{s/u}}\mathrm{K}_1\left(\frac{\sqrt{s\alpha}}{\sqrt{u}}\right)$
26	$\sqrt{t}e^{-\frac{t}{4\alpha}}$	$\frac{\sqrt{\pi u}}{2s^{3/2}}(1+\sqrt{s\alpha/u})e^{-\sqrt{s\alpha/u}}$

(continued)

Table 4 (continued)

S. No	$f(t)$	$\mathbb{N}[f(t)]$
27	$\dfrac{e^{-\frac{t}{4\alpha}}}{\sqrt{t}}$	$\dfrac{\sqrt{\pi}}{u\sqrt{s/u}}e^{-\sqrt{s\alpha/u}}$
28	$\dfrac{e^{-\frac{t}{4\alpha}}}{t^{3/2}}$	$\dfrac{2\sqrt{\pi}}{u\alpha}e^{-\sqrt{s\alpha/u}}$
29	$t^{v-1}e^{-\frac{t}{4\alpha}}$	$\dfrac{@}{u}(s/4\alpha u)^{v/2}\mathrm{K}_v(\sqrt{s\alpha/u})$
30	$\dfrac{(e^{-\frac{t}{4\alpha}}-1)}{\sqrt{t}}$	$\dfrac{\sqrt{\pi}}{u\sqrt{s/u}}(e^{\sqrt{s\alpha/u}}-1)$
31	$e^{-2\sqrt{\alpha t}}$	$\dfrac{1}{s}-\dfrac{\sqrt{\alpha\pi u}}{s^{3/2}}e^{\alpha u/s}\mathrm{Erfc}(\sqrt{\alpha u/s})$
32	$\sqrt{t}e^{-2\sqrt{\alpha t}}$	$-\dfrac{u\sqrt{\alpha}}{s^2}+\dfrac{\sqrt{\pi}s^{3/2}}{u^{3/2}}(\alpha+s/2u)e^{\alpha u/s}$ $\times\ \mathrm{Erfc}(\sqrt{\alpha u/s})$
33	$\dfrac{e^{-2\sqrt{\alpha t}}}{\sqrt{t}}$	$\dfrac{\sqrt{\pi}}{u\sqrt{s/u}}e^{\alpha u/s}\mathrm{Erfc}(\sqrt{\alpha u/s})$
34	$\dfrac{e^{-2\sqrt{\alpha t}}}{\sqrt{2t}}$	$\dfrac{1}{u}\sqrt{\alpha u/2s}e^{\alpha u/2s}\mathrm{K}_{\frac{1}{4}}(\alpha u/2s)$
35	$(2t)^{v-1}e^{-2\sqrt{\alpha t}}$	$\dfrac{u^{v-1}\Gamma(2v)}{s^v}e^{s/2\alpha u}\mathrm{D}_{-2v}(\sqrt{2\alpha u/s})$
36	$\exp(-\alpha e^{-t})$	$\dfrac{1}{u\alpha^{s/u}}\gamma(s/u,\alpha)$
37	$\exp(-\alpha e^{t})$	$\dfrac{\alpha^s}{u}\mathrm{F}(s/u,\alpha)$
38	$(1-e^{-t})^{v-1}\exp(-\alpha e^{-t})$	$\dfrac{\Gamma(v)\Gamma(s/u)}{u\Gamma(v+s/u)}\alpha^{-v/2-s/2u}e^{\alpha/2}$ $\times\ \mathrm{M}_{v/2-s/2u,v/2+s/2u-1/2}(\alpha)$
39	$(1-e^{-t})^{v-1}\exp(-\alpha e^{t})$	$\dfrac{\Gamma(v)}{u}\alpha^{-1/2-s/2u}e^{\alpha/2}$ $\times\ \mathrm{W}_{1/2-s/2u-v,-s/2u}(\alpha)$
40	$\dfrac{(1-e^{-t})^{v-1}}{(1-\lambda e^{-t})^{\mu}}\exp(-\alpha e^{-t})$	$\dfrac{\Gamma(v)\Gamma(s/u)}{u\Gamma(v+s/u)}\Phi_1(s/u.\mu, v;\lambda,\alpha)$
41	$(e^t-1)^{v-1}\exp(-\alpha/e^t-1)$	$\dfrac{1}{u}\mathrm{F}(s/u-v+1)e^{\alpha/a}\alpha^{v/2-1/2}$ $\times\ \mathrm{W}_{v/2-1/2-s/u,v/2}(\alpha)$

References

1. Z.H. Khan, W.A. Khan, N-transform properties and applications. NUST J. Eng. Sci. **1**(1), 127–133 (2008)
2. R. Silambarasan, F.B.M. Belgacem, Applications of the natural transform to Maxwell's equations, in *PIERS Proceedings in Suzhou, China. Sept 12–16* (2011), pp 899–902
3. F.B.M. Belgacem, R. Silambarasan, Theory of natural transform. Int. J. Math. Eng. Sci. Aeorosp. (MESA) **3**(1), 99–124 (2012)
4. F.B.M. Belgacem, R. Silambarasan, Maxwell's equations solutions by means of the Natural transform. Spec. Iss. Complex Dyn. Syst. Nonlinear Methods Math. Models Thermodyn. Int. J. Math. Eng. Sci. Aerosp. (MESA), **3**(3), 313–323 (2012)
5. F.B.M. Belgacem, R. Silambarasan, The Generalized nth order Maxwell's equations, in *PIERS Proceedings Held in Moscow Technical University of Radio Engineering, Electronics and Automatics (MIREA)* (Moscow, Russia, 2012), pp 500–503
6. F.B.M. Belgacem, R. Silambarasan, Advances in the natural transform. AIP Conf. Proc. **1493**(1), 106–110 (2012)
7. D. Loonker, P.K. Banerji, Solution of fractional ordinary differential equations by natural transform. Int. J. Math. Eng. Sci. **2**(12), 1–7 (2013)
8. D. Loonker, P.K. Banerji, Natural transform for distribution and Boehmian spaces. Math. Eng. Sci. Aerosp. **4**(1), 69–76 (2013)
9. S.K.Q. Al-Omari, On the applications of natural transform. Int. J. Pure Appl. Math. **85**(4), 729–744 (2013)
10. D. Loonker, P.K. Banerji, Natural transform and solution of integral equations for distribution spaces. American. J. Math. Sci. **3**(1), 65–72 (2014)
11. S. Maitama, A new approach to linear and nonlinear Schr$\ddot{o}$dinger equations using the natural decomposition method. Int. Math. Forum **9**(17), 835–847 (2014)
12. S.K.Q. Al-Omari, On some q-analogues of the natural transform and further investigations, arXiv:1505.02179v1 [math.CA] (2015), pp. 1–16
13. D. Poltem, P. Totassa, A. Wiwatwanich, Application of decomposition method for natural transform. Asian. J. of Appl. Sci. **3**(6), 905–909 (2015)
14. K. Shah, M. Junaid, N. Ali, Extraction of laplace, Sumudu, fourier and Mellin transform from the natural transform. J. Appl. Environ. Biol. Sci. **5**(9), 108–115 (2015)
15. A.S. Abdel-Rady, S.Z. Rida, A.A.M. Arafa, H.R. Abedl-Rahim, Natural transform for solving fractional models. J. Appl. Math. Phys. **3**, 1633–1644 (2015)
16. M. Junaid, Application of natural transform to Newtonian fluid problems. Eur. Int. J. Sci. Technol. **5**(4), 138–147 (2016)
17. S. Maitama, A hybrid Natural transform Homotopy perturbation method for solving fractional partial differential equations. Int. J. Differ. Equ. **2016**, 1–7 (2016)
18. M. Omran, A. Kiliman, Natural transform of fractional order and some properties, in *Cogent Mathematics* (2016), pp. 1–8
19. S. Maitama, A new analytical approach to linear and nonlinear partial differential equations. Nonlinear Stud. **23**(4), 675–684 (2016)
20. S. Maitama, Explicit solution of solitary wave equation with compact support by natural homotopy perturbation method. Math. Eng. Sci. Aerosp. **7**(4), 625–635 (2016)
21. F.B.M. Belgacem, A.A. Karaballi, S.L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations. Math. Probl. Eng. (MPE) **3**, 103–118 (2003)
22. F.B.M. Belgacem, A.A. Karaballi, Sumudu transform fundamental properties investigations and applications. J. Appl. Math. Stoch. Anal. (JAMSA) **91083**, 1–23 (2005)
23. F.B.M. Belgacem, Introducing and analysing deeper Sumudu properties. Nonlinear Stud. J. (NSJ) **13**(1), 23–41 (2006)
24. M.G.M. Hussain, F.B.M. Belgacem, Transient solutions of Maxwell's equations based on Sumudu transform. Progr. Electromagn. Res. **74**, 273–289 (2007)

25. F.B.M. Belgacem, Sumudu transform applications to Bessel's functions and equations. Appl. Math. Sci. **4**(74), 3665–3686 (2010)
26. Q.K. Katatbeh, F.B.M. Belgacem, Applications of the Sumudu transform to fractional differential equations. Nonlinear Stud. (NSJ) **18**(1), 99–112 (2011)
27. P. Goswami, F.B.M. Belgacem, Fractional differential equations solutions through a Sumudu rational. Nonlinear Stud. J. (NSJ) **19**(4), 591–598 (2012)
28. F.B.M. Belgacem, R. Silambarasan, Laplace transform analytical restructure. Appl. Math. (AM) **4**, 919–932 (2013)
29. H. Bulut, H.M. Baskonus, F.B.M. Belgacem, The analytical solution of some fractional ordinary differential equations by the Sumudu transform method. J. Abstr. Appl. Anal. Article ID 203875, 1–6 (2013)
30. Z. Hammouch, T. Mekkaoui, F.B.M. Belgacem, Numerical simulations for a variable order fractional Schnakenberg model. AIP Conf. Proc. **1637**, 1450–1455 (2014)
31. S.T. Demiray, H. Bulut, F.B.M. Belgacem, Sumudu transform method for analytical solutions of fractional type ordinary differential equations. J. Math. Probl. Eng. (2014)
32. D.S. Tuluce, H. Bulut, F.B.M. Belgacem, Sumudu transform method for analytical solutions of fractional type ordinary differential equations. Math. Probl. Eng. Spec. Iss. Par. Fract. Equ. Appl. 1–6 (2014)
33. T. Mekkaoui, H. Zakia, F.B.M. Belgacem, A. El Abbassi, Fractional-order nonlinear systems: chaotic dynamics, numerical simulation and circuits design, chapter, in *Fractional Dynamics* (2015), pp. 343–356
34. F.B.M. Belgacem, V. Gulati, P. Goswami, A. Aljouiee, On generalized fractional differential equations solutions: Sumudu transform solutions and applications, Chap. 22 in De Gruyter open, in *Fractional Dynamics* (2015), pp. 382–393
35. H. Zakia, T. Mekkaoui, F.B.M. Belgacem, Double-diffusive natural convection in porous cavity heated by an internal boundary. Math. Eng. Sci. Aerosp. **7**(3), 453–466 (2016)
36. F.B.M. Belgacem, C. Cattani, Sumudu transform of Weierstrass functions, in *CMES Conference* (2016)
37. F.B.M. Belgacem, R. Silambarasan, Further distinctive investigations of the Sumudu transform, in *Submitted to ICNPAA Conference, France, and Accepted in AIP Conferences Proceedings* (2016)
38. F.B.M. Belgacem, R. Silambarasan, Sumudu transform of Dumont bimodular Jacobi elliptic functions for arbitrary powers, in *Submitted to ICNPAA Conference, France, and Accepted in AIP Conferences Proceedings* (2016)
39. F.B.M. Belgacem, R. Silambarasan, A distinctive Sumudu treatment of trigonometric functions. J. Comput. Appl. Math. **312**, 74–81 (2017)
40. F.B.M. Belgacem, E.H. Al-Shemas, R. Silambarasan, Sumudu computation of the transient magnetic field in a lossy medium. Appl. Math. Inf. Sci. **11**(1), 209–217 (2017)
41. A. Erdélyi, *Tables of Integral Transforms*, vol. 1 (McGraw-Hill Inc., 1954)
42. T. Myint-U, L. Debnath, *Linear Partial Differential Equations for Scientists and Engineers* (Birkhäuser Inc., Boston, 2007)
43. L. Debnath, D. Bhatta, *Integral Transforms and their Applications* (Chapman & Hall/CRC, Boca Raton, 2007)
44. L. Bernardin et al., *Maple Programming Guide* (Waterloo Maple Inc., 2011)
45. F.B.M. Belgacem, Sumudu applications to Maxwell's equations. PIERS Online **5**(4), 355–360 (2009)

On Uncertain-Fractional Modeling: The Future Way of Modeling Real-World Problems

Abdon Atangana and Ilknur Koca

Abstract It has been a long time a challenge for many researchers to give a real interpretation of derivatives with fractional order. Some researchers said, fractional derivative is the shadow on the wall. This interpretation was wrong since the shadow of any object does not provide the real properties of the real object, for instance a black man has the same shadow with a white man. Using the definition and applications of a convolution, we gave new interpretation of derivative with fractional order. We gave specific interpretation for Caputo and Caputo–Fabrizio types as the fractional order changes. It was long believed that, the derivative with fractional order portray the effect of memory, this was only proved to be true in theory of elasticity and nowhere else. In this chapter, we introduced a new operator called uncertain derivative capable or portraying the memory effect in almost all situation. In order to include into mathematical formulation, the real rate of change and also the effect of memory, we introduced a new way of modeling real-world problem called uncertain-fractional modeling (UFM) and applied it to advection dispersion model. Numerical simulations of the new model show that real-world observation. This method will be the future way of modeling real-world problem efficiently.

Keywords Interpretation of fractional derivatives · Uncertain derivative · Uncertain-fractional modeling · Advection dispersion model · Numerical analysis

A. Atangana
Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences,
University of the Free State, 9300 Bloemfontein, South Africa
e-mail: abdonatangana@yahoo.fr

I. Koca (✉)
Department of Mathematics, Faculty of Sciences, Mehmet Akif Ersoy University,
15100 Burdur, Turkey
e-mail: ikoca@mehmetakif.edu.tr

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_7

1 Introduction

Modeling a real-world problem requires a well knowledge within and around this problem. There are at least three elements that are very important in the process of modeling. The first thing is understanding and interpretation of the in order to convert it into mathematical equation. Secondly, the derivative used to describe the rate of change, as the change occurs in time and space. The last thing is perhaps the coefficient from the physical problem introduced in the mathematical equation. For the derivative, we have four types, the local derivative [1–3], fractional derivatives [4–9], variable order derivatives [10–12], and uncertain derivative [13, 14]. With proof in the literature, it was revealed that the local derivative was not suitable for modeling some kind of problems. However, the fractional derivatives were selected as suitable derivatives for modeling some complex problems [15–18]; nonetheless, for highly complex problems only derivative with variable order was best candidates [19, 20].

For the coefficient, we can, for example, speak of the advection dispersion equation, where it is always assumed that the advection and dispersion coefficient are both constants in the geological formation called aquifer [20–24], which is not the case in practice as from one point of the aquifer to another, properties may change; this is, for example, a wrong interpretation of the physical problem and the conversion into mathematical equation. In many research papers in the literature, one will observe that researchers are focused only in using fractional derivative to better describe the rate of change; however, the coefficients used in the model are always neglected. Since uncertain derivative is suitable in describing accurately, the physical parameters introduced in the mathematical equation and the fractional derivative aim to provide the effect of memory or filter, it is perhaps better to combine both while modeling real-world problem.

The aim of this work was to propose a new way to model real-world problem. This chapter will be structured as follows: The concept of fractional derivatives and recent trends will be presented in Sect. 2. A novel interpretation of fractional derivative will be presented in Sect. 3, this will be followed by the new development of uncertain derivative and it properties in Sect. 4. We will carry on with the novel approach of modeling real-world problem, for example, advection dispersion equation in Sect. 5; the approach is called uncertain-fractional modeling (UFM). Section 6 will be devoted to the analysis of existence and uniqueness of the novel advection equation; finally, numerical simulations will be presented in Sect. 7.

2 Fractional Derivatives

In this section, we present some information about some derivative with fractional

Definition 1 ([15–18]) The Riemann–Liouville fractional derivative, according to Riemann–Liouville, the fractional derivative of a function says f is given as

$$D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-x)^{n-\alpha-1} f(x)dx, \quad n-1 < \alpha \leq n \tag{1}$$

Definition 2 The Riemann–Liouville fractional integral, according to Riemann–Liouville, the fractional integral that is considered as anti-fractional derivative of a function f is

$$I_t^{\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx, \ x > a \tag{2}$$

Definition 3 Caputo fractional derivative, according to Caputo, the fractional derivative of a continuous and n-time differentiable function f is given as

$${}^{C}D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \left(\frac{d}{dx}\right)^n f(x)dx, \quad n-1 < \alpha \leq n \tag{3}$$

Definition 4 The modified Riemann–Liouville fractional derivative of a function f is given as

$$D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-x)^{n-\alpha-1} [f(x) - f(a)]dx, \quad n-1 < \alpha \leq n \tag{4}$$

There are other definitions that are not mentioned below.

Definition 5 Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$, then the new Caputo derivative of fractional order is defined as

$$D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(x) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \tag{5}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. However, if the function does not belong to $H^1(a,b)$, then the derivative can be reformulated as

$$D_t^{\alpha}(f(t)) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(x)) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx$$

Remark 1 The initiators witnessed that if $\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty)$, $\alpha = \frac{1}{1+\sigma} \in [0,1]$, then Eq. (5) assumes the form

$$D_t^{\alpha} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t f^{'}(x) \exp\left[-\frac{t-x}{\sigma}\right] dx, \quad N(0) = N(\infty) = 1 \tag{6}$$

In addition,

$$\lim_{\tau \to 0} \frac{1}{\sigma} Exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t) \tag{7}$$

3 New Physical Interpretation of Fractional Derivative

In the last past years, the big problem faced by researchers within the field of fractional calculus is the physical meaning of derivative with fractional order. Some fewer researchers have tried to answer this question; however, many other researchers around science were not satisfied with their demonstration. For instance, Tavassoli et al., suggested that we quote "We conclude that the product of fractional order derivative with the correspondent area is constant, so the fractional derivative produces the change in the area of the triangle enclosed by the tangent line at particular point and vertical line passing through this point and above X-axes with respect to fractional gradient line" [25]. The question we ask here is that: how can we then use this to portray the flow of groundwater within a geological formation? What about in epidemiology what can we say when we are describing the spread of the disease? Podlubny suggested that the geometry interpretation of the fractional derivative is the shadows on the walls [26]. According to the dictionary, the shadow is, we quote "is a region where light from a light source is obstructed by opaque object. It occupies all of the three-dimensional volume behind an object with light in front". The shadow on the walls! It is a very big philosophical term, but the problems with the shadow are the following (Fig. 1):

1. The shadow does not always represent the real shape of an object.
2. The shadow cannot tell the physical properties of an object, let us look at the following shadows and the real persons.

From the shadow, we can say the two persons in this picture are very tall, which may not be the case. In addition to this, we cannot say what clothes they are wearing, neither we can tell their race. From theirs shadows, there are many physical properties cannot be identified. If the geometric interpretation of fractional derivative is really the shadow on the wall, then it is not worth using fractional calculus to model real-world problems, as the results will never be accurate.

Let us forget about all this speculations around the geometric interpretation of fractional derivative. Let us face the reality, mathematical formulas sometime described accurately some physical problems without any physical interpretation, maybe this is the case of derivative with fractional order. Nevertheless, it was revealed by some authors in the literature that the derivative with fractional order describes the

Fig. 1 Objects and their shadows

memory effect [15–18]. Ah, this is another problem with fractional derivative! In the theory of elasticity, Yes, this can be acceptable. Nonetheless, in groundwater studies, the memory effect cannot be described, because with the fractional derivative, for a farmer that observed the pollution in his borehole would not be able to use the fractional derivative to say where the pollution is coming from. In epidemiology, using fractional derivative, the modeler will not be able to tell when the person was infected. This simply tells us that the derivative with fractional order will not always portray the memory effect.

Let us look at the mathematical formulation of derivative with fractional order, well the with Caputo derivative with fractional order we have a convolution of the local derivative with the power function. With the Riemann–Liouville derivative with fractional order, we have the derivative of a convolution of a given function and the power function. With the Caputo–Fabrizio derivative with fractional order, we have a convolution of local derivative of a given function and the function exponent. Now what is the convolution? What are the applications of convolution?

Convolution and associated functions are found in many applications in science, engineering, and mathematics. The applications of convolutions can be found in [27] and are listed below from (a) to (g).

(a) In image processing

In digital image processing, convolutional filtering plays an important role in many important algorithms in edge detection and related processes.

In optics, an out-of-focus photograph is a convolution of the sharp image with a lens function. The photographic term for this is bokeh.

In image processing applications such as adding blurring.

(b) In digital data processing

In analytical chemistry, Savitzky–Golay smoothing filters are used for the analysis of spectroscopic data. They can improve signal-to-noise ratio with minimal distortion of the spectra.

In statistics, a weighted moving average is a convolution.

(c) In acoustics

Reverberation is the convolution of the original sound with echoes from objects surrounding the sound source.

In digital signal processing, convolution is used to map the impulse response of a real room on a digital audio signal.

In electronic music, convolution is the imposition of a spectral or rhythmic structure on a sound. Often, this envelope or structure is taken from another sound. The convolution of two signals is the filtering of one through the other [15].

(d) In electrical engineering

The convolution of one function (the input signal) with a second function (the impulse response) gives the output of a linear time-invariant system (LTI). At any given moment, the output is an accumulated effect of all the prior values of the input function, with the most recent values typically having the most influence (expressed as a multiplicative factor). The impulse response function provides that factor as a function of the elapsed time since each input value occurred.

(e) In physics

Wherever there is a linear system with a "superposition principle," a convolution operation makes an appearance. For instance, in spectroscopy line broadening due to the Doppler effect on its own gives a Gaussian spectral line shape and collision broadening alone gives a Lorentzian line shape. When both effects are operative, the line shape is a convolution of Gaussian and Lorentzian, a Voigt function.

In time-resolved fluorescence spectroscopy, the excitation signal can be treated as a chain of delta pulses, and the measured fluorescence is a sum of exponential decays from each delta pulse.

In computational fluid dynamics, the large eddy simulation (LES) turbulence model uses the convolution operation to lower the range of length scales necessary in computation thereby reducing computational cost.

(f) In probability theory

The probability distribution of the sum of two independent random variables is the convolution of their individual distributions.

In kernel density estimation, a distribution is estimated from sample points by convolution with a kernel, such as an isotropic Gaussian (Diggle 1995).

(g) In radiotherapy

Treatment planning systems, most part of all modern codes of calculation, apply a convolution-superposition algorithm.

The above applications of a convolution show that the fractional derivative as convolution has multiple purposes, it can portray the memory as in the case of theory of elasticity, and it can be considered as a filter, in particular the Caputo and Caputo–Fabrizio type can be viewed as a filter of local derivative with power and exponent functions. In the following figures, we show the different between the Caputo–Fabrizio and Caputo filter as function of layer and fractional order. Physical interpretation of filters are given as follows, the Caputo–Fabrizio filter will constantly get rid of impurities from the local derivative from the first layer to the last layer no matter the fractional order. On the other hand, the Caputo filter will only get rid of the as the layer becomes dense.

The aim a filter is to get rid of impurities and produce only the real product, the fractional derivative is there for in some case the real velocity.

Figures 2 and 3 show that the Caputo and Fabrizio derivative is a low-pass filter for alpha greater than half; it is a band-pass filter when alpha is half, and finally, it is a high-pass filter when alpha is greater than half. It is important to recall that the low-pass filter is a filter that passes signals with a frequency lower than a certain cutoff frequency and attenuates signal with frequencies higher than cutoff frequency [28]. The high-pass filter is the opposite of low-pass filter. The band-pass filter is the combination of a low-pass filter and a high-pass filter.

Nevertheless with Caputo derivative, Figs. 4 and 5 show t hat the Caputo derivative is a high-pass filter for alpha greater than half; it is a band-pass filter when alpha

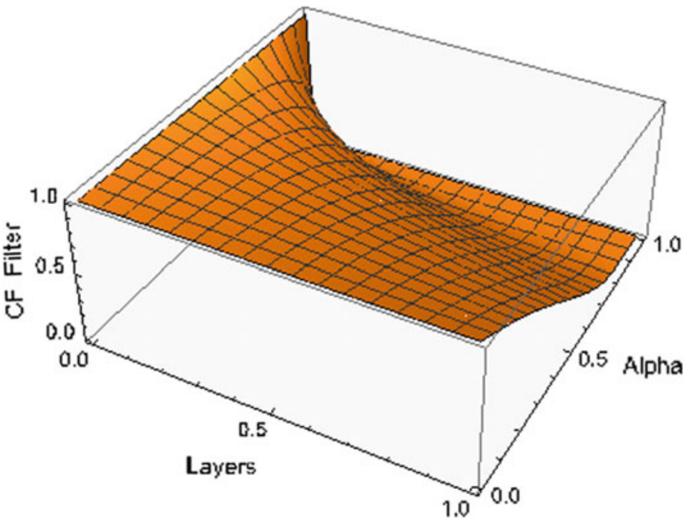

Fig. 2 Caputo–Fabrizio filter as function of space/time and alpha

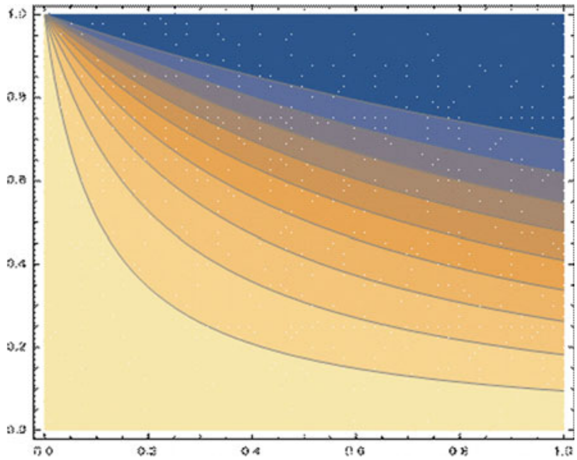

Fig. 3 Contour-plot for Caputo–Fabrizio filter

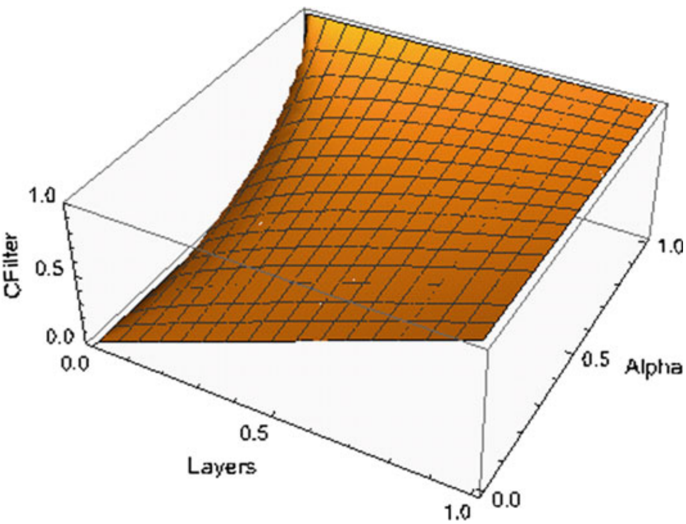

Fig. 4 Caputo filter as function of space/time and alpha

is half, and finally, it is a low-pass filter when alpha is greater than half. Therefore, we can see that both derivatives play the same role; however, the new derivative with fractional order does not have singularity, which is a greater advantage of this derivative over existing derivatives.

As we discussed earlier, the fractional derivative cannot inform us about the history of a given pollution within the geological formation called aquifer. Or the fractional derivative cannot allow us obtaining the function of dispersion and convection in

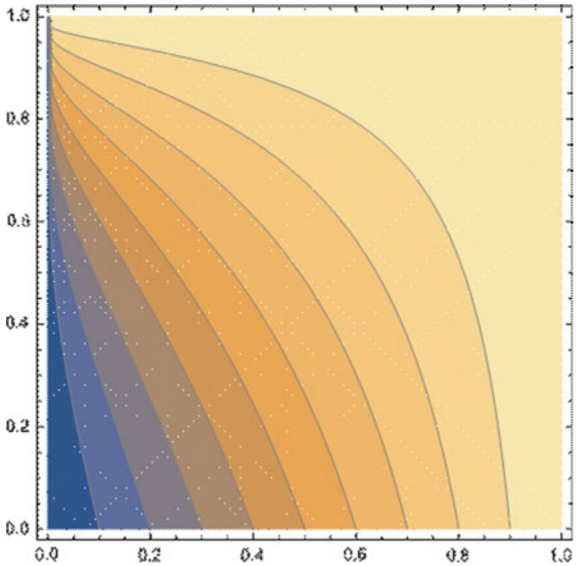

Fig. 5 Contour-Plot for Caputo filter

space and time when apply it to modified the advection-convection equation. It is therefore important to introduce a new derivative having the ability to first represent the nature in its real form which is nonlinearity and also be able to include the memory effect in a given real-world problem. We will introduce in the next section a derivative that introduces the effect of memory.

4 Uncertain Derivative

In mathematics, a dynamic scheme is a tuple (D, h, B) with D a manifold which can be a locally Banach space or Euclidean space, B the domain for time which is a set of nonnegative real, and h is an evolution rule $t \rightarrow f^t$ the range is of course a diffeomorphism of a manifold to itself.

Definition 6 Let D be a dynamic system with domain B (time or space), let u a positively defined function called uncertainty function of D within the domain B, then if $h \in D$, the uncertain derivative of a function h denoted by $U^u(f)$ is defined as

$$U^{u(t)}(h(t)) = (1 - u(t))h(t) + u(t)h^{'}(t) \tag{8}$$

Remark 2 If $u = 1$, we recover the first derivative (Local derivative), and if $u = 0$, we recover the initial function; this is conformable with the primary law of derivative.

4.1 Properties of Uncertain Derivative

Addition:

$$U^{u(t)}(ah(t)+bf(t)) = aU^{u(t)}(h(t)) + bU^{u(t)}(f(t)) \tag{9}$$

Multiplication:

$$U^{u(t)}(fg(t)) = f(t)U^{u(t)}(g(t)) + u(t)g(t)f^{'}(t) \tag{10}$$

Proof:

$$\begin{aligned}
U^{u(t)}(fg(t)) &= (1-u(t))f(t)g(t) + u(t)\left(f(t)g(t)\right)' \\
U^{u(t)}(fg(t)) &= (1-u(t))f(t)g(t) + u(t)[f(t)g'(t) + g(t)f'(t)] \\
U^{u(t)}(fg(t)) &= f(t)U^{u(t)}(g(t)) + u(t)g(t)f^{'}(t)
\end{aligned}$$

Division:

$$U^{u(t)}\left(\frac{f(t)}{g(t)}\right) = (1-u(t))\frac{f(t)}{g(t)} + u(t)\left(\frac{f(t)g'(t) - g(t)f'(t)}{g^2(t)}\right) \tag{11}$$

$$U^{u(t)}(\frac{f(t)}{g(t)}) = \left(\frac{g(t)U^{u(t)}(f(t)) - u(t)f(t)g'(t)}{g^2(t)}\right)$$

Lipchitz condition: Let $f(t)$ and $g(t)$ be two functions, then

$$\begin{aligned}
&\left\|U^{u(t)}(g(t)) - U^{u(t)}(f(t))\right\| \\
&\leq \left\|(1-u(t))g(t) + u(t)g'(t) - (1-u(t))f(t) - u(t)f'(t)\right\| \\
&\leq |1-u(t)|\,\|g(t)-f(t)\| + |u(t)|\left\|g'(t) - f'(t)\right\| \\
&\leq a\,\|g(t)-f(t)\| + b\left\|g'(t) - f'(t)\right\| \\
&\leq a\,\|g(t)-f(t)\| + b\alpha\,\|g(t)-f(t)\| \\
&\leq H\,\|g(t)-f(t)\|
\end{aligned}$$

This proves that the uncertain derivative possess the Lipchitz condition.

5 Fractional-Uncertain Modeling: Example Advection Equation

The groundwater is a very important source of drinkable water; in fact, it was revealed in many studies that 80% of fresh water is found in the geological formation called aquifers. This water is in high risk of pollution and the mathematical equation describing the migration of pollution underground is the well-known advection dispersion equation given as

$$D\frac{\partial^2 c}{\partial x^2} - v\frac{\partial c}{\partial x} - R\frac{\partial c}{\partial t} = 0 \tag{12}$$

In the above equation, D is the dispersion coefficient, v is the advection velocity, and R is the retardation factor. The above equation also tells us that all over a given aquifer, the dispersion, advection, and retardation factors are the same. This is not practically correct because the properties of souls change from one point of the aquifer to another that means the dispersion; advection and retardation factor must vary in space and time. With the variability of the dispersion coefficient, the retardation factor and advection, we will be able to trace the movement of the pollution from a certain point of time to another. To achieve this, we apply the uncertain derivative in space and time as follows:

$$DU^{u(x)}\left(U^{u(x)}\left(C(x,t)\right)\right) - vU^{u(x)}\left(C(x,t)\right) - RU^{u(t)}\left(C(x,t)\right) = 0 \tag{13}$$

Replacing the derivative by its definition and simplifying, we obtain

$$\begin{aligned} &D\left\{[(1-u(x))^2 - u'(x)u(x)]C(x,t) + \left\{\begin{array}{c} 2(1-u(x))u(x) \\ +u(x)u'(x) \end{array}\right\}\frac{\partial C(x,t)}{\partial x}\right\} \\ &+u^2(x)\frac{\partial^2 C(x,t)}{\partial x^2} - v\left\{(1-u(x))\,C(x,t) + u(x)\frac{\partial C(x,t)}{\partial x}\right\} - \\ &R\left\{(1-f(t))\,C(x,t) + f(t)\frac{\partial C(x,t)}{\partial x}\right\} \\ &= 0 \end{aligned} \tag{14}$$

Assuming that the uncertain order respect to time is small, then we divide on both sides with $1 - f(t)$, and using some asymptotic technique, the above equation can be approximated as follows:

$$\begin{aligned} &D\left\{\begin{array}{c} [(1-u(x)+u(t))^2 - u'(x)(u(x)+f(t))]C(x,t)+ \\ \left\{\begin{array}{c} 2\,(1-u(x)+f(t))\,(u(x)+f(t)) \\ +\,(u(x)+f(t))\,u'(x) \end{array}\right\}\frac{\partial C(x,t)}{\partial x} \end{array}\right\} \\ &+(u(x)+f(t))^2\frac{\partial^2 C(x,t)}{\partial x^2} \\ &-v\left\{(1-u(x)+f(t))\,C(x,t) + (u(x)+f(t))\right\}\frac{\partial C(x,t)}{\partial x} - RC(x,t) \\ &+f(t)R\frac{\partial C(x,t)}{\partial t} \\ &= 0 \end{aligned} \tag{15}$$

Rearranging the above equation, we obtain the following:

$$\left\{\begin{array}{l} \left[D\begin{pmatrix}(1-u(x)+u(t))^2\\ -u'(x)(u(x)+f(t))\end{pmatrix}\right. \\ \left. \quad -v\,(1-u(x)+f(t)) \right] C(x,t) \\ +\left\{D\begin{pmatrix}2\,(1-u(x)+f(t))\,(u(x)+f(t))\\ +(u(x)+f(t))u'(x)\\ -v\,\{(u(x)+f(t))\}\end{pmatrix}\right\}\dfrac{\partial C(x,t)}{\partial x}\end{array}\right\} \tag{16}$$
$$+D(u(x)+f(t))^2\frac{\partial^2 C(x,t)}{\partial x^2}-RC(x,t)+f(t)R\frac{\partial C(x,t)}{\partial t}$$
$$=0$$

And finally, we can reduce the above to

$$H(x,t)C(x,t)+v(x,t)\frac{\partial C(x,t)}{\partial x}+D(x,t)\frac{\partial^2 C(x,t)}{\partial x^2}=R(t)\frac{\partial C(x,t)}{\partial x} \tag{17}$$

The above equation shows that the advection and dispersion are functions of time and space, while the retardation function is a function of time. However, there is a new force H is viewed as the proportion that allows the value of that chemical concentration to remember its trajectory in the geological formation system and the time where it was retarded since its departure from the point of injection [29]. It is also possible for a given portion of pollution to remember where it was retarded in the aquifer. In order to include into mathematical equation the filter effect in time, meaning in order to have an accurate representation of the change in time of concentration of pollution within the geological formation, we introduce the Caputo–Fabrizio derivative with fractional order into equation to obtain

$$H(x,t)C(x,t)+v(x,t)\frac{\partial C(x,t)}{\partial x}+D(x,t)\frac{\partial^2 C(x,t)}{\partial x^2}=R(t){}_0^{CF}D_t^{\alpha}\left(C(x,t)\right) \tag{18}$$

The above equation is the result of UFM. It is clear that the above equation is more descriptive that the fractional advection dispersion equation that was proposed by many scholars.

6 Numerical Analysis

In this section, we will discuss the numerical solution of the Eq. (18). To do this, we first present the numerical approximation of the Caputo–Fabrizio derivative with fractional order [30]. For some positive integer N, the grid sizes in time for finite difference technique I are defined by

$$k = \frac{1}{N} \tag{19}$$

The grid points in the time interval $[0, T]$ are labeled $t_n = nj$, $n = 0, 1, 2, \ldots, TN$. The value of the function f at the grid point is $f_i = f(t_i)$. A discrete approximation to the Caputo–Fabrizio derivative with fractional order can be obtained by simple quadrature formula as follows:

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_n)) = \frac{M(\alpha)}{1-\alpha} \int_{a}^{t_n} f^{'}(x) \exp\left[-\alpha \frac{t_n - x}{1-\alpha}\right] dx \tag{20}$$

The above equation can be modified using the first-order approximation to

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_j)) = \frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \int_{(j-1)k}^{jk} \left(\frac{f^{j+1} - f^{j}}{\Delta t} + O(k)\right) \exp\left[-\alpha \frac{t_j - x}{1-\alpha}\right] dx \tag{21}$$

Before integration, we obtain the following expression:

$$\frac{M(\alpha)}{1-\alpha} \sum_{j=1}^{n} \left(\begin{matrix} \frac{f^{j+1} - f^{j}}{\Delta t} \\ +O(\Delta t) \end{matrix}\right) \int_{(j-1)k}^{jk} \exp\left[-\alpha \frac{t_n - x}{1-\alpha}\right] dx. \tag{22}$$

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_j)) = \frac{M(\alpha)}{\alpha} \sum_{j=1}^{n} \left(\frac{f^{j+1} - f^{j}}{\Delta t} + O(\Delta t)\right) d_{j,k}$$

where

$$d_{j,k} = \exp\left[-\alpha \frac{k}{1-\alpha}(n-j+1)\right] - \exp\left[-\alpha \frac{k}{1-\alpha}(n-j)\right] \tag{23}$$

We have finally that

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_n)) = \frac{M(\alpha)}{\alpha} \sum_{j=1}^{n} \left(\frac{f^{j+1} - f^{j}}{\Delta t}\right) d_{j,k} + \frac{M(\alpha)}{\alpha} \sum_{j=1}^{n} d_{j,k} O(\Delta t)$$

derivative at a point t_n is

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_n)) = \frac{M(\alpha)}{\alpha} \sum_{j=1}^{n} \left(\frac{f^{j+1} - f^{j}}{\Delta t}\right) d_{j,k} + O\left((\Delta t)^2\right) \tag{24}$$

Replacing the above together with first and second approximations of local derivative, we obtain

$$
\begin{aligned}
& H_i^j \left\{ \frac{C_i^{j+1} - C_i^j}{2} \right\} - v_i^j \left\{ \frac{\left(C_{i+1}^{j+1} - C_{i-1}^{j+1}\right) - \left(C_{i+1}^j - C_{i-1}^j\right)}{4\triangle x} \right\} \\
& + D_i^j \left\{ \frac{\left(C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}\right) - \left(C_{i+1}^j - 2C_i^j + C_{i-1}^j\right)}{2\left(\triangle x\right)^2} \right\} \\
= {} & R^j \frac{M(\alpha)}{\alpha} \sum_{k=1}^{j} \left(\frac{C_i^{k+1} - C_i^k}{\triangle t} \right) d_{j,k}
\end{aligned}
\tag{25}
$$

For simplicity, we let

$$
a_i^j = \frac{H_i^j}{2},\ b_i^j = \frac{v_i^j}{4\triangle x},\ c_i^j = \frac{D_i^j}{2\left(\triangle x\right)^2},\ d_i^j = R^j \frac{M(\alpha)}{\alpha \triangle t}
$$

then, Eq. (25) becomes

$$
\begin{aligned}
& a_i^j \left(C_i^{j+1} - C_i^j\right) - b_i^j \left(C_{i+1}^{j+1} - C_{i-1}^{j+1} - C_{i+1}^j + C_{i-1}^j\right) + \\
& c_i^j \left(\left(C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}\right) + \left(C_{i+1}^j - 2C_i^j + C_{i-1}^j\right)\right) \\
= {} & d_i^j \left(C_i^{j+1} - C_i^j\right) d_{i,j} + d_i^j \sum_{k=1}^{j-1} \left(C_i^{k+1} - C_i^k\right) d_{k,j}
\end{aligned}
\tag{26}
$$

Then,

$$
\begin{aligned}
& \left(a_i^j - 2c_i^j - d_i^j d_{i,j}\right) C_i^{j+1} \\
= {} & \left(a_i^j + 2c_i^j - d_i^j d_{i,j}\right) C_i^j \\
& + b_i^j \left(C_{i+1}^{j+1} - C_{i-1}^{j+1} - C_{i+1}^j + C_{i-1}^j\right) \\
& + c_i^j \left(\left(C_{i+1}^{j+1} + C_{i-1}^{j+1}\right) + \left(C_{i+1}^j + C_{i-1}^j\right)\right) \\
& + d_i^j \sum_{k=1}^{j-1} \left(C_i^{k+1} - C_i^k\right) d_{k,j}
\end{aligned}
\tag{27}
$$

We shall now present the stability analysis of the numerical scheme for solving the modified model.

6.1 Stability Analysis of the Numerical Scheme

The aim of this section is to show the efficiency of the numerical scheme via the stability analysis. To achieve this, we assume that $g_i^j = C_i^j - y_i^j$ where y_i^j is the approximate solution of the modified equation at the given point in time and space (x_i, t_j), $(i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M)$, also the error for approximation is given as $g^j = \left[g_1^j, g_2^j, \ldots, g_N^j\right]$. The error committed while solving the new advection equation is given as

$$\begin{aligned}\left(a_i^j - 2c_i^j - d_i^j d_{i,j}\right) g_i^{j+1} = &\left(a_i^j + 2c_i^j - d_i^j d_{i,j}\right) g_i^j \\ &+ b_i^j \left(g_{i+1}^{j+1} - g_{i-1}^{j+1} - g_{i+1}^j + g_{i-1}^j\right) \\ &- c_i^j \left(\left(g_{i+1}^{j+1} + g_{i-1}^{j+1}\right) + \left(g_{i+1}^j + g_{i-1}^j\right)\right) \\ &+ d_i^j \sum_{k=1}^{j-1} \left(g_i^{k+1} - g_i^k\right) d_{k,j}\end{aligned} \tag{28}$$

to study the stability, we let

$$g_m(x,t) = \exp[at]\exp[ik_m x] \tag{29}$$

In our study, the stability characteristics can be studied using just the above form for error with no loss in generality

$$\begin{aligned} g_n^j &= \exp[at]\exp[ik_m x], \\ g_n^{j+1} &= \exp[a(t+\Delta t)]\exp[ik_m x], \\ g_{n+1}^j &= \exp[at]\exp[ik_m(x+\Delta x)], \\ g_{n-1}^j &= \exp[at]\exp[ik_m(x-\Delta x)], \\ g_{n+1}^{j+1} &= \exp[a(t+\Delta t)]\exp[ik_m(x+\Delta x)], \\ g_{n-1}^{j+1} &= \exp[a(t+\Delta t)]\exp[ik_m(x-\Delta x)], \\ g_{n-1}^{j-1} &= \exp[a(t-\Delta t)]\exp[ik_m(x-\Delta x)] \end{aligned} \tag{30}$$

where $k_m = \frac{\pi m}{L}$, $m = 1, 2, \ldots, M = \frac{L}{\Delta x}$. Now replacing the above in Eq. (28), we obtain

$$\begin{aligned}
&\left(a_i^j+2c_i^j-d_i^j d_{i,j}\right)\exp[a\,(t+\Delta t)]\exp[ik_m x] \\
&=\left(a_i^j-2c_i^j-d_i^j d_{i,j}\right)\exp[at]\exp[ik_m x]- \\
&b_i^j\begin{pmatrix}\exp[a\,(t+\Delta t)]\exp[ik_m\,(x+\Delta x)] \\ -\exp[a\,(t+\Delta t)]\exp[ik_m\,(x-\Delta x)]- \\ \exp[at]\exp[ik_m\,(x+\Delta x)]+\exp[at]\exp[ik_m\,(x-\Delta x)]\end{pmatrix}+ \\
&+c_i^j\begin{pmatrix}\begin{pmatrix}\exp[a\,(t+\Delta t)]\exp[ik_m\,(x+\Delta x)] \\ +\exp[a\,(t+\Delta t)]\exp[ik_m\,(x-\Delta x)]\end{pmatrix} \\ +\left(\exp[at]\exp[ik_m\,(x+\Delta x)]+\exp[at]\exp[ik_m\,(x-\Delta x)]\right)\end{pmatrix} \\
&d_i^j\sum_{k=1}^{j-1}\left(\left(\exp[a\,(t+\Delta t)]\exp[ik_m x]-\exp[a\,(\Delta t)]\exp[ik_m x]\right)\right)d_{k,j}
\end{aligned} \tag{31}$$

After simplification, we obtain the following

$$\begin{aligned}
&\left(a_i^j+2c_i^j-d_i^j d_{i,j}\right)\exp[a\,(\Delta t)] \\
&=\left(a_i^j-2c_i^j-d_i^j d_{i,j}\right)- \\
&b_i^j\begin{pmatrix}\exp[a\,(\Delta t)]\exp[ik_m\,(\Delta x)] \\ -\exp[a\,(\Delta t)]\exp[ik_m\,(-\Delta x)]- \\ \exp[ik_m\,(\Delta x)]+\exp[ik_m\,(-\Delta x)]\end{pmatrix}+ \\
&c_i^j\begin{pmatrix}\begin{pmatrix}\exp[a\,(\Delta t)]\exp[ik_m\,(\Delta x)] \\ +\exp[a\,(\Delta t)]\exp[ik_m\,(-\Delta x)]\end{pmatrix} \\ +\left(\exp[ik_m\,(\Delta x)]+\exp[ik_m\,(-\Delta x)]\right)\end{pmatrix} \\
&+d_i^j\sum_{k=1}^{j-1}\left(\left(\exp[a\,(\Delta t)]-1\right)\right)d_{k,j}
\end{aligned} \tag{32}$$

Rearranging the above, we obtain

$$\begin{aligned}
&\left\{\begin{matrix}\left(a_i^j+2c_i^j-d_i^j d_{i,j}\right)+b_i^j\begin{pmatrix}\exp[ik_m\,(\Delta x)] \\ -\exp[ik_m\,(-\Delta x)]\end{pmatrix} \\ -c_i^j\left(\begin{pmatrix}\exp[ik_m\,(\Delta x)] \\ +\exp[ik_m\,(-\Delta x)]\end{pmatrix}\right)-jd_i^j d_{k,j}\end{matrix}\right\}\exp[a\,(\Delta t)] \\
&=\left(a_i^j-2c_i^j-d_i^j d_{i,j}\right)+b_i^j\left(\exp[ik_m\,(\Delta x)]+\exp[ik_m\,(-\Delta x)]\right)+ \\
&\quad c_i^j\left(\exp[ik_m\,(\Delta x)]+\exp[ik_m\,(-\Delta x)]\right)-jd_i^j d_{i,j}
\end{aligned} \tag{33}$$

Then,

$$
\begin{aligned}
&\left(a_i^j - 2c_i^j - d_i^j d_{i,j}\right) - b_i^j \left(\exp[ik_m(\Delta x)] + \exp[ik_m(-\Delta x)]\right) + \exp[a(\Delta t)] \\
&= \frac{-c_i^j \left(\left(\exp[ik_m(\Delta x)] + \exp[ik_m(-\Delta x)]\right)\right) - j d_i^j d_{i,j}}{\begin{Bmatrix} \left(a_i^j + 2c_i^j - d_i^j d_{i,j}\right) + b_i^j \left(\exp[ik_m(\Delta x)] - \exp[ik_m(-\Delta x)]\right) \\ -c_i^j \left(\left(\exp[ik_m(\Delta x)] + \exp[ik_m(-\Delta x)]\right)\right) - j d_i^j d_{k,j} \end{Bmatrix}}
\end{aligned} \tag{34}
$$

Note that the condition for stability analysis is given by the following inequality

$$
\frac{g_i^{j+1}}{g_i^j} = \exp[a(\Delta t)]
$$

Thus, if

$$
\left|\frac{g_i^{j+1}}{g_i^j}\right| \le 1
$$

From Eq. (33), we have the following

$$
\begin{aligned}
\left|\frac{g_i^{j+1}}{g_i^j}\right| &= \left|\exp[a(\Delta t)]\right| \\
&= \left| \frac{\begin{matrix} \left(a_i^j - 2c_i^j - d_i^j d_{i,j}\right) - b_i^j \left(\exp[ik_m(\Delta x)] + \exp[ik_m(-\Delta x)]\right) + \\ + c_i^j \left(\left(\exp[ik_m(\Delta x)] + \exp[ik_m(-\Delta x)]\right)\right) - j d_i^j d_{i,j} \end{matrix}}{\begin{Bmatrix} \left(a_i^j + 2c_i^j - d_i^j d_{i,j}\right) + b_i^j \left(\exp[ik_m(\Delta x)] - \exp[ik_m(-\Delta x)]\right) \\ -c_i^j \left(\left(\exp[ik_m(\Delta x)] + \exp[ik_m(-\Delta x)]\right)\right) - j d_i^j d_{k,j} \end{Bmatrix}} \right|
\end{aligned}
$$

$$
\cos[k_m \Delta x] = \frac{\exp[ik_m x] + \exp[-ik_m x]}{2}, \quad \sin^2[k_m \Delta x] = \frac{1 - \cos[2k_m \Delta x]}{2}
$$

Then, the condition for stability is given as

$$
b_i^j \le c_i^j
$$

Thus from that above statement, we can present the following theorem.

Theorem 1 *The Crank–Nicholson scheme for solving the uncertain-fractional advection dispersion equation is stable providing that the following inequality is satisfied*

$$
\frac{v_i^j}{D_i^j} \le \frac{2}{\Delta x}
$$

7 Numerical Simulations

In this section, we present the numerical simulations of the resulted model from the uncertain-fractional advection dispersion equation. In this simulation, we will choose the uncertain derivative orders to be $u(x) = 2 + \sin(x + \frac{\pi}{3})$, $f(t) = 1 + \cos(t + \frac{\pi}{2})$, we consider the dispersion coefficient to be 0.96, the retardation coefficient to be 2, and the advection coefficient to be 0.74. The numerical simulation will be done for different values of the fractional order derivative; we will also alter the uncertain functions to see the effectiveness of the input. The numerical results are depicted in Figs. 6, 7, and 8.

In this simulation, we will choose the uncertain derivative orders to be $u(x) = 2 + 2\sin(x + \frac{\pi}{6})$, $f(t) = 2 - \cos(t + \frac{\pi}{5})$. The numerical simulations are therefore depicted in Figs. 9, 10, and 11.

It is clear from above Figs. 6, 7, 8, 9, 10, and 11 that both fractional derivative and uncertain derivative play major role in simulation or prediction. One of the big challenges faced by those researchers modeling the movement of plume via geological formation is perhaps the fingering effect, which is actually what we usually observe in real-world problem. Many research have been developed in trying to produce a mathematical equation that will best predict this physical occurrence; however, no sound equation was found suitable for this task. The fingering effect is a proof that the properties of the geological formation via which the plume is moving are not the same. In Figs. 6, 7, and 8, we used different uncertain functions for time

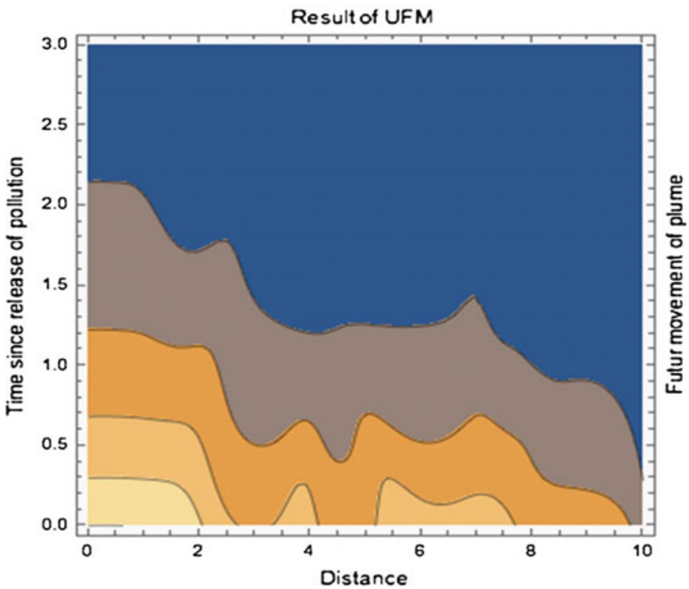

Fig. 6 Simulation of plume for alpha = 0.95

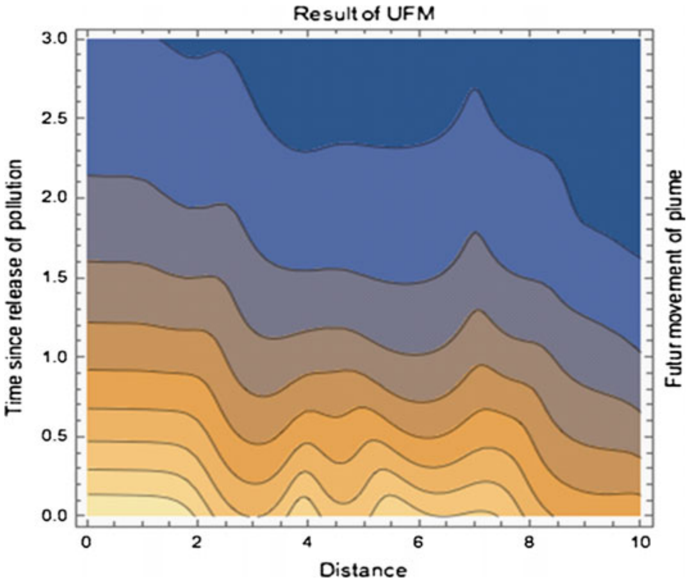

Fig. 7 Simulation of plume for alpha = 0.5

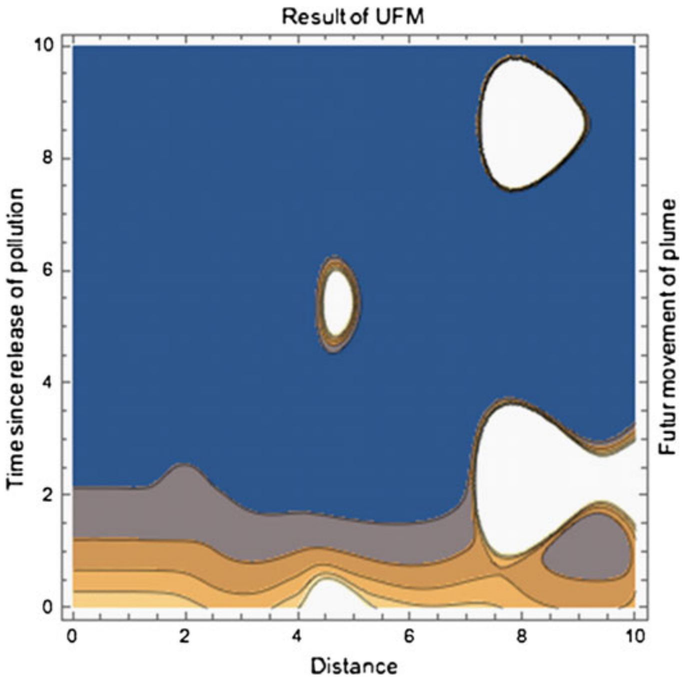

Fig. 8 Simulation of plume for alpha = 0.3

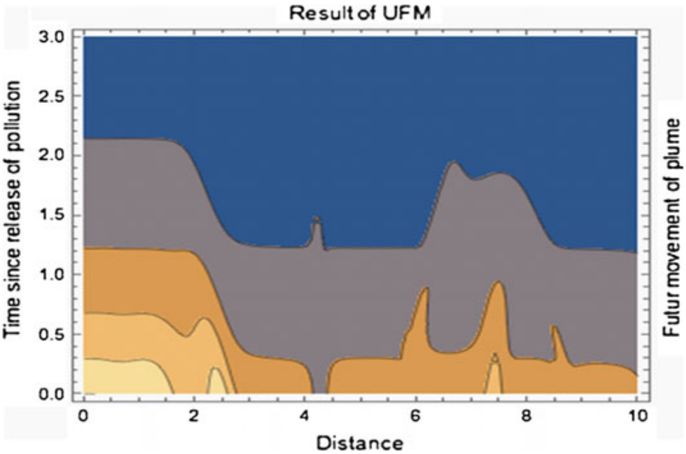

Fig. 9 Simulation of plume for alpha = 0.95

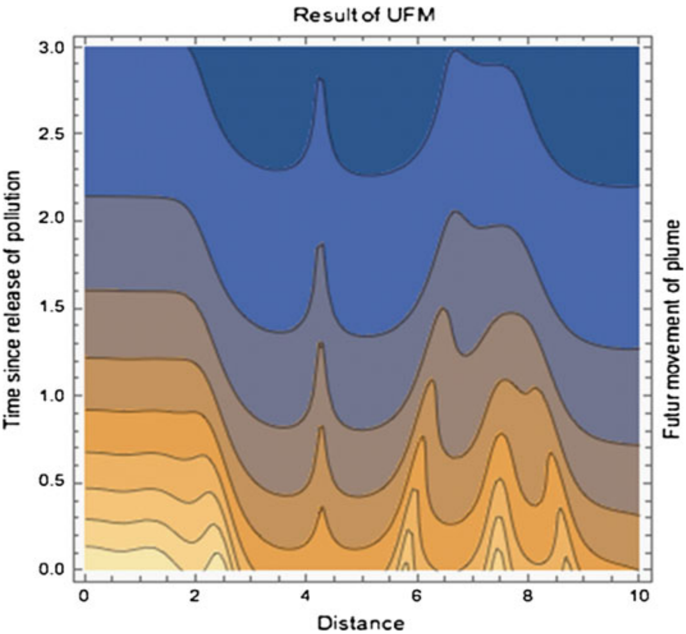

Fig. 10 Simulation of plume for alpha = 0.5

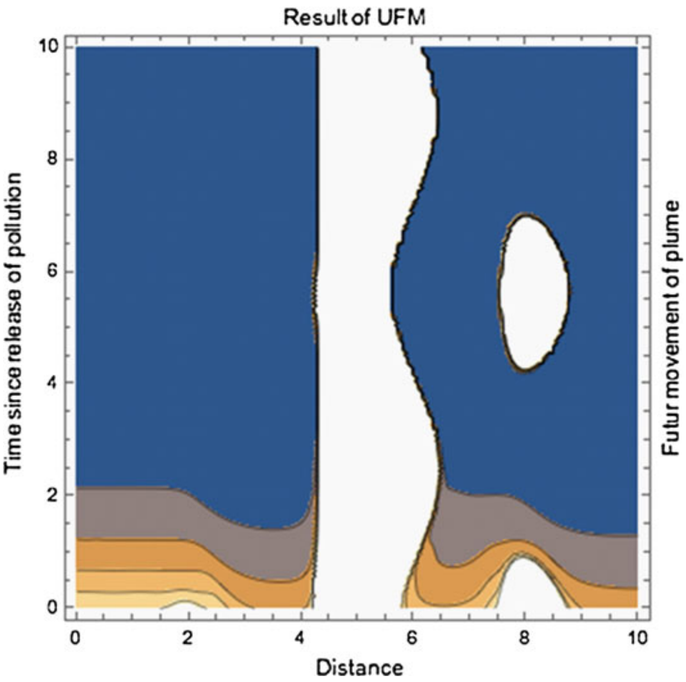

Fig. 11 Simulation of plume for alpha = 0.3

and space, and then, we observed a kind of fingering effect as the fractional order changes from 0.95 to 0.3. In Fig. 8, in particular we observed that there will be some places in the aquifer where there will be no pollution at all, this cannot be described via neither the advection dispersion model nor fractional advection dispersion model. In Figs. 9, 10, and 11, we changed uncertain function and observed different kinds of fingering effects.

8 Conclusion

To have a good prediction of natural occurrence, two important aspects are required. The first is perhaps the observations and the second one is the interpretation of the observation as mathematical formula. The local derivative was first introduced to portray the rate of change; latter on this derivative faced many challenges to model real-world problems due to their complexity. The concept of derivative with fractional order was later introduced and used to enhance the field of modeling. However, these derivatives with fractional order faced lot of controversies, as their physical interpretations were not fully understood. Some researchers said their physical interpretation is the shadow on the wall. However, when looking at the shadow of an object, it is sometime impossible to identify the real object; therefore, their interpretation was not correct. In many research papers in the literatures, it is claimed that the fractional derivatives portray the memory effect. Now our question was, can the fractional derivative allow the pollution to remember its path in the geological formation? Or when a given disease affects an individual can the fractional derivative be able to trace the history of the infection? The answer is no; therefore, a need of an operator able to do this job is at hand. In this chapter, we have using the concept of convolution provided a suitable interpretation of derivative with fractional order. We provided a derivative able to describe the effect of memory and used it to model the advection dispersion problems. We solved the new equation numerically. We presented the stability analysis of the used scheme and some numerical simulations.

References

1. F. Cajori, The history of notations of the calculus. Ann. Math. **25**(1), 1–46 (1923)
2. P. Leonid, J. Lebedev, M. Cloud, Approximating Perfection: a Mathematician's Journey into the World of Mechanics, *The Tools of Calculus* (Princeton University Press, Princeton, 2004)
3. J. Albers, Donald, D. Richard, O. Anderson, L. Don, ed. Undergraduate Programs in the Mathematics and Computer Sciences: The 1985-1986 Survey, Mathematical Association of America No. 7, 1986
4. B. Ross, A brief history and exposition of the fundamental theory of fractional calculus, Fractional calculus and its applications. Lect. Notes Math. **457**, 1–36 (1975)
5. L. Debnath, A brief historical introduction to fractional calculus. Int. J. Math. Educ. Sci. Technol. **35**(4), 487–501 (2004)
6. M. Caputo, Linear model of dissipation whose Q is almost frequency independent-II. Geophys. J. R. Astr. Soc. **13**, 529–539 (1967)
7. D. Benson, S. Wheatcraft, M. Meerschaert, Application of a fractional advection-dispersion equation. Water Resour. Res **36**, 1403–1412 (2000)
8. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel. Progr. Fract. Differ. Appl. **1**(2), 73–85 (2015)
9. J. Losada, J.J. Nieto, Properties of a new fractional derivative without singular kernel. Progr. Fract. Differ. Appl. **1**(2), 87–92 (2015)
10. S.G. Samko, Fractional integration and differentiation of variable order. Anal. Math. **21**, 213–236 (2005)
11. G.R.J. Cooper, D.R. Cowan, Filtering using variable order vertical derivatives. Comput. Geosci. **30**, 455–459 (2004)
12. H. Sun, W. Chen, Y. Chen, Variable-order fractional differential operatorsin anomalous diffusion modeling. Phys. A **388**(21), 4586–4592 (2009)
13. A. Atangana, A generalized advection dispersion equation. J. Earth Syst. Sci. **123**(1), 101–108 (2014)
14. A. Atangana, S.C. Oukouomi Noutchie, A modified groundwater flow model using the space time Riemann-Liouville fractional derivatives approximation, Abstr. Appl. Anal. 2014, Article ID 498381, 7 pp (2014)
15. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993)
16. M. Davison, C. Essex, Fractional differential equations and initial value problems. Math.Sci. **23**(2), 108–116 (1998)
17. A. Atangana, *Local derivative with new parameter: Theory, Methods and Applications, ISBN 978-0-08-100644-3* (Academic press, Elsevier, 2015)

18. A.V. Chechkin, R. Gorenflo, I.M. Sokolov, Fractional diffusion in inhomogeneous media. J. Phys. A **38**(42), L679–L684 (2005)
19. H.G. Sun, W. Chen, Y.Q. Chen, Variable order fractional differential operators in anomalous diffusion modeling. Phys. A **388**(21), 4586–4592 (2009)
20. S. Umarov, S. Steinberg, Variable order differential equations with piecewise constant order-function and diffusion with changing modes. Zeitschrift für Analysis und ihre Anwendungen **28**(4), 431–450 (2009)
21. M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations. J. Comput. Appl. Math. **172**(1), 65–77 (2004)
22. C. Tadjeran, M.M. Meerschaert, H.P. Scheffler, A second-order accurate numerical approximation for the fractional diffusion equation. J. Comput. Phys. **213**(1), 205–213 (2006)
23. M.T. van Genuchten, W.J. Alves, Analytical Solutions of One Dimensional Convective-Dispersive Solute Transport Equations, Technical Bulletin, no. 1661, United State Department of Agriculture (1982)
24. D.A. Benson, R. Schumer, M.M. Meerschaert, S.W. Wheatcraft, Fractional dispersion, Lévy motion, and the MADE tracer tests. Transp. Porous Media **42**(1–2), 211–240 (2001)
25. M.H. Tavassoli, A. Tavassoli, M.R. Ostad Rahimi, The geometric and physical interpretation of fractional order derivatives of polynomial functions. Differ. Geom. Dyn. Syst. **15**, 93–104 (2013)
26. I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. **5**(4), 367–386 (2002)
27. Z. Udo, ed. by DAFX:Digital Audio Effects, pp. 48–49 (2002)
28. J. Watkinson, The art of sound reproduction. Focal Press **268**, 479 (2009)
29. A. Atangana, A. Kilicman, On the generalized mass transport equation to the concept of variable fractional derivative. Math. Prob. Eng. **2014**, 9 (2014)
30. A. Atangana, J.J. Nieto, Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel. Adv. Mech. Eng. **7**(10), 1–7 (2015)

Abdon Atangana obtained his B.S. degree in pure mathematics. He obtained his honors degree and M.S. degree at the Department of Applied Mathematics with distinction. He obtained his Ph.D. degree in applied mathematics at the Institute for Groundwater Studies, University of the Free State, Bloemfontein, South Africa. He is serving as editor of some international journals and also reviewer of more than 100 international accredited journals. He has been lead and guest editor of some special issues. He has presented and participated in more than 15 international conferences. His research interests are methods and applications of partial and ordinary differential equations, fractional differential equations, perturbations methods, asymptotic methods, iterative methods, and groundwater modeling. He is the author of the book with title Derivative with new parameter: theory, methods and applications published by Academic Press Elsevier.

İlknur Koca obtained her master's and Ph.D. in applied mathematics in Ankara University. She is currently working as Assistant Professor at the Department of Mathematics of Burdur University. She is interesting in the use of fractional derivative to model real-world problems and methods of fractional differential equations.

Quadratic Reciprocity and Some "Non-differentiable" Functions

Kalyan Chakraborty and Azizul Hoque

Abstract Riemann's non-differentiable function and Gauss's quadratic reciprocity law have attracted the attention of many researchers. In [28] (Proc Int Conf–Number Theory 1, 107–116, 2004), Murty and Pacelli gave an instructive proof of the quadratic reciprocity via the theta transformation formula and Gerver (Amer J Math 92, 33–55, 1970) [12] was the first to give a proof of differentiability/non-differentiability of Riemann's function. The aim here is to survey some of the work done in these two directions and concentrates more onto a recent work of the first author along with Kanemitsu and Li (Res Number Theory 1, 14, 2015) [5]. In that work (Kanemitsu and Li, Res Number Theory 1, 14, 2015) [5], an integrated form of the theta function was utilised and the advantage of that is that while the theta function $\Theta(\tau)$ is a dweller in the upper half-plane, its integrated form $F(z)$ is a dweller in the extended upper half-plane including the real line, thus making it possible to consider the behaviour under the increment of the real variable, where the integration is along the horizontal line.

Keywords Quadratic reciprocity · Theta transformation · Non-differentiable function

2010 Mathematics Subject Classification Primary: 11A15 · Secondary: 11F27

1 Introduction

In the early part of the nineteenth century, many mathematicians believed that a continuous function has derivative in a reasonably large set. A.M. Ampére in his paper in 1806 tried to give a theoretical justification for this based of course on the knowledge at that time. In a presentation before the Berlin Academy on July 18,

K. Chakraborty (✉) · A. Hoque
Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi 211019, Allahabad, India
e-mail: kalyan@hri.res.in

A. Hoque
e-mail: azizulhoque@hri.res.in

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_8

1872, K. Weierstrass kind of shocked the mathematical community by proving this assertion to be false! He presented a function which was everywhere continuous but differentiable nowhere. We will talk about this function of Weierstrass in the later sections in some detail. This example was first published by du Bois-Reymond in 1875. Weierstrass also mentioned Riemann, who apparently had used a similar construction (without proof though!) in his own lectures in 1861. However, it seems like neither Weierstrass nor Riemann was first to get such examples. The earliest known example is due to B. Bolzano, who in the year 1830 exhibited (published in the year 1922 after being discovered a few years earlier) a continuous nowhere differentiable function. Around 1860, the Swiss mathematician, C. Cellérier, discovered such a function, but unfortunately it was not published then and could be published only in 1890 after his death. To know more about the interesting history and details about such functions, the reader is referred to the excellent Master's thesis of J. Thim [32].

Riemann, as mentioned in the earlier paragraph, opined that the function,

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$$

is nowhere differentiable. K. Weierstrass (in 1872) tried to prove this assertion, but could not resolve it. He could construct another example of a continuous nowhere differentiable function

$$\sum_{n=0}^{\infty} \cos(b^n \pi x)$$

where $0 < a < 1$ and b is a positive integer such that

$$ab > 1 + 3/2\pi.$$

G.H. Hardy [16] showed that Weierstrass function has no derivative at points of the form $\xi\pi$ with ξ is either irrational or rational of the form $2A/(4B+1)$ or $(2A+1)/(2B+2)$. Much later in 1970, J. Gerver [12] disproved Riemann's assertion by proving that his function is differentiable at points of the form $\xi\pi$, where ξ is of the form $(2A+1)/(2B+1)$, with derivative $-1/2$. Arthur, a few years later in 1972, used Poisson's summation formula and properties of Gauss sums to deduce Gerver's result and thus established a link between Riemann's function and quadratic reciprocity (via Gauss sums). Interested reader can also look into two excellent expositions of Riemann's function by E. Neuenschwander [29] and that of S.L. Segal [30] for further enhancement in knowledge regarding this problem. This problem was explored by many other authors and among them a few references could be [13, 14, 16, 21, 23].

In an interesting work in [5], the authors observed that Riemann's function $f(x)$ is really an integrated form of the classical θ function. Then, they make the link to quadratic reciprocity from an exposition of M.R. Murty and A. Pacelli [28], who (following Hecke) showed that the transformation law for the theta function can be used to derive the law of quadratic reciprocity. The goal of [5] was to combine these two ideas and derive both the differentiability of $f(x)$ at certain points and the law of quadratic reciprocity at one go.

An identity of Davenport and Chowla arose our interest in Riemann's function. The identity is

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} \psi(nx) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n^2 x}{n^2}. \tag{1}$$

The notations are standard, i.e.

$$\lambda(n) = (-1)^{\Omega(n)}$$

with $\Omega(n)$ denotes the total number of distinct prime factors of n. Also,

$$\psi(x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n}$$

is the saw-tooth Fourier series, i.e. it is the Fourier series expansion of the "saw-tooth" function:

$$f(x) = \begin{cases} 1/2(\pi - x), & \text{if} \quad 0 < x \le 2\pi; \\ f(x + 2\pi), & \text{otherwise.} \end{cases}$$

We would like to spare some discussion for this identity. On the one hand in (1), there is the Liouville function, a prime number theoretic entity. On the other hand, one has Riemann's example of an interesting function. The integrated identity can be derived from the functional equation only, but to differentiate it, one needs the estimate for the error term for the Liouville function. This is as deep as the prime number theorem and is known to be very difficult.

The situation is similar to Ingham's handling [20] of the prime number theorem. First, one applies the Abelian process (integration) and then Tauberian process (differencing) which needs more information. A huge advantage of this process in [5] is that while the theta function $\Theta(\tau)$ dwells in the upper half-plane, its integrated form $F(z)$ is a dweller in the extended upper half-plane which includes the real line. This makes it possible to consider the behaviour under the increment of the real variable, where the integration is along the horizontal line. The elliptic theta function $\theta(s) = \Theta(-i\tau)$ is a dweller in the right half-plane $\{\sigma > 0\}$, where the integration is along the vertical line. In terms of Lambert series, an idea of Wintner deals with limiting the behaviour of the Lambert series on the circle of convergence, i.e. radial integration. Here, it corresponds to integration along an arc.

One can think of it as two apparently disjoint aspects merging on the real line as limiting behaviours of zeta and that of theta functions. In [5], the main observation was that the right-hand side may be viewed as the imaginary part of the integrated theta series. It seems that the uniform convergence of the left-hand side and the differentiability of the right-hand side merge as the limiting behaviour of a sort of modular function and that of the Riemann zeta function.

2 Weierstrass's Non-differentiable Function

We begin with the following function which is due to Weierstrass:

$$f(x) = \sum a^n \cos b^n \pi x.$$

In 1875, Weierstrass proved that $f(x)$ has no differential coefficient for any value of x with restrictions that b is an odd integer,

$$0 < a < 1 \tag{2}$$

and

$$ab > 1 + \frac{3}{2}\pi. \tag{3}$$

This result has been generalised by many mathematicians (for details, see [6, 10, 25, 26, 35]) by considering functions of more general forms

$$C(x) = \sum a_n \cos b_n x \tag{4}$$

and

$$S(x) = \sum a_n \sin b_n x \tag{5}$$

where a_n and b_n are positive, the series $\sum a_n$ is convergent, and the sequence $\{b_n\}$ increases steadily with more than a certain rapidity. In 1916, G.H. Hardy with a new idea developed a powerful method to discuss the differentiability of Weierstrass's function. This method is easy to apply to find very general conditions for the non-differentiability of the type of series (4) and (5). The known results concerning the series (4) are, so far we are aware, as follows: K. Weierstrass gave the condition (3) and only improvement to this is

$$ab > 1 + \frac{3}{2}\pi(1 - a). \tag{6}$$

This was due to T.J. Bromwich [2]. The conditions (3) and (6) debar the existence of a finite (or infinite) differential coefficient. For the non-existence of a finite differential coefficient, there are alternative conditions which were independently given by U. Dini, M. Lerch, and T.J. Bromwich. The conditions given by U. Dini are

$$ab \geq 1, \qquad ab^2 > 1 + 3\pi^2 \tag{7}$$

and that are given by M. Lerch:

$$ab \geq 1, \qquad ab^2 > 1 + \pi^2. \tag{8}$$

Finally, T.J. Bromwich provided the following conditions for the same

$$ab \geq 1, \qquad ab^2 > 1 + \frac{3}{4}\pi^2(1-a). \tag{9}$$

All these conditions though supposed that b is an odd integer. However, U. Dini [7] showed that if the condition (3) is replaced by

$$ab > 1 + \frac{3}{2}\pi\frac{1-a}{1-3a} \tag{10}$$

or the condition (7) by

$$ab > 1, \qquad ab^2 > 1 + 15\pi^2\frac{1-a}{5-21a} \tag{11}$$

then the restriction "odd" on b may be removed. It is naturally in built in the condition (10) that $a < \frac{1}{3}$ and in the condition (11) that $a < \frac{5}{21}$.

The conditions (6)–(11) look superficial though. It is hard to find any corroboration between these conditions as to why they really correspond to any essential feature of the problems arise in discussion of Weierstrass function. They appear merely as a consequence of the limitations of the methods that were employed. There is in fact only one condition which suggests itself naturally and seems truly relevant to the situation at hand, namely:

$$ab \geq 1.$$

The main results that were proved by G.H. Hardy [16] concerning Weierstrass function and the corresponding function defined by a series of sines and cosines are summarised below. It is interesting to note that b has no more restriction to be an integer in the next two results.

Theorem 2.1 (Hardy) *The functions*

$$C(x) = \sum a^n \cos b^n \pi x, \qquad S(x) = \sum a^n \sin b^n \pi x,$$

(with $0 < a < 1,\ b > 1$*) have no finite differential coefficient at any point whenever* $ab \geq 1$.

Remark 2.1 The above Theorem 2.1 is **not true** if the word "finite" is omitted.

Theorem 2.2 (Hardy) *Let $ab > 1$ and so $\xi = \frac{\log(1/a)}{\log b} < 1$. Then, each of the functions in the previous theorem satisfies*

$$f(x+h) - f(x) = O\big(|h|^{\xi}\big),$$

for each value of x. Neither of them satisfy

$$f(x+h) - f(x) = o\big(|h|^{\xi}\big),$$

for any x.

Hardy proved these theorems in two steps. In the first step, he considered b an integer, and then in the second step, he extended his proof for general case. In the next two subsections, we give the outline of the proof of these theorems.

2.1 b Is an Integer

Let us substitute $\theta = \pi x$ and then the function of Weierstrass becomes a Fourier series in θ. Following which he defines a harmonic function $G(r, \theta)$ by the real part of the power series:

$$\sum a_n z^n = \sum a_n r^n e^{ni\theta}.$$

This series is convergent when $r < 1$. One further supposes that $G(r, \theta)$ is continuous for $r \leq 1$, and that

$$G(1, \theta) = g(\theta).$$

Let us first recall some results concerning the function $G(r, \theta)$ under the above assumptions. We also use the familiar Landau symbol:
$f(n) = o(g(n))$ which means that for all $c > 0$ there exists some $k > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq k$. The value of k must not depend on n, but may depend on c. The first lemma can be proved by considering $\theta_0 = 0$.

Lemma 2.1 *Let*

$$g(\theta) - g(\theta_0) = o(|\theta - \theta_0|)$$

where $0 < \alpha < 1$ and $\theta \to \theta_0$. Then

$$\frac{\delta G(r, \theta_0)}{\delta \theta_0} = o(1-r)^{(1-\alpha)}$$

whenever $r \to 1$.

The next lemma is a well-known result, and interested reader can find a proof of it in [11].

Lemma 2.2 *Suppose $g(\theta)$ has a finite differential coefficient $g'(\theta_0)$ for $\theta = \theta_0$. Then*

$$\frac{\delta G}{\delta \theta_0} \to g'(\theta_0)$$

with $r \to 1$.

The next result is a special case of a general theorem proved by J.E. Littlewood and G.H. Hardy in [17].

Lemma 2.3 *Let $f(y)$ be a real or complex valued function of the real variable y, possessing a p^{th} differential coefficient $f^{(p)}(y)$ which is continuous in $(0, y_0]$. Let $\lambda \geq 0$ and that*

$$f(y) = o(y^{-\lambda})$$

whenever $\lambda > 0$ and

$$f(y) = A + o(1)$$

whenever $\lambda = 0$. Also, in either cases that

$$f^{(p)}(y) = O(y^{-p-\lambda}).$$

Then

$$f^{(q)}(y) = o(y^{-q-\lambda})$$

for $0 < q < p$.

Now by setting $e^{-y} = u,\ f(y) = \sum a_n u^n$ and then applying

$$a_0 + a_1 + \cdots + a_n = s_n = 1 + b^{\rho} + b^{2\rho} + \cdots + b^{\nu\rho}$$

for $b^{\nu} \leq n < b^{\nu+1}$, one can easily get:

Lemma 2.4 *Let us suppose $\rho > 0$ and that $f(y) = \sum b^{n\rho} e^{-b^n y}$. Then*

$$f(y) = O(y^{-\rho})$$

as $y \to 0$.

The next result is also not difficult to prove.

Lemma 2.5 *Let*

$$\sin b^n \pi x \to 0$$

as $n \to 0$. Then, $x = \frac{p}{b^q}$ for some integers p and q.

Remark 2.2 It is clear from the above lemma that $\sin b^n \pi x = 0$ for $n \geq q$.

To state the next lemma, one needs the following notations which were introduced by G.H. Hardy and J.E. Littlewood in [18]. The notation $f = \Omega(\phi)$ basically signifies the negation of $f = o(\phi)$, that is, to say as asserting the existence of a constant K such that $|f| > K\phi$ for some special sequence of values whose limit is that to which the variable is supposed to tend. The sequence that one can use to prove the following lemma is the values of y, that is, $y = \frac{\rho}{b^m}$ for $m = 1, 2, 3, \ldots$

Lemma 2.6 *Suppose that*

$$f(y) = \sum b^{n\rho} e^{-b^n y} \sin b^n \pi x,$$

where $y > 0$*, and that*

$$x \neq \frac{p}{b^q}$$

for any integral values of p *and* q*. Then*

$$f(y) = \Omega(y^{-\rho})$$

for all sufficiently large values of ρ*.*

We are now in a position to give outline of the proofs of Theorems 2.1 and 2.2. We give the proof for cosine series and then we provide the outline of the same for the sine series. We begin the proof with the following conditions:

$$ab > 1 \tag{12}$$

and

$$x \neq \frac{p}{b^q}. \tag{13}$$

Let us suppose

$$f(x) = \sum a_n \cos b^n \pi x = \sum a_n \cos b^n \theta = g(\theta)$$

satisfy the condition

$$f(x+h) - f(x) = o(|h|^\xi).$$

That is,

$$g(\theta + h) - f(\theta) = o(|h|^\xi) \tag{14}$$

with

$$\xi = \frac{\log(1/a)}{b} < 1.$$

Then if

$$G(r,\theta) = \sum a_n r^{b^n} \cos b^n\theta = \sum a_n e^{-b^n y} \cos b^n \pi x,$$

we have (using Lemma 2.1),

$$\begin{aligned} F(y) = \frac{\delta_G}{\delta_\theta} &= -\sum (ab)^n e^{-b^n y} sin b^n \pi x \\ &= -\sum b^{(1-\xi)n} e^{-b^n y} sin b^n \pi x \\ &= o(y^{\xi-1}) \end{aligned}$$

when $r \to 1,\ y \to 0$.

Again using Lemma 2.4, one has,

$$\begin{aligned} F^{(p)}(y) &= (-1)^{p+1} \sum (ab^{p+1})^n e^{-b^n y} \sin b^n \pi x \\ &= O\left(\sum b^{(p+1-\xi)n} e^{-b^n y}\right) \\ &= O(y^{\xi-p-1}) \end{aligned}$$

for all positive values of p. It follows from Lemma 2.3 that

$$F^{(q)}(y) = o(y^{\xi-q-l})$$

for $0 < q < p$, and thus for all positive values of q. But this contradicts the assertion of Lemma 2.6, if q is sufficiently large. Hence, the conditions (14) cannot be satisfied. The case in which

$$ab = 1, \quad \xi = 1,$$

may be treated in the same manner. The only difference is that one should use Lemma 2.2 instead of Lemma 2.1, and that the final conclusion is that:
$f(x)$ cannot possess a finite differential coefficient for any value of x which is not of the form $\frac{p}{b^q}$.

This approach though fails in the case when $x = \frac{p}{b^q}$. These values of x need to be treated differently. In this case,

$$\cos\{b^n\pi(x+h)\} = \cos(b^{n-q} p\pi + b^n\pi h) = \pm \cos b^n \pi h$$

for $n > q$. One takes negative sign if both b and p are odd, and positive sign otherwise. Therefore, the properties of the function in the neighbourhood of such a value of x are the same, for the present purpose, as those of the function

$$f(h) = \sum a^n \cos b^n \pi h$$

when $h \to 0$. Thus

$$\begin{aligned} f(h) - f(0) &= -2\sum a^n \sin^2 \frac{1}{2} b^n \pi h \\ &= -2\sum (f_1 + f_2). \end{aligned}$$

Here,

$$f_1 = \sum_0^{\nu} a^n \sin^2 \frac{1}{2} b^n \pi h \quad \text{and} \quad f_2 = \sum_{\nu+1}^{\infty} a^n sin^2 \frac{1}{2} b^n \pi h.$$

We now choose ν in such a way that

$$b^{\nu}|h| \leq b^{\nu+1}|h|. \tag{15}$$

Then

$$\begin{aligned} f_1 + f_2 > f_1 > \sum_0^{\nu} a^n (b^n h)^2 = h^2 \frac{(ab^2)^{\nu+1} - 1}{ab^2 - 1} \\ > K h^2 (ab^2)^{\nu} > K a^{\nu} > K b^{-\xi\nu} > K|h|^{\xi} \end{aligned}$$

where the Ks are constants. Therefore,

$$f(h)) - f(0) \neq o(|h|^{\xi}).$$

This completes the proof when $ab > 1, \ \xi < 1$. In this case, the graph of $f(h)$ has a cusp (pointing upward) for $h = 0$, and that of Weierstrass's function has a cusp for $x = \frac{p}{b^q}$. On the other hand, if $ab = 1, \ \xi = 1$, then it is proved that

$$\overline{\lim}_{h \to 0+} \frac{f(h) - f(0)}{h} < 0,$$

and

$$\underline{\lim}_{h \to 0-} \frac{f(h) - f(0)}{h} > 0,$$

so that $f(h)$ has certainly no finite differential coefficient for $h = 0$, nor the Weierstrass's function has for $x = \frac{p}{b^q}$. This completes the proof of Theorems 2.1 and 2.2 in so far as they relate to the cosine series and are of a negative character. Only part remains is to show that, when $\xi < 1$, Weierstrass's function satisfies the condition

$$f(x + h) - f(x) = O(|h|^{\xi})$$

for all values of x. One starts with the left-hand side:

$$f(x+h)-f(x) = -2\sum a^n \sin\{b^n\pi(x+h)\}\sin\frac{1}{2}b^n\pi h$$
$$= O\left(\sum a^n|\sin\frac{1}{2}b^n\pi h\right).$$

Again choose ν as in (15) and then we have

$$f(x+h)-f(x) = O\left(|h|\sum_0^{\nu} a^n b^n + \sum_{\nu+1}^{\infty} a^n\right)$$
$$= O(a^\nu b^\nu|h| + a^\nu)$$
$$= O(a^\nu)$$
$$= O(|h|^\xi).$$

Hence, the condition is satisfied, and in fact, it holds uniformly in x. It is observed that the above argument fails when $ab = 1,\ \xi = 1$. In this case though one can only say that

$$f(x+h)-f(x) = O(\nu|h|I + a^\nu) = O\left(|h|\log\frac{1}{|h|}\right).$$

It is also observed that the argument of this paragraph applies to the cosine series as well as to the sine series. This is indeed independent of the restriction that b is an integer.

The proof of Theorems 2.1 and 2.2 is now complete so far as the cosine series is concerned. The corresponding proof for the sine series differs only in detail. The subsidiary results required are the same except that Lemma 2.5 is being replaced by the following one.

Lemma 2.7 *If*

$$\cos b^n\pi x \to 0,$$

then b must be odd and

$$x = \frac{p+\frac{1}{2}}{b^q};$$

so that $\cos b^n\pi x = 0$ *from a particular value of n onward. Also, the corresponding changes must be made in Lemma 2.6.*

If the value of x is not exceptional (i.e. one of those as is specified in Lemma 2.7), one can repeat the arguments that were used in the case when (12) and (13) hold. Thus, it is only necessary to discuss the exceptional values, which can exist only when b is odd. In this case, we have,

$$\sin\{b^n\pi(x+h)\} = \sin\left(b^{n-q}p\pi + \frac{1}{2}b^{n-q}\pi + b^n\pi h\right)$$
$$= \pm\sin\left(\frac{1}{2}b^{n-q}\pi + b^n\pi h\right),$$

for $n > q$, the sign has to be fixed as in the case when $x = \frac{p}{b^q}$. The last function is numerically equal to $\cos b^n\pi h$. It always has the same sign as $\cos b^n\pi h$, or always the opposite sign, if $b = 4k+1$. While whenever b is of the form $4k+3$, the corresponding signs agree and differ alternatively. Therefore we are reduced this case to discuss the function

$$f(h) = \sum a^n \cos b^n\pi h$$

near $h = 0$, or to discuss the function

$$f(h) = \sum(-a)^n \cos b^n\pi h.$$

The need is to show that

$$f(h) - f(0) \neq o(|h|^\xi)$$

if $\xi < 1$, and that $f(h)$ has no finite differential coefficient for $h = 0$, if $\xi = 1$.

To do this, let us consider the special sequence of values

$$h = \frac{2}{b^\nu} \qquad (\nu = 1,\ 2,\ 3, \ldots).$$

Then, we have

$$f(h) - f(0) = -2\sum_0^{\nu-1}(-a)^n \sin^2\frac{1}{2}b^n\pi h$$
$$= (-1)^\nu 2a^{\nu-1}\sum_1^\nu\left(-\frac{1}{a}\right)^n \sin^2\frac{\pi}{b^n}.$$

Now

$$\sum_1^\nu\left(-\frac{1}{a}\right)^n \sin^2\frac{\pi}{b^n} \to \sum_1^\infty\left(-\frac{1}{a}\right)^n \sin^2\frac{\pi}{b^n} = S \text{ (say)}.$$

Now as S is the sum of an alternating series of decreasing terms, it is positive. Again, we have

$$a^\nu = b^{-\xi\nu} = \left(\frac{1}{2}h\right)^\xi.$$

Thus

$$|f(h) - f(0)| > ch^\xi,$$

for some constant c and is alternately positive and negative. This completes the proof of Theorems 2.1 and 2.2.

Now the time is to come back to Remark 2.1, that is, the question remains whether an equally comprehensive result holds for infinite differential coefficients. The result that includes the Remark 2.1 shows that the answer to this question is negative.

Theorem 2.3 *If*

$$ab \geq 1 \ \ and \ \ a(b+a) < 2$$

then the sine series has the differential coefficient $+\infty$ *for* $x = 0$. *If* $b = 4k + 1$ *and* $x = \frac{1}{2}$, *then the same is true of the cosine series.*

It is enough to prove the first statement. The second one then follows by the transformation $x = \frac{1}{2} + y$.

We have,

$$\begin{aligned}\frac{f(h) - f(0)}{h} &= \frac{1}{h}\sum a^n \sin b^n \pi h \\ &= \frac{1}{h}\sum_{0}^{\nu-1} a^n \sin b^n \pi h + \frac{1}{h}\sum_{\nu}^{\infty} a^n \sin b^n \pi h \\ &= f_1 + f_2 \text{ (say).}\end{aligned}$$

Here, ν has to be chosen so that

$$b^{\nu-1}|h| \leq \frac{1}{2} < b^{\nu}|h|. \tag{16}$$

We first suppose that $ab > 1$ and then,

$$f_1 > 2\sum_{0}^{\nu-1}(ab)^n = 2\frac{(ab)^{\nu} - 1}{ab - 1} \tag{17}$$

and

$$|f_2| < \frac{1}{|h|}\sum_{\nu}^{\infty} a^n = \frac{a^{\nu}}{(1-a)|h|}. \tag{18}$$

Now it is clear that

$$a(b+1) < 2, \qquad 1 - a > ab - 1$$

and thus

$$\frac{1-a}{ab-1} = 1 + \delta \tag{19}$$

where $\delta > 0$. Without loss of generality, one can assume h is so small (or ν is so large) so that

$$\frac{(ab)^{\nu}-1}{(ab)^{\nu}} > \frac{2+\delta}{2(1+\delta)}. \tag{20}$$

Then, from (16)–(20), it follows that

$$\frac{|f_2|}{f_1} < \frac{(ab)^{\nu}(ab-1)}{\{(ab)^{\nu}-1\}(1-a)} < \frac{1}{1+\frac{1}{2}+\delta}.$$

Thus

$$f_1 + f_2 > c_1 f_1 \text{ or } c_2(ab)^{\nu},$$

for some constants c_i, $i = 1, 2$. Thus, we have

$$\frac{f(h)-f(0)}{h} \to +\infty \tag{21}$$

as $h \to 0$.

Next, if $ab = 1$, then $|f_2| < k$, a constant, and that

$$f_1 > 2\nu \to +\infty.$$

Hence, (21) remains true in this case too.

A number $\alpha(b)$ exists when b is given, and it is simply the least number such that the condition

$$ab > \alpha(b).$$

This debars the existence of a differential coefficient whether finite or infinite. At present, all that one can say about $\alpha(b)$is that

$$\frac{2}{b+1} \leq \alpha(b) \leq \frac{1+\frac{3}{2}\pi}{b+\frac{3}{2}\pi}.$$

2.2 *b Is Not an Integer*

One needs to discuss everything those are stated in previous subsection with b is no more an integer and thus the series are no longer Fourier series, and one can no longer have the luxury to employ Poisson's integral associated with $G(r, \theta)$.

The job is naturally to construct a new formula to replace the Poisson's one. Once it is has been achieved, further modifications of the argument are needed. This is because of the lack of any simple result corresponding to Lemmas 2.5 and 2.7, and the difficulty of determining precisely the exceptional values of x for which $\sin b^n \pi x \to 0$ or $\cos b^n \pi x \to 0$. The beauty of the argument is that, however, it will be found that no fundamental change in the method is necessary. Also that the additional analysis required is not complicated.

Let b is any number greater than 1. Also

$$s = \sigma + it,$$

(usual in the theory of Dirichlet series), and that

$$f(s) = \sum_{1}^{\infty} a^n e^{-b^n s} = G(\sigma, t) + iH(\sigma, t), \qquad (\sigma \geq 0)$$

with the condition that

$$G(0, t) = g(t).$$

Then, one can show that:

Lemma 2.8 *Let* $\sigma > 0$*. Then*

$$G(\sigma, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma g(u)}{\sigma^2 + (u-t)^2} du.$$

First, let us set $ab > 1$. In this case, one uses the following ones instead of Lemma 2.1:

Lemma 2.9 *If*

$$g(t) - g(t_0) = o(|t - t_0|^{\alpha}),$$

where $0 < \alpha < 1$*, when* $t \to t_0$*. Then*

$$\frac{\delta G(\sigma, t_0)}{\delta t_0} = o(\sigma^{\alpha - 1}),$$

whenever $\sigma \to 0$.

Lemma 2.10

$$\frac{\delta G(\sigma, t_0)}{\delta \sigma} = o(\sigma^{\alpha - 1})$$

under the same conditions as in the previous Lemma.

The proofs of these lemmas are very similar, and the first is similar to that of Lemma 2.1. One can consult [16] for detail of the proofs.

We now discuss the exceptional values of t. Suppose that

$$g(t + h) - g(t) = o(|h|^{\xi}).$$

Then, by Lemmas 2.9 and 2.10, we have

$$\frac{\delta G}{\delta t} = -\sum (ab)^n e^{-b^n\sigma} \sin b^n t = o(\sigma^{\xi-1})$$

and

$$\frac{\delta G}{\delta \sigma} = -\sum (ab)^n e^{-b^n\sigma} \cos b^n t = o(\sigma^{\xi-1}).$$

Therefore, we have

$$f(y) = \sum (ab)^n e^{-b^n(\sigma+it)} = o(\sigma^{\xi-1}).$$

One can now obtain a contradiction by employing the same argument as used earlier when we consider (12) and (13). It is only necessary to observe that Lemma 2.3 holds for complex as well as for real functions of a real variable. Also instead of Lemma 2.7, one has to use the following proposition.

Proposition 2.1 *If*

$$f(y) = \sum b^{n\rho} e^{-b^n(\sigma+it)}, \qquad (\sigma > 0),$$

then

$$f(y) = \Omega(\sigma^{-\rho})$$

for all sufficiently large values of ρ.

There is no longer any question of exceptional values of t as $|e^{-b^n it}| = 1$.

Next, we treat when $ab = 1$. In this case instead of Lemma 2.9, one uses the following result (which corresponds to Lemma 2.2).

Lemma 2.11 *Let* $g(t)$ *possesses a finite differential coefficient* $g'(t_0)$ *for* $t = t_0$. *Then*

$$\frac{\delta G(\sigma, t_0)}{\delta t_0} \to g'(t_0)$$

when $\sigma \to 0$.

The proof of this lemma is no more difficult. One needs to keep in mind though that it is not necessarily true that

$$\frac{\delta G(\sigma, t_0)}{\delta t_0}$$

tends to a limit. Thus, it is necessary to follow a slightly different argument from that of when we treated the exceptional values of t.

Lemma 2.12 *Under the same conditions as those of Lemma 2.11, we have*

$$\frac{\delta^2 G(\sigma, t_0)}{\delta t_0^2} = o\left(\frac{1}{\sigma}\right).$$

Suppose that $g(t)$ has a finite differential coefficient $g'(t)$, and write

$$f(\sigma) = \frac{\delta G}{\delta t} = -\sum e^{-b^n\sigma} \sin b^n t.$$

Then, by Lemma 2.11, we have

$$f(\sigma) = g'(t) + o(1)$$

when $\sigma \to 0$. But by Lemma 2.4, we have,

$$f''(\sigma) = -\sum b^{2n} e^{-b^n\sigma} sinb^n t = O\left(\frac{1}{\sigma^2}\right).$$

Therefore, by Lemma 2.3,

$$f'(\sigma) = \sum b^n e^{-b^n\sigma} \sin b^n t = o\left(\frac{1}{\sigma}\right). \tag{22}$$

On the other hand, by Lemma 2.12, we have

$$\frac{\delta^2 G}{\delta t^2} = -\sum b^n e^{-b^n\sigma} cosb^n t = o\left(\frac{1}{\sigma}\right). \tag{23}$$

Now from (22) and (23), it follows that

$$F(\sigma) = \sum b^n e^{-b^n(\sigma+it)} = o\left(\frac{1}{\sigma}\right).$$

Also, by Lemma 2.4, we have

$$F^{(p)}(\sigma) = (-1)^p \sum b^{(p+1)n} e^{-b^n(\sigma+it)} = o\left(\frac{1}{\sigma^{p+1}}\right),$$

for all values of p. Thus, it follows that the O can be replaced by o, and this leads to a contradiction as before.

Finally, the following remark completes the proof.

Remark 2.3 The above argument has been stated in terms of Weierstrass's cosine series. The same arguments apply to the sine series, as there are now no "exceptional values". It was only the existence of such values which differentiated the two cases in second subsection. The positive statement in Theorem 2.2 has already been proved, applying to all values of b.

3 Some More Non-differentiable Functions

In this section, we discuss some more non-differentiable functions which are available in [16].

3.1 A Function Which Doesn't Satisfy a Lipschitz Condition of Any Order

It is interesting to give an example of an absolutely convergent Fourier series whose sum does not satisfy any condition of the following type:

$$f(x+h) - f(x) = O(|h|^{\alpha}), \qquad (\alpha > 0)$$

for any value of x. An interesting example of such a function is

$$f(x) = \sum \frac{\cos b^n \pi x}{n^2}.$$

It is in fact easy to prove, by the methods used in the previous section, that

$$f(x+h) - f(x) \neq o\left(\frac{1}{|\log|h||}\right)^2.$$

However, a somewhat less simpler example may be found by simply combining remarks made by G. Faber and G. Landsberg. In [10], G. Faber defined

$$F(x) = \sum 10^{-n} \phi(2^{n!x}), \tag{24}$$

where

$$\phi(x) = \begin{cases} x, & for 0 \leq x \leq 1/2; \\ 1-x, & for 1/2 \leq x \leq 1. \end{cases}$$

He showed that

$$F(x+h) - F(x) \neq O\left(\frac{1}{|\log|h||}\right).$$

On the other hand, G. Landsberg [25] used the expansion of a function, which is in a Fourier series equivalent to $\phi(x)$. In fact,

$$\phi(x) = \frac{1}{4} - \frac{2}{\pi^2} \sum \frac{\cos 2\nu\pi x}{\nu^2} \qquad (\nu = 1, 3, 5, \cdots).$$

If we substitute this expansion in (24), we obtain an expansion of $F(x)$ as an absolutely convergent Fourier series, and thus is an example of the kind we are looking for.

3.2 A Theorem of S. Bernstein

It is natural to suggest the following theorem of S. Bernstein [1] in this connection. This can be proved similarly as is being done in the previous section.

Theorem 3.1 *If $f(x)$ satisfies a Lipschitz condition of order α (> 2) in $(0,\ 1)$, i.e. if*

$$|f(x+h) - f(x)| < K|h|,$$

where K is an absolute constant. Then, the Fourier series of $f(x)$ is absolutely convergent. Also, $\frac{1}{2}$ is the least number which has this property.

Proof We assume that $2\pi x = \theta$ and that

$$f(x) = g(\theta) = \frac{1}{2}a_0 + \sum(a_n \cos n\theta + b_n \sin n\theta).$$

Also, let

$$G(r, \theta) = \frac{1}{2}a_0 + \sum r^n(a_n \cos n\theta + b_n \sin n\theta) \text{ if } < 1,$$

and

$$G(1, \theta) = g(\theta).$$

Then, $G(r, \theta)$ is continuous for

$$0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

It follows from a simple modification (i.e. O in place of o) of Lemma 2.1, that

$$\frac{\delta G}{\delta \theta} = -\sum nr^{n-1}(a_n \sin n\theta - b_n \cos n\theta) = O\{(1-r)^{\alpha-1}\},$$

uniformly in θ. Squaring, and integrating from $\theta = 0$ to $\theta = 2\pi$, one can obtain

$$\sum n^2 r^{2n}(|a_n|^2 + |b_n|^2) = O(1-r)^{2\alpha-1}.$$

Hence, by putting $r = 1 - (1/\nu)$, one can obtain

$$\sum_1^{\nu} n^2(|a_n|^2 + |b_n|^2) = O(\nu^{2-2\alpha}),$$

and so, by Schwarz's inequality,

$$\sum_{1}^{\nu} n(|a_n| + |b_n|) = O(\nu^{\frac{3}{2}\alpha}).$$

Thus, it is easy to deduce that the series

$$\sum_{1}^{\nu} n^{\beta}(|a_n| + |b_n|)$$

is convergent if $\beta < \alpha - \frac{1}{2}$.

This establishes the first part of Bernstein's Theorem (indeed more!). The second part is shown by the following example:

$$g(\theta) = \sum n^{-b} \cos(n^a + n\theta),$$

where $0 < a < 1,\ 0 < b < 1$. In this case, $G(r, \theta)$ is the real part of

$$F(z) = F(re^{i\theta}) = \sum n^{-b} e^{in^a} z^n.$$

This function is continuous (see [15]) for $|z| \geq 1$ if

$$\frac{1}{2}a + b > 1;$$

and it is not difficult to go further, and to show that $g(\theta)$ satisfies a Lipschitz condition of order $\frac{1}{2}a + b - 1$.

Now let α be any number less than $\frac{1}{2}$. Then, one can choose numbers a and b, each less than 1 in such a way that

$$\frac{1}{2}a + b - 1 > \alpha.$$

Then, the function $g(\theta)$ satisfies a Lipschitz condition of order greater than α, but its Fourier series is not absolutely convergent. □

4 Riemann's Non-differentiable Function Revisited

Riemann is reported to have stated [8, 16], but never proved, that the continuous function,

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$$

is nowhere differentiable. In 1872, K. Weierstrass [34] tried to prove this assertion but could not and instead constructed his own example of a continuous nowhere differentiable function which is $a^n \cos b^n \pi x$ along with the conditions (2) and (3). J.P. Kahane [23] renewed the interest in this classical problem in connection with lacunary series, and refers to K. Weierstrass [34].

Riemann's assertion was partially confirmed by G.H. Hardy [16], who proved that the above function $f(x)$ has no finite derivative at any point $\xi\pi$, where ξ is

(i) irrational;
(ii) rational of the form $\frac{2A}{4B+1}$, where A and B are integers;
(iii) rational of the form $\frac{2A+1}{2(2B+1)}$.

We provide an outline of Hardy's method in this case. Suppose that Riemann's function is differentiable for certain values of x, then by Lemma 2.2,

$$\sum r^{n^2} \cos n^2 \pi x = A + o(1),$$

where A is a constant, as $r \to 1$. However,

$$\sum r^{n^2} \cos n^2 \pi x = \Omega\{(1-r)^{-\frac{1}{4}}\}$$

if x is irrational, and

$$\sum r^{n^2} \cos n^2 \pi x = \Omega\{(1-r)^{-\frac{1}{2}}\}$$

if x is a rational of the form $\frac{2\lambda+1}{2\mu}$ or $\frac{2\lambda}{4\mu+1}$. Therefore, Riemann's function is certainly not differentiable for any irrational (and some rational) values of x. It is easy, by using Lemma 2.1, instead of Lemma 2.2, to show that Riemann's function cannot satisfy the condition

$$f(x+h) - f(x) = o(|h|^{\frac{3}{4}})$$

for any irrational values of x. In this context, Hardy [16] proved the following theorem:

Theorem 4.1 *None of the functions*

$$f_{c,\alpha}(x) = \sum \frac{\cos n^2 \pi x}{n^\alpha}$$

and

$$f_{s,\alpha}(x) = \sum \frac{\sin n^2 \pi x}{n^\alpha}$$

where $\alpha < \frac{5}{2}$, is differentiable for any irrational value of x.

Proof Suppose that $f_{s,\alpha}$ is differentiable, and consequently, Lemma 2.12 would imply,

$$\sum n^{2-\alpha} r^{n^2} \cos n^2\pi x = A + o(1),$$

or

$$f(y) = \sum n^{2-\alpha} e^{-n^2 y} \cos n^2\pi x = A + o(1).$$

But

$$\begin{aligned} f^{(p)}(y) &= (-1)^p \sum n^{2p+2-\alpha} e^{-n^2 y} \cos n^2\pi x \\ &= O\big(\sum n^{2p+2-\alpha} e^{-n^2 y}\big) = O(y^{-p-\frac{3}{2}+\frac{\alpha}{2}}). \end{aligned}$$

Hence, by the theorem of Hardy and Littlewood [18], we have

$$f^{(q)}(y) = o\left(y^{-\frac{q}{p}(p+\frac{3}{2}-\frac{\alpha}{2})}\right).$$

Here, $0 < q < p$, and in particular

$$f'(y) = o\left(y^{-1-\frac{3}{2p}+\frac{\alpha}{2p}}\right). \tag{25}$$

Again, it is easy to prove that

$$f'(y) = -\sum n^{4-\alpha} e^{-n^y} \cos 2\pi x = \Omega(y^{-\frac{9}{4}+\frac{\alpha}{2}}). \tag{26}$$

From (25) and (26), it follows that

$$1 + \frac{3}{2p} - \frac{\alpha}{2p} > \frac{9}{4} - \frac{\alpha}{2}.$$

But this is not possible if $\alpha < \frac{5}{2}$ and p is sufficiently large. It is clear that the series $f_{c,\beta}$ and $f_{s,\beta}$ with $0 < \beta < \frac{1}{2}$ are not Fourier's series. For if the first one is a Fourier's series, then the sum of the integrated series $f_{s,2+\beta}$ would be a function of limited total fluctuation, and would therefore be differentiable almost everywhere.

It is easy to prove directly that the function $f_{s,\alpha}$, where $2 < \alpha < \frac{5}{2}$, has the differential coefficient $+\infty$ for $x = 0$. A similar direct method could no doubt be applied to an everywhere dense set of rational values of x. □

In 1970, J. Gerver [12] proved that Riemann's assertion is false, by proving the following result.

Theorem 4.2 *The derivative of the following function*

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$$

exists and is equal to $-\frac{1}{2}$ *at any point* $\frac{(2A+1)\pi}{2B+1}$, *where* A *and* B *are integers.*

In the same paper, J. Gerver [12] extended G.H. Hardy's results [16] by proving the following:

Theorem 4.3 *The derivative of the Riemann functions does not exist at any point* $\frac{(2A+1)\pi}{2^N}$*, where* N *is an integer* ≥ 1 *and* A *is any integer.*

One can consult [12] for detailed proof of Theorems 4.2 and 4.3.

In 1971, J. Gerver further proved some results concerning the non-differentiability of Riemann's function. More precisely, he proved the following:

Theorem 4.4 *The function*

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$$

is not differentiable at any point $\frac{2A\pi}{2B+1}$ *or* $\frac{\pi(2A+1)}{2B}$*, where* A *and* B *are integers.*

This result together with Hardy's result [16] that the function is not differentiable at any irrational multiple of π, completely solves the problem of differentiability.

In 1972, A. Smith [31] extended the above results to the remaining cases. He also discussed the existence of finite left-hand and right-hand derivatives at certain rationals, and proved that these derivatives exist at all rationals if the values $\pm\infty$ were allowed. He gave completely elementary and fairly short proof of all the above assertions. J. Gerver's proof was extremely long, and G.H. Hardy obtained his results indirectly. A. Smith worked with the following function

$$g(x) = x + 2\sum_{n=1}^{\infty} \frac{\sin n^2 \pi x}{\pi n^2},$$

so that one can verify that $g'(x)$ exists and is zero whenever x is of the form $\frac{2A+1}{2B+1}$ for some integers A and B.

The following lemmas are required to obtain expansions for $g(x)$ about a rational point x, which using properties of Gaussian sums reveal the properties of the derivatives.

Lemma 4.1 *Let* ϕ *be a continuous function in* $L_1(-\infty, \infty)$*. Suppose that the series for* $Q(\alpha)$ *(defined below) converges uniformly in every finite* α *interval, for each fixed* $h > 0$*. Let*

$$\hat{\phi}(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} \phi(x) dx$$

and assume that $|y|^\beta |\hat{\phi}(y)|$ *is bounded for some fixed* $\beta > 1$*. Then, for any real constant* α*, as* $h \to 0+$*,*

$$Q(\alpha) = \sum_{k=-\infty}^{\infty} h\phi(hk + h\alpha) = \hat{\phi}(0) + O(h^\beta).$$

Proof The conditions on ϕ allow one to apply the Poisson summation formula to

$$\sum_{k=-\infty}^{\infty} h\phi(hk + h\alpha)$$

to obtain

$$\sum_{k=-\infty}^{\infty} h\phi(hk + h\alpha) = \sum_{k=-\infty}^{\infty} e^{2\pi i k\alpha}\hat{\phi}\left(\frac{k}{n}\right)$$

provided this series converges absolutely. The condition on $\hat{\phi}$ gives, for $k \neq 0$,

$$e^{2\pi i k\alpha}\hat{\phi}\left(\frac{k}{n}\right) = O\left(\frac{h^{\beta}}{|k|^{\beta}}\right)$$

which shows that the above sum, leaving out the $k = 0$ term, converges absolutely and is $O(h^{\beta})$. Thus

$$\sum_{k=-\infty}^{\infty} e^{2\pi i k\alpha}\hat{\phi}\left(\frac{k}{n}\right) = \hat{\phi}(0) + O(h^{\beta}).$$

□

Lemma 4.2 *Let*

$$\phi_1(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

$$\phi_2(x) = \begin{cases} \frac{1-\cos \pi x}{\pi x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then, Lemma 4.1 with $\beta = 2$ applies to the functions $\psi_i(x) = \psi_i(x^2),\ i = 1, 2$, and

$$\sum_{k=-\infty}^{\infty} h\psi(hk + h\alpha) = 2^{1/2} + O(h^2),\ i = 1, 2.$$

The following lemma is straightforward.

Lemma 4.3 *Assume that $x = \frac{r}{s}$ and that $(r, s) = 1$. Let us define*

$$G(x) = \sum_{t=0}^{s-1} e^{i\pi t^2 x} = C(x) + iS(x) \equiv \sum_{t=0}^{s-1} \cos \pi t^2 x + i \sum_{t=0}^{s-1} \sin \pi t^2 x;$$

then
(a) when $r \equiv 0 \pmod 2$,

$$G(x) = \begin{cases} (\frac{r}{2s})s^{1/2} = 1, & s \equiv 1 \pmod 4, \\ i(\frac{r}{2s})s^{1/2} = i, & s \equiv 3 \pmod 4; \end{cases}$$

(b) when $s \equiv 0 \pmod 2$,

$$G(x) = \begin{cases} (\frac{r}{2s})\sqrt{\frac{s}{2}}(1+i), & r \equiv 1 \pmod 4, \\ (\frac{r}{2s})\sqrt{\frac{s}{2}}(1-i), & r \equiv 3 \pmod 4; \end{cases}$$

where $(\frac{a}{b})$ *denotes the Jacobi symbol;*
(c) when $rs = 0 \pmod 2$,

$$|G(x)| = s^{1/2}.$$

We are now in a position to discuss the derivative of $g(x)$ at rational and at some other points.

4.1 The Derivative at Rational Points

We begin with the following assumptions:

$$x = \frac{r}{s}, \ (r, \ s) = 1, \ rs \equiv 0 \pmod 2.$$

We have

$$\begin{aligned} g(x+h^2) + g(x-h^2) &= 2x + 4\sum_{n=1}^{\infty} \frac{\sin \pi n^2 x}{\pi n^2} \cos \pi n^2 h^2 \\ &= 2g(x) - 2h^2 \sum_{n=-\infty}^{\infty} \sin \pi n^2 x \psi_2(ph). \end{aligned}$$

Let us write $n = ks + t$ with $0 \le t \le s-1$. Note that $\sin \pi(ks+t)^2 x = \sin \pi t^2 x$, since $rs \equiv 0 \pmod 2$. Then

$$\begin{aligned} g(x+h^2) + g(x-h^2) &= 2g(x) - 2h^2 \sum_{t=0}^{s-1} \sum_{k=-\infty}^{\infty} \sin \pi t^2 x \psi_2(khs + ht) \\ &= 2g(x) - 2\frac{h}{s} \sum_{t=0}^{s-1} \sin \pi t^2 x \{2^{1/2} + O(h^2)\} \end{aligned}$$

$$= 2g(x) - 2^{3/2}S(x)\frac{h}{s} + O(h^3).$$

Note that Lemma 4.2 is used in the penultimate line.

Similarly, we have

$$g(x + h^2) - g(x - h^2) = 2^{3/2}C(x)\frac{h}{s} + O(h^3).$$

Adding and subtracting these two equations, we obtain

$$g(x \pm h^2) = g(x) - 2^{1/2}\{S(x) \mp C(x)\}\frac{h}{s} + O(h^3) \tag{27}$$

We now assume that $rs \equiv 1 \pmod 2$. One can easily verify the relation

$$g(x) = 1 + \frac{1}{2}g(4x) - g(x+1)$$

which is then used in (27) to deduce that

$$g(x \pm h^2) = g(x) - 2^{1/2}\{S(4x) - S(x+1) \mp [C(4x) - C(x+1)]\}\frac{h}{s} + O(h^3). \tag{28}$$

The properties of Jacobi symbols provide

$$\left(\frac{2r}{s}\right) = \left(\frac{2r+2s}{s}\right) = \left(\frac{4((r+s)/2)}{s}\right) = \left(\frac{(r+s)/2}{s}\right),$$

since 4 is the square of the prime 2 and $s \equiv 1 \pmod 2$. This immediately simplifies (28) to

$$g(x \pm h^2) = g(x) + O(h^3).$$

Thus, when $r \equiv s \equiv 1 \pmod 2$, we see that $g'(x)$ exists and is 0, since the right-hand derivative

$$g'_+(x) = \lim_{h^2 \to \infty} \frac{g(x+h^2) - g(x)}{h^2}$$

and the left-hand derivative

$$g'_-(x) = \lim_{h^2 \to \infty} \frac{g(x) - g(x-h^2)}{h^2}$$

both exist and are 0. In this case, it follows that the symmetric derivative

$$g'_0(x) = \lim_{h^2 \to \infty} \frac{g(x+h^2) - g(x-h^2)}{2h^2}$$

also exists and is 0. When $rs \equiv 0 \pmod 2$, the relation (27) shows that $g'(x)$ is finite if and only if $G(x) = 0$. However, by Lemma 4.3, $G(x)$ is not 0. Hence, $g'(x)$ is not finite when rs even. One can easily verify that $g_+(x)$ when $r \equiv 1 \pmod 4$, $g'_-(x)$ when $r \equiv 3 \pmod 4$, and $g_0(x)$ when $s \equiv 3 \pmod 4$ are all 0, but in other cases, these derivatives are infinite.

4.2 Derivatives at Other Points

At negative rationals, the results of the preceding section carry over, since g is an odd function.

We now assume that x is irrational, which without loss generality we take to be positive. Let $\{q_k\}$ be a strictly increasing sequence of positive integers, and let p_k be the least integer such that $x_k = \frac{2p_k}{4q_k+1} > x$. Then, $x_k - x < \frac{2}{4q_k+1}$ and $x_k \to x$ as $k \to \infty$. From (27) and condition (a) of Lemma 4.3, we have

$$\left|\frac{g(x)-g(x_k)}{x-x_k}\right| = \left\{\frac{1}{2}(4q_k+1)(x_k-x)\right\}^{-1/2} + O((x_k-x)^{1/2}).$$

Therefore,

$$\lim_{k\to\infty} \inf \left|\frac{g(x)-g(x_k)}{x-x_k}\right| \geq 1.$$

Let $y_k = x_k + \frac{1}{4q_k+1} = \frac{2p_k+1}{4q_k+1}$. Then, $y_k \to x$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{g(x)-g(y_k)}{x-y_k} = 0.$$

From these two equations, we obtain Hardy's result that g does not have a finite or infinite derivative at the irrational point x.

In 1981, S. Itatsu [21] gave a short proof of the differentiability as well as a finer estimate of the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$$

at points of rational multiple of π. Namely, he proved the following result.

Theorem 4.5 *The function*

$$F(x) = \sum_{n=1}^{\infty} \frac{e^{in^2\pi x}}{in^2\pi}$$

have the following behaviour near $x = \frac{q}{p}$, *where* p *is a positive integer and* q *is an integer such that* $\frac{q}{p}$ *is an irreducible fraction,*

$$F(x+h) - F(x) = R(p,q)p^{-1/2}e^{\frac{i\pi}{4}sgn\ h}|h|^{1/2}sgn\ h - \frac{h}{2} + O(|h|^{1/2})$$

as $h \to 0$ *where* $sgn\ h = \frac{h}{|h|}$ *if* $h \neq 0$*,* $sgn\ h = 0$ *if* $h = 0$*, and* $R(p,q)$ *is a constant defined by*

$$R(p,q) = \begin{cases} (\frac{q}{p})e^{\frac{-\pi i}{4}(p-1)}, & \text{if } p \text{ is odd and } q \text{ even,} \\ (\frac{p}{|q|})e^{\frac{\pi i}{4}q}, & \text{if } p \text{ is even and } q \text{ odd,} \\ 0, & \text{if } p \text{ and } q \text{ are odd,} \end{cases}$$

with the Jacobi's symbol $(\frac{p}{q})$*.*

5 Quadratic Reciprocity and Riemann's Function

Here, we discuss the recent work of Chakraborty et al. [5] who gave a combined proof of both, that is, the quadratic reciprocity law and the differentiability/non-differentiability of Riemann's function.

Let p be a natural number and $\mathfrak{z} = h + i\epsilon \in \mathcal{H}$ tending to 0. We denote the upper half-plane by $\mathcal{H}$. Also, let for $z \in \mathcal{H} \cup \mathbb{R}$,

$$F(z) = \sum_{n=1}^{\infty} \frac{e^{\pi i n^2 z}}{\pi i n^2} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{\pi i n^2 z}}{\pi i n^2}.$$

Let us denote by $S(b,a)$ the quadratic Gauss sum defined by

$$S(b,a) = \sum_{j=0}^{b-1} e^{2\pi i j^2 \frac{a}{b}}$$

for a natural number b. One extends the definition for nonzero integral values b by,

$$S(b,a) = S(|b|, \operatorname{sgn}(b)a).$$

We note that $S(|b|, -a) = \overline{S(|b|, a)}$ and $S(ka, kb) = S(a, b)$.

We begin with the following result:

Theorem 5.1 *For any integers* $p > 0$*,* q*, we have*

$$F\left(\frac{2q}{p} + \mathfrak{z}\right) - F\left(\frac{2q}{p} + i\epsilon\right) = S(p,q)\frac{e^{-\pi i/4}}{p}\sqrt{\mathfrak{z}} - \frac{1}{2}h + O(\mathfrak{z}^2) \tag{29}$$

where for a nonzero integer p*, the coefficient is to be understood as* $S(|p|, \operatorname{sgn}(p)q)$*.*

Proof Let b be an arbitrary real number. One can obtain by using Euler–Maclaurin summation formula as in Lemma 4 in [28] (the resulting integral can be evaluated as in [24] (pp. 20–22)):

$$\sum_{n=-\infty}^{\infty} e^{(b+pn)^2 i\mathfrak{z}} = \frac{2\sqrt{\pi}}{p} e^{-\pi i/4}\sqrt{\mathfrak{z}} + O(\mathfrak{z}) \tag{30}$$

where the branch of $\sqrt{\mathfrak{z}}$ is chosen so that it is positive for $\mathfrak{z} > 0$.

We integrate this along the line segment parallel to the real axis, say over $[\mathfrak{z}', \mathfrak{z}]$ with $\mathfrak{z} - \mathfrak{z}' = h$. Now after separating the case $(b, n) = (0, 0)$, the integrated form of (30) becomes,

$$\begin{aligned} h + \sum_{\substack{n=-\infty \\ (n,b)\neq(0,0)}}^{\infty} \frac{e^{(b+pn)^2 i\mathfrak{z}}}{i(b+pn)^2} - \sum_{\substack{n=-\infty \\ (n,b)\neq(0,0)}}^{\infty} \frac{e^{(b+pn)^2 i(h'+i\epsilon)}}{i(b+pn)^2} \\ = \frac{2\sqrt{\pi}}{p} e^{-\pi i/4}\sqrt{\mathfrak{z}} + O(\mathfrak{z}^2). \end{aligned} \tag{31}$$

This can be rewritten as

$$T(\mathfrak{z}) - T(i\epsilon) = \frac{2\sqrt{\pi}}{p} e^{-\pi i/4}\sqrt{\mathfrak{z}} - h(1 + o(1)) + O(\mathfrak{z}^2). \tag{32}$$

Here

$$T(\mathfrak{z}) = T(\mathfrak{z}, b) = \sum_{\substack{n=-\infty \\ (n,b)\neq(0,0)}}^{\infty} \frac{e^{(b+pn)^2 i\mathfrak{z}}}{i(b+pn)^2}.$$

Then, by the decomposition into residue classes,

$$\begin{aligned} F\left(\frac{2q}{p} + \mathfrak{z}\right) &= \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{\pi i n^2\left(\frac{2q}{p}+\mathfrak{z}\right)}}{\pi i n^2} \\ &= \frac{1}{2} \sum_{b=0}^{p-1} e^{2\pi i b^2 \frac{q}{p}} \sum_{\substack{n\equiv b \pmod q \\ (n,b)\neq(0,0)}} \frac{e^{\pi i n^2 \mathfrak{z}}}{\pi i n^2} \\ &= \frac{1}{2} \sum_{b=0}^{p-1} e^{2\pi i b^2 \frac{q}{p}} \frac{1}{\pi} T(\pi\mathfrak{z}, b). \end{aligned}$$

Now using (32),

$$F\left(\frac{2q}{p}+\mathfrak{z}\right)=\frac{1}{2}\sum_{b=0}^{p-1}e^{2\pi i b^2\frac{q}{p}}\frac{1}{\pi}\left(\frac{2\sqrt{\pi}}{p}e^{-\pi i/4}\sqrt{\pi\mathfrak{z}}-\frac{1}{2}h+T(i\epsilon,b)\right)+O(\mathfrak{z}^2)$$

$$=\frac{1}{2}S(p,q)\left(\frac{2\sqrt{\pi}}{p}e^{-\pi i/4}\sqrt{\mathfrak{z}}-\mathfrak{z}\right)+\frac{1}{2\pi}\sum_{b=0}^{p-1}e^{-2\pi i b^2\frac{q}{p}}T(i\epsilon,b)+O(\mathfrak{z}^2)$$

$$=S(p,q)\left(\frac{1}{p}e^{-\pi i/4}\sqrt{\mathfrak{z}}-\frac{1}{2}h\right)+F\left(\frac{2q}{p}+i\epsilon\right)+O(\mathfrak{z}^2).$$

□

In (31), the variable can be $\frac{2q}{p}+\mathfrak{z}$ and $\frac{2q}{p}+\mathfrak{z}\prime$, and then instead of h, we would have $\mathfrak{z}-\mathfrak{z}\prime$. This will be used in deriving (Theorem 5.3).

The relation (29) in this form is essentially Theorem 1 of S. Itatsu [21], and from here, non-differentiability of Riemann's function can be deduced. Indeed, let $\mathfrak{z}=h+i\epsilon$ and let $\epsilon\to 0+$, in which we have to pay attention to the sign $\operatorname{sgn} h$ of h. Then

$$F\left(\frac{2q}{p}+h\right)-F\left(\frac{2q}{p}\right)=S(p,q)\frac{e^{-\pi i/4\,\operatorname{sgn} h}}{p}\sqrt{|h|}-\frac{1}{2}h+O(h^2). \tag{33}$$

Hence, differentiability follows only in the case $S(p,q)=0$ with differential coefficient $-\frac{1}{2}$. This will be done in the next section appealing to Corollary 1. At the same time, this is an elaboration of [28, (47)] (on the right-hand side of which the factor $\sqrt{\pi}$ is to be deleted). Arguing as in [28] using the theta transformation formula, we may deduce the Landsberg–Schaar identity, from which the quadratic reciprocity may be deduced.

Remark 5.1 We would like to make a few comments on the work of J.J. Duistermaat [9]. In [9, p. 4, $\ell\ell$. 1–2], J.J. Duistermaat says that "this self-similarity formula was just an integrated version of the well-known transformation formula (35)". By this, [9, Theorem 4.2] is meant. The Eq. (3.4) (was already proved by Cauchy [4, pp. 157–159]) [9] for $r=\frac{q}{p}$ becomes

$$\begin{aligned}\mu_\gamma(x)&=e^{\frac{\pi}{4}m}p^{-\frac{1}{2}}(x-r)^{-\frac{1}{2}}\\&=e^{\frac{\pi}{4}}p^{-1}S(2p,q)(x-r)^{-\frac{1}{2}}.\end{aligned}$$

Incorporating this in [9, (4.1)], we see that it refers to the case $S(2p,q)$ of our Theorem 5.1. Hence, by Corollary 1, differentiability of Riemann's function can be read off.

Further on [9, p. 9, ℓ 7 from below], the relation (47) in [28] is stated in the form

$$\Theta\left(\frac{2q}{2p}+i\epsilon\right)\sim\frac{1}{p\sqrt{\epsilon}}S(2p,q),\quad \epsilon\to 0+.$$

Thus, we could say that [9] also gives material to deduce the reciprocity law. In [9, Theorem 3.4], Duistermaat states that

$$\Theta(z) = \begin{cases} \Theta(\gamma z)\, e^{\frac{\pi i}{4} p} \left(\frac{-q}{p}\right) p^{-\frac{1}{2}} (z-r)^{-\frac{1}{2}} & p \text{ odd} \\ \Theta(\gamma z)\, e^{\frac{\pi i}{4}(q+1)} \left(\frac{p}{|q|}\right) p^{-\frac{1}{2}} (z-r)^{-\frac{1}{2}} & q \text{ odd} \end{cases} \tag{34}$$

From (34), the reciprocity law follows. However, it is used in its proof, and thus unfortunately, this does not lead to the proof of reciprocity law.

5.1 Reciprocity Law

The well-known law of quadratic reciprocity has had numerous proofs. Gauss, who first discovered the law, gave several proofs in his book, *Disquitiones Arithmeticae*. We recall the statement of the law of quadratic reciprocity. For a given pair of distinct primes p and q, one can define the Legendre symbol $\left(\frac{p}{q}\right)$ to be $+1$ if the quadratic congruence $x^2 \equiv p \pmod{q}$ has a solution; the symbol to be -1 if the quadratic congruence has no solution.

Theorem 5.2 *(Quadratic Reciprocity Law)*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}.$$

This theorem is remarkable in many ways, the most notable being the relationship between the solvability of the congruence $x^2 \equiv q \pmod{p}$ to that of the congruence $x^2 \equiv p \pmod{q}$. Let us denote for $z \in \mathcal{H}$,

$$\Theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z}$$

and then the classical theta function for $\operatorname{Re} z > 0$ is

$$\theta(z) = \Theta(iz) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 z}.$$

At this point, we note down the theta transformation formula:

$$\Theta(z) = e^{\frac{\pi i}{4}} z^{-\frac{1}{2}} \Theta\left(-\frac{1}{z}\right). \tag{35}$$

We now prove the reciprocity law.

Theorem 5.3 *Let $p \in \mathbb{N}$ and $(0 \neq)q \in \mathbb{Z}$. Then*

$$S(p,q) = e^{\frac{\pi}{4}\operatorname{sgn}(q)i}\left(\frac{p}{2|q|}\right)^{1/2} S(4|q|, -\operatorname{sgn}(q)p).$$

Proof Let us first note that $F(z)$ is essentially the integral of $\Theta(z)$:

$$\begin{aligned}\int_0^z \theta(-iz)\,\mathrm{d}z &= \int_0^z \Theta(z)\,\mathrm{d}z \\ &= z + 2\left(\sum_{n=1}^{\infty}\frac{e^{\pi i n^2 z}}{\pi i n^2} - \sum_{n=1}^{\infty}\frac{e^{\pi i n^2 z}}{\pi i n^2}\right) \\ &= z + 2(F(z) - F(0)). \qquad (36)\end{aligned}$$

In particular, for $z = x + u + i\epsilon \in \mathbb{C}$ (with $\epsilon > 0$) and $u \in (0, h)$, the above relation (36) becomes

$$\begin{aligned}\int_x^{x+h} \theta(\epsilon - iu)\,\mathrm{d}u &= \int_{x+i\epsilon}^{x+h+i\epsilon} \Theta(z)\,\mathrm{d}z \\ &= h + 2(F(x+h+i\epsilon) - F(x+i\epsilon)). \qquad (37)\end{aligned}$$

The theta transformation formula (35) with $y > 0$ gives

$$\begin{aligned}\theta(y - iu) &= e^{\frac{\pi}{4}i}\frac{1}{\sqrt{u+iy}}\theta\left(\frac{i}{u+iy}\right) \\ &= e^{\frac{\pi}{4}i}\frac{1}{\sqrt{u+iy}}\sum_{n=-\infty}^{\infty} e^{\frac{i\pi n^2}{u+iy}}.\end{aligned}$$

We now make the following change of variable:

$$\frac{i}{u+i\epsilon} = \frac{i}{x+v+i\epsilon} = \tau + \frac{1}{x}i$$

i.e.

$$\tau = \frac{\epsilon - iv}{x(x+v+i\epsilon)} \sim \frac{\epsilon - iv}{x^2}.$$

Now with this change, the integral in (37) becomes

$$\int_x^{x+h} \theta(\epsilon - iu)\,\mathrm{d}u = -ie^{\pi/4}\int_{\frac{i}{x+h+i\epsilon}-\frac{i}{x}}^{\frac{i}{x+i\epsilon}-\frac{i}{x}} \frac{1}{\left(\tau+\frac{1}{x}i\right)^{\frac{3}{2}}}\theta\left(\tau + \frac{1}{x}i\right)\mathrm{d}\tau. \qquad (38)$$

The following relation is useful (which is in fact equivalent to (37)) in applying integration by parts:

$$\int \theta\left(\tau+\frac{i}{x}\right)\,\mathrm{d}u=\tau-2iF\left(-\frac{1}{x}+i\tau\right)+C.$$

Using this, we may evaluate (38) and it becomes

$$\begin{aligned}
&\int_{\frac{i}{x+h+i\epsilon}-\frac{i}{x}}^{\frac{i}{x+i\epsilon}-\frac{i}{x}} \frac{1}{\left(\tau+\frac{1}{x}i\right)^{\frac{3}{2}}}\theta\left(\tau+\frac{1}{x}i\right)\,\mathrm{d}\tau\\
=&\left[(\tau+\frac{i}{x})^{3/2}\left(\tau-2iF\left(-\frac{1}{x}+i\tau\right)\right)\right]_{\frac{i}{x+i\epsilon}-\frac{i}{x}}^{\frac{i}{x+h+i\epsilon}-\frac{i}{x}}\\
=&\left(\frac{x+h+i\epsilon}{i}\right)^{3/2}\left(\frac{i}{x+h+i\epsilon}-\frac{i}{x}-2iF\left(-\frac{1}{x+h+i\epsilon}\right)\right)\\
&-\left(\frac{x+i\epsilon}{i}\right)^{3/2}\left(\frac{i}{x+i\epsilon}-\frac{i}{x}-2iF\left(-\frac{1}{x+i\epsilon}\right)\right)+O(h).
\end{aligned}\tag{39}$$

At this point, we note that

$$\frac{-1}{x+h+i\epsilon}=-\frac{1}{x}+\frac{1}{x^2}(\mathfrak{z}(1+o(1))).\tag{40}$$

Using (40), the main term in (39) is

$$\begin{aligned}
&-2e^{\frac{\pi}{4}i}\left((x+h+i\epsilon)^{3/2}F\left(-\frac{1}{x+h+i\epsilon}\right)-(x+i\epsilon)^{3/2}F\left(-\frac{1}{x+i\epsilon}\right)\right)\\
=&\,2e^{\frac{\pi}{4}i}(x+i\epsilon)^{3/2}\left(F\left(-\frac{1}{x}+\frac{1}{x^2}\mathfrak{z}'\right)-F\left(-\frac{1}{x}+\frac{1}{x^2}\epsilon'\right)\right)+O(h),
\end{aligned}\tag{41}$$

where we have used

$$\mathfrak{z}'=\mathfrak{z}(1+o(1))\ \text{ and }\ \epsilon'=\epsilon(1+o(1)).$$

Now, we specify $x=\frac{2q}{p}$ and apply Theorem 5.1. Under this specification, (41) takes the shape

$$\begin{aligned}
&=2e^{\frac{\pi}{4}i}\left(\frac{2q}{p}+i\epsilon\right)^{3/2}\left(F\left(-\frac{2p}{4q}+\left(\frac{p}{2q}\right)^2\mathfrak{z}'\right)-F\left(-\frac{2p}{4q}+\left(\frac{p}{2q}\right)^2\epsilon'\right)\right)+O(h)\\
&=2e^{\frac{\pi}{4}i}\left(\frac{2q}{p}+i\epsilon\right)^{3/2}S(4q,-p)e^{-\frac{\pi}{4}i}\frac{1}{4|q|}\left|\frac{p}{2q}\right|\sqrt{\mathfrak{z}'}+O(h)
\end{aligned}$$

$$= \left(\frac{p}{2|q|}\right)^{1/2} \frac{1}{p} S(4|q|, -\operatorname{sgn}(q)p)\sqrt{\mathfrak{z}'} + O(h).$$

Now on letting $\epsilon \to 0$, we get the desired result. □

As a corollary, we note that:

Corollary 1 *Let $x = \frac{q}{p}$ be of the form $\frac{2A+1}{2B+1}$, i.e. p, q both being odd. Then*

$$R(2A+1, 2B+1) = S(2p, q) = 0 \tag{42}$$

where R is the coefficient in the forthcoming Eq. (43).

Proof

$$\begin{aligned} S(2p, q) &= e^{\frac{\pi}{4}i} \left(\frac{p}{2|q|}\right)^{1/2} S(4|q|, 2\operatorname{sgn}(q)p) \\ &= e^{\frac{\pi}{2}i} \left(\frac{p}{2|q|}\right)^{1/2} \left(\frac{4|q|}{2|2p|}\right)^{1/2} S(4 \cdot 2p, 2\operatorname{sgn}(q)|q|) \\ &= e^{\frac{\pi}{2}i} \frac{p}{|q|} \sqrt{\operatorname{sgn}(q)} S(2p, \operatorname{sgn}(q)|q|). \end{aligned}$$

We now conclude (42) by simply noting that $\operatorname{sgn}(q)|q| = q$. □

Remark 5.2 The relation in Theorem 5.3 leads to the so-called Landsberg–Schaar identity (see [28, (5)]) if we take p and q to be co-prime positive integers. This is

$$\frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} e^{2\pi i j^2 \frac{q}{p}} = \frac{e^{\frac{\pi}{4}} i}{\sqrt{2q}} \sum_{j=0}^{2q-1} e^{2\pi i j^2 \frac{p}{2q}}.$$

The following result will be required to complete the proof of the differentiability of Riemann's function.

Lemma 5.1 *For a natural number p,*

$$S(p, q) = \varepsilon(p) \left(\frac{q}{p}\right) \sqrt{p}$$

where $\left(\frac{q}{p}\right)$ indicates the Jacobi symbol and

$$\varepsilon(p) = \begin{cases} 1 & p \equiv 1 \bmod 4 \\ i & p \equiv 3 \bmod 4 \end{cases}$$

We are now ready to state a seemingly more general version of Theorem 5.1. This implies differentiability of Riemann’s function at the rational point $\frac{2A+1}{2B+1}$ on putting $\mathfrak{z} = h + i\epsilon$ and $\epsilon \to +0$. We note that similar result is also obtained in [9, Theorem 4.2].

Corollary 2

$$F\left(\frac{q}{p} + \mathfrak{z}\right) - F\left(\frac{q}{p} + i\epsilon\right) = R(p, q)\frac{e^{-\pi i/4}}{p}\sqrt{\mathfrak{z}} - \frac{1}{2}h + O(\mathfrak{z}^2),$$

where

$$\begin{aligned} R(p, 2q) &= S(p, q) = \varepsilon(p)\left(\frac{q}{p}\right)\sqrt{p}, \\ R(2p, q) &= S(4p, q) = e^{\frac{\pi}{4}i}\sqrt{2p}\left(\frac{-p}{q}\right) \\ R(2B+1, 2A+1) &= 0. \end{aligned} \tag{43}$$

Proof Only the case $R(2p, q)$ needs to be considered (by Corollary 1). Now by Theorem 5.3, we have,

$$\begin{aligned} R(2p, q) &= S(4p, q) = e^{\frac{\pi}{4}i}\left(\frac{4p}{2|q|}\right)^{1/2} S(4|q|, -4\operatorname{sgn}(q)p) \\ &= e^{\frac{\pi}{4}i}\left(\frac{2p}{|q|}\right)^{1/2} S(|q|, -\operatorname{sgn}(q)p) \\ &= e^{\frac{\pi}{4}i}\left(\frac{2p}{|q|}\right)^{1/2}\sqrt{|q|}\varepsilon(|q|)\left(\frac{-\operatorname{sgn}(q)p}{|q|}\right) \\ &= e^{\frac{\pi}{4}i}\sqrt{2p}\left(\frac{-p}{q}\right). \end{aligned}$$

□

Remark 5.3 We make a historical remark on Riemann’s function. [3] contains an almost complete list of references up to 1986. One addition is a correction to [31] in 1983. After this, the review of [14] contains an almost complete list after [3] except for [27] (esp. 619) and [33]. Among the papers listed in the review of [14], we mention [19] and [22] for consideration from the point of wavelets and [9] for self-similarity.

References

1. S. Bernstain, Sur la convergence absolue des séries trigonométriques. Communications de la Société mathématique de Kharkow **14**, 139–144 (1914)
2. T.J. Bromwich, Infinite Series, pp. 490–491; Dini, Grundlagen, pp. 205 et seq.; Hobson, Functions of a real variable, pp. 620 et seq
3. P.L. Butzer, E.L. Stark, "Riemann's example" of a continuous non-differentiable function in the light of two letters (1865) of Christoffel to Prym. Bull. Soc. Math. Belg. Sèr. A **38**, 45–73 (1986)
4. A.L. Cauchy, Methode simple et nouvelle pour la determination complete des sommes alternee, formees avec les racines primitives des equattions binomes. J. Math. Pure Appl. (Liouvilee) **5**, 154–183 (1840)
5. K. Chakraborty, S. Kanemitsu, H. Long Li, Quadratic reciprocity and Riemann's non differentiable function. Res. Number Theory **1**, 14 (2015)
6. J.S. Darboux, Mémoire sur les fonctions discontinues. Annales del' Ecole Normale, ser 2 **4**, 57–112 (1875)
7. U. Dini, Sulle funzioni finite continue di variabili reali che non hanno mai derivata [On finite continuous functions of real variables that have no derivatives]. Atti Reale Accad. Lineci S'er 3 **1**, 130–133 (1877). (Italian)
8. P. du Bois-Reymond, Versuch einer classification der willkürlichen functionen reeller argumente nch ihren Änderungen in den kleinsten intervallen. J. für Mathematik **79**, 28 (1875)
9. J.J. Duistermaat, Self-similarity of "Riemann's non-differentiable function". Nieuw Arch. Wisk. **9**, 303–337 (1991)
10. G. Faber, Einfaches beispiel einer stetigen nirgends differentiirbaren funktion. Jahresbericht der Deutschen Mathe. Ver. **16**, 538–540 (1907)
11. P. Fatou, Séries trigonométriques et s'(e)ries de Taylor. Acta, Math. **9**, 335–400 (1906)
12. J. Gerver, The differentiability of the Riemann function at certain rational multiples of π. Amer. J. Math. **92**, 33–55 (1970)
13. J. Gerver, More on the differentiability of the Riemann function. Amer. J. Math. **93**, 33–41 (1970)
14. J. Gerver, On cubic lacunary Fourier series. Trans. Amer. Math. Soc. **355**, 4297–4347 (2003)
15. G.H. Hardy, A theorem concerning Taylor series. Q. J. Math. **44**, 147–160 (1913)
16. G.H. Hardy, Weierstrass non-differentiable functions. Trans. Amer. Math. Soc. **17**, 301–325 (1916)
17. G.H. Hardy, J.E. Littlewood, Contributions to the arithmetic theory of series. Proc. London Math. Soc., Ser. 2, **11**, 411–478 (1912)
18. G.H. Hardy, J.E. Littlewood, Some problems of diophantine approximation (II). Acta Math. **37**, 193–238 (1914)
19. M. Holschneider, P. Tchamichian, Pointwise analysis of Riemann's non differentiable function. Invent. Math. **105**, 157–275 (1991)
20. A.E. Ingham, *The Distribution of Prime Numbers*, vol. 30, Cambridge Tracts in Mathematics and Mathematical Physics (Stechert-Hafner, Inc., New York, 1964)
21. S. Itatsu, Differentiability of Riemann's function. Proc. Japan Acad. Ser. A Math. Sci. **57**, 492–495 (1981)
22. S. Jaffard, The spectrum of singularities of Riemann's function. Rev. Mat. Iberoamericana **12**, 441–460 (1996)
23. J.P. Kahane, Lacunary Taylor and Fourier series. Trans. Amer. Math. Soc. **79**, 199–213 (1964)
24. S. Kanemitsu, H. Tsukada, *Contributions to the Theory of Zeta-functions–modular Relation Supremacy* (World Scientific, London etc, 2014)
25. G. Landsberg, Über differentiirbarkeit stetiger funktionen. ibid. **17**, 46–51 (1908)
26. M. Lerch, Über die Nichtdifferentiirbarkeit gewisser functionen. Journal für Mathematik **103**, 126–138 (1888)
27. Y. Meyer, Le traitment du signal et l'analyze mathématique. Ann. Inst. Fourier Grenoble **50**, 593–632 (1994)

28. R. Murty, A. Pacelli, Quadratic reciprocity via theta functions. Proc. Int. Conf.–Number Theory **1**, 107–116 (2004)
29. E. Neuenschwander, Riemann’s example of a continuous “non-differentiable” function. Math. Intell. **1**(1), 40–44 (1978)
30. S.L. Segal, Riemann’s example of a continuous “non-differentiable” function continued. Math. Intell. **1**(2), 81–82 (1978)
31. A. Smith, The differentiability of Riemann’s function, *Proc. Amer. Math. Soc.* **34**, 463–468 (1972); Correction to “The differentiability of Riemann’s function”, ibid. **89**, 567–568 (1983)
32. J. Thim, Continuous Nowhere Differentiable functions, Master’s Thesis, Dept. Of Mathematics, LULEA University of Technology, 2003:320 CIV
33. P. Ulrich, Anmerkungen zum “Riemannschen Beispiel” einer stetigen, nicht differenzierbaren Funktion. Res. Math. **31**, 245–265 (1997)
34. K. Weierstrass, Über continuerliche funktionen einer reellen arguments die fur keinen werth des letzeren einen bestimmten differentialquotienten bestizen. Mathematische Werk **II**, 71–74 (1872)
35. H. Wiener, Geometrische und analytische untersuchung der Weierstrass’schen function. Journal für Mathematik **90**, 221–252 (1881)

Survey on Metric Fixed Point Theory and Applications

Yeol Je Cho

Abstract Fixed Point Theory is divided into the following three major areas:

- **Topological Fixed Point Theory**, which came from Brouwer's fixed point theorem in 1912;
- **Metric Fixed Point Theory**, which came from Banach's fixed point theorem in 1922;
- **Discrete Fixed Point Theory**, which came from Tarski's fixed point theorem in 1955.

In this chapter, we focus on recent topics on metric fixed point theory and its applications, which will be very helpful to beginners and specialists of metric fixed point theory and its applications. In fact, since Banach's fixed point theorem in metric spaces, because of its simplicity, usefulness and applications, it has become a very popular tools in solving the existence problems in many branches of mathematical analysis and applied sciences. Recently, Banach's fixed point theorem has been applied to Economics, Chemical Engineering Sciences, Medicine, Image Recovery, Electric Engineering, Game Theory and other applied sciences by many authors.

Keywords Picard's convergence theorem · Banach's fixed point theorem · Contractive mapping · Multiplicative metric space · Weakly commuting mapping and compatible mapping · (CLR_g)-property · Posedness · Stability · The limit shadowing property · Picard iteration · Mann iteration · Best proximity point · n-cyclically monotone mapping · Maximal n-cyclically monotone mapping · Cyclically firmly nonexpansive mapping

Mathematics Subject Classification 47H09 · 47H10 · 54H25 · 37C25

Y.J. Cho (✉)
Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 660-701, Korea
e-mail: yjcho@gnu.ac.kr

Y.J. Cho
Center for General Education, China Medical University, Taichung 40402, Taiwan

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_9

1 Introduction

In 1912, Brouwer [1] proved the following fixed point theorem, which is called *Brouwer's Fixed Point Theorem*:

Theorem B. *Every continuous mapping from the unit ball of* $\mathbb{R}^n$ *into itself has a fixed point.*

Since Brouwer's fixed point theorem, some authors, Schauder [2], Tychonoff [3], Kakutani [4] and many others have improved and generalized this theorem in several ways. In fact, Schauder's fixed point theorem is an extension of Brouwer's fixed point theorem to topological vector spaces and, also, there are several entirely different ways to prove Brouwer's fixed point theorem by some authors.

In 1955, Tarski [5] proved the following fixed point theorem, which is called *Tarski's Fixed Point Theorem*:

Theorem T. *If* F *is a monotone function on a nonempty complete lattice, then the set of fixed points of* F *forms a nonempty complete lattice.*

Note that the least fixed point of the mapping f is the least element x such that $f(x) = x$ or, equivalently, such that $f(x) \leq x$ and the greatest fixed point is the greatest element x such that $f(x) = x$ or, equivalently, such that $f(x) \leq x$. Consequently, from Theorem T, f has the greatest fixed point $\overline{u}$ and the least fixed point $\underline{u}$ and, moreover, for all $x \in L$, $x \leq f(x)$ implies $x \leq \overline{u}$, whereas $f(x) \leq x$ implies $\underline{u} \leq x$.

Example T1. Let $a, b \in \mathbb{R}$ with $a \leq b$, where $\leq$ is the usual order of real numbers. Since the closed interval $[a, b]$ is a complete lattice, every monotone increasing mapping $f : [a, b] \to [a, b]$ has the greatest fixed point and the least fixed point. Here the mapping f need not be continuous.

Since Tarski's fixed point theorem, many authors, for example, Hayashi [6], Heikkila [7], Schröder [8], Jachymski et al. [9], Uhl [10], Ok [11] and many others, have improved and generalized this theorem in several ways. Recently, Theorem T has many applications in theoretical computer science and others.

Especially, in [12], Davis proved the converse of Theorem T, that is, if every order preserving function $f : L \to L$ has a fixed point, then L is a complete lattice. Also, Theorem T can be used for a simple proof of the Cantor-Bernstein-Schroeder theorem (see Example 3 in Uhl [10]) in set theory, that is, if there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijective function $h : A \to B$.

Note that, since famous Brouwer's and Tarski's fixed point theorems have been studied by many authors, in this chapter, we don't mention any more about these two theorems.

Now, we introduce recent results on metric fixed point theory and its applications as follows:

[A] Picard's Convergence Theorem

In 1890, Picard [13] proved the following theorem to show the existence of solutions for nonlinear equations.

Theorem P. *Let* $T : [a, b] \to \mathbb{R}$ *be a continuous function and* $T : (a, b) \to \mathbb{R}$ *be differentiable. If there exists* $L < 1$ *such that*

$$|T'(x)| \leq L \tag{PC}$$

for all $x \in (a, b)$, *then the sequence* $\{x_n\}$ *in* (a, b) *defined by*

$$x_{n+1} = Tx_n \tag{P}$$

for all $n \geq 0$ *converges to a solution of the equation* $Tx = x$.

The iterative sequence $\{x_n\}$ defined by (**P**) is called the *Picard iterative sequence*.

[B] Banach's Fixed Point Theorem

In 1922, Banach [14] proved a theorem, which is well known as "*Banach's fixed point theorem*" to establish the existence of solutions for integral equations.

Theorem B. *Let* (E, d) *be a complete metric space and* $T : E \to E$ *be a contractive mapping* (*that is, there exists* $L \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq Ld(x, y) \tag{BC}$$

for all $x, y \in E$). *Then, we have the following*:

(1) T *has a unique fixed point* $z \in E$;
(2) *Furthermore, for each* $x_0 \in E$, *the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = Tx_n$$

for each $n \geq 0$ *converges to the fixed point* z *of* T, *that is*, $Tz = z$.

Note that the following conditions are equivalent:

(1) In Picard's theorem, there exists a number $L < 1$ such that

$$|T'(x)| \leq L \tag{PC}$$

for all $x \in (a, b)$.

(2) In Banach's fixed point theorem, there exists $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y) \tag{BC}$$

for all $x, y \in E$ (that is, T is a contractive mapping).

Further, since Banach's fixed point theorem, because of its simplicity, usefulness, and applications, it has become a very popular tool in solving the existence problems in many branches of mathematical analysis. Recently, many authors have improved, extended, and generalized Banach's fixed point theorem in the following ways.

First, how to generalize Banach's contraction? Second, how to extend Banach's fixed point theorem in metric spaces to the large class of various spaces? Third, how to extend Banach's fixed point theorem for single-valued mappings to multi-valued mappings? Fourth, how to show the existence of common fixed points for two nonlinear mappings? Fifth, how to improve Banach's fixed point theorem in several ways? Sixth, how to generalize the Picard iterative sequence? Seventh, how to apply Banach's fixed point theorem to applied mathematics and others? Eighth, dose the converse of Banach's fixed point theorem hold? Ninth, we introduce a conjecture of Banach's fixed point theorem, which is called Generalized Banach's Fixed Point Theorem; Tenth, we introduce some relations between best proximity point theorems and Banach's fixed point theorem in metric spaces; Finally, eleventh, we introduce some better nonlinear mappings than Banach's contraction.

2 Generalizations of Contractive Mappings

Recently, many authors have introduced many kinds of contractive mappings (or Banach's contraction) in metric spaces and generalized metric spaces as follows:

(1) In 1969, *Meir–Keeler's contraction* [15]: For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon \tag{MK}$$

Note that if T satisfies Meir–Keeler's contraction (MK), then T is Banach's contraction, that is,

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. For more details, see Park and Rhoades [16, 17].

(2) In 1976, *Caristi's contraction* [18]:

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \tag{CC}$$

where $\phi : X \to [0, \infty)$ is a lower semi-continuous function.

Note that every Banach's contraction T satisfies Caristi's contraction if, for some $L \in [0, 1)$,

$$\phi(x) = \frac{d(x, Tx)}{1 - L}.$$

(3) Banach's contraction (BC) can be expressed as follows:

$$d(Tx, Ty) \leq d(x, y) - qd(x, y),$$

where $L = 1 - q$ with $q \in [0, 1)$. Thus, we can define a new contraction T, which is called a *weakly contraction*, as follows:

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{WC1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$, $\phi(t) > 0$ for all $t \in (0, \infty)$ and $\lim_{t \to 0} \phi(t) = \infty$.

Note that, in (WC1), if $\phi(t) = (1 - L)t$, then we can get Banach's contraction (BC).

Also, Banach's contraction (BC) can be expressed as follows:

$$d(Tx, Ty) \leq (1 + q)d(x, y) - (1 - q)d(x, y),$$

where $L = 2q$ with $q \in [0, \frac{1}{2})$. Thus, we can define a new contraction T, which is called a $(\phi - \psi)$-*weak contraction*, as follows:

$$d(Tx, Ty) \leq \phi(d(x, y)) - \psi(d(x, y)), \tag{WC2}$$

where $\phi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous and nondecreasing function and $\psi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous and nonincreasing function satisfying the following conditions:

(a) $\phi(0) - \psi(0) = 0$;
(b) $\phi(t), \psi(t) > 0$ for all $t \in (0, \infty)$;
(c) $\phi(t) - \psi(t) < t$ for all $t \in (0, \infty)$.

Note that, in the condition (WC2), if $\phi(t) = t$ for all $t \in [0, \infty)$, then we have the condition (WC1).

(4) From Banach's contraction (BC), it follows that the mapping T is continuous. Further, we use the continuity of the mapping T to prove Banach's fixed point theorem. Thus, it is natural to consider the following question:

Do there exist some contractive conditions which do not force the mapping T to be continuous?

The answer for this question was positive by Kannan [19] in 1968 who proved *Kannan's fixed point theorem* for the following contractive condition, which is called *Kannan's contraction*:

Theorem K. *Let (E, d) be a complete metric space and $T : E \to E$ be a mapping such that there exists a number $h \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq h[d(Tx, x) + d(Ty, y)] \tag{KC}$$

for all $x, y \in X$. Then, T has a unique fixed point in E.

Now, we give one example that a mapping T is not continuous, but the mapping T is Kannan's contraction:

Example K. Let $X = \mathbb{R}$ be a usual metric space and $T : X \to X$ be a mapping defined by

$$Tx = \begin{cases} 0, & \text{if } x \in (-\infty, 2], \\ \frac{1}{2}, & \text{if } x \in (2, +\infty). \end{cases}$$

Then, T is not continuous on $\mathbb{R}$, but it satisfies Kannan's contraction (KC) with $k = \frac{1}{5}$.

(5) In 1972, Chatterjea [20] introduced the following contractive condition: there exists a number $h \in [0, \frac{1}{2})$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq h[d(Tx, y) + d(Ty, x)]. \tag{CHC}$$

Note that Banach's contraction (BC), Kannan's contraction (KC), and Chatterjea's contraction (CHC) are *independent* (see Rhoades' paper [21]).

(6) In 2004, Berinde [22] introduced the following contractive condition: There exist $h \in [0, 1)$ and $L \geq 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq hd(x, y) + Ld(y, Tx) \tag{VBC}$$

(7) In 1971, Reich [23] introduced the following contractive condition: There exist nonnegative numbers $q, r, s \in [0, \infty)$ such that $q + r + s < 1$ and

$$d(Tx, Ty) \leq qd(x, y) + rd(x, Tx) + sd(y, Ty) \tag{RC}$$

for all $x, y \in X$.

(8) In 1971, Ćirić [24] introduced the following contractive condition: There exist nonnegative numbers $q, r, s, t \in [0, \infty)$ such that $q + r + s + 2t < 1$ and

$$\begin{aligned} &d(Tx, Ty) \\ &\leq qd(x, y) + rd(x, Tx) + sd(y, Ty) + t[d(x, Ty) + d(y, Tx)] \end{aligned} \tag{CRC1}$$

for all $x, y \in X$.

(9) In 1972, Zamfirescu [25] introduced the following contractive condition:

$$\begin{aligned} &d(Tx, Ty) \\ &\leq \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\} \end{aligned} \tag{ZC}$$

for all $x, y \in X$.

(10) In 1973, Hardy and Rogers [26] introduced the following contractive condition: There exist nonnegative numbers $a_1, a_2, a_3, a_4, a_5 \in [0, \infty)$ such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and

$$\begin{aligned} &d(Tx, Ty) \\ &\le a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Tx) + a_5 d(x, y) \end{aligned} \tag{HRC}$$

for all $x, y \in X$.

(11) In 1974, Ćirić [27] introduced the following contractive condition: There exists $h \in [0, 1)$ such that

$$\begin{aligned} &d(Tx, Ty) \\ &\le h \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\} \end{aligned} \tag{CRC2}$$

for all $x, y \in X$.

(12) Let I be a closed interval in $\mathbb{R}$ and $T : I \to I$ be differentiable with $|T'(t) < 1$ for all $t \in I$. Then, by the mean value theorem, we have

$$|T(x) - T(y) < |x - y| \tag{C}$$

for all $x, y \in I$ with $x \neq y$. Then, the following functions satisfy the condition (C):

(a) $T(x) = x + \frac{1}{x}$ on $I = [1, +\infty)$;
(b) $T(x) = \sqrt{x^2 + 1}$ on $I = \mathbb{R}$;
(c) $T(x) = \ln(1 + e^x)$ on $I = \mathbb{R}$.

In each case, $T(x) > x$ and so none of these functions has a fixed point in I.

Despite such examples, in 1962, Edelstein [28] proved fixed point theorems by using the following contraction, which is called *Edelstein's contraction* or *strictly contraction*:

$$d(f(x), f(y) < d(x, y) \tag{EC}$$

for all $x, y \in X$ with $x \neq y$ provided the space X is compact.

Theorem ES. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying the following*:

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then, T has a unique fixed point in X.

Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following:

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then, the mapping T has no fixed point in X as in the following example:

Example EC. Let $X = [1, \infty)$ be the set of real numbers with the usual metric and define a mapping $T : X \to X$ by

$$Tx = x + \frac{1}{x}$$

for all $x \in X$. Then, for all $x, y \in$ with $x \neq y$,

$$d(Tx, Ty) = \left|\left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right)\right| < d(x, y).$$

However, $Tx = x + \frac{1}{x} \neq x$, that is, T has no fixed point in X.

(13) In 1965, Prešić [29] generalized Banach's fixed point theorem in product spaces and proved the following theorem:

Theorem P. *Let (X, d) be a complete metric space, k be a positive integer, and $T : X^k \to X$ be a mapping satisfying the following contractive type condition*:

$$\begin{aligned} &d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \\ &\leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \cdots + q_k d(x_k, x_{k+1}) \end{aligned} \tag{PC}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$, where $q_1, q_2, \ldots, q_k$ are nonnegative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then, there exists a unique point $x \in X$ such that $T(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and, for each $n \geq 1$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and

$$\lim_{n\to\infty} x_n = T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_n).$$

Example PC. ([30]) Let $I = [0, 1]$ be the unit interval with the usual Euclidean norm and $f : I^3 \to I$ be defined by

$$f(x, y, z) = \frac{2x + y + 2z}{5}$$

for all $x, y, z \in I$. Then, f satisfies the condition (PC).

Note that, from (PC), we can consider the following contractions:

(a) There exists $\lambda \in (0, 1)$ such that

$$\begin{aligned} &d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \\ &\leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} \end{aligned} \tag{PC1}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$ with $x_1 \leq x_2 \leq \cdots \leq x_{k+1}$;

(b) There exists $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi : \mathbb{R}_+^k \to \mathbb{R}_+$ and $\lambda \in (0, 1)$ such that

$$\begin{aligned} &\phi(d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1}))) \\ &\leq \lambda\psi(\phi(d(x_1, x_2)), \phi(d(x_2, x_3)), \dots, \phi(d(x_k, x_{k+1}))) \end{aligned} \tag{PC2}$$

for all $x_1, x_2, \dots, x_{k+1} \in X$ with $x_1 \leq x_2 \leq \cdots \leq x_{k+1}$, where two functions ϕ and ψ satisfy some conditions.

Recently, some authors generalized Prešić's fixed point theorem in several ways (see [31–36]). In particular, George et al. [37], Khan and Samanipour [38], Malhotra et al. [39] and Khan et al. [40] studied the *cone metric version* of Prešić's fixed point theorem, and H. Fukhar-Ud-Din et al. [30] studied fixed point approximations of Prešić nonexpansive mappings in product of CAT(0) spaces.

(14) Also, in 2005, Zhu et al. [41] gave some equivalent contractive conditions in symmetric spaces.

3 Extensions of Banach's Fixed Point Theorem in Metric Spaces to other Spaces

Recently, some authors have introduced some generalizations of metric spaces in several ways and have studied fixed point theory and it applications:

Cone metric spaces, partially ordered metric spaces, fuzzy metric spaces, complex-valued metric spaces, probabilistic metric spaces, random normed spaces, ordered Banach spaces, b-metric spaces, 2-metric spaces, G-metric spaces, M-metric spaces, S-metric spaces, and other spaces

In this section, we introduce *multiplicative metric spaces*, *partial metric spaces*, and *M-metric spaces* and study fixed point theory and its applications in these metric spaces.

(I) Fixed Point Theorems in Multiplicative Metric Spaces

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^+$ is called a *multiplicative metric* (see Bashirov et al. [42]) if the following conditions are satisfied: For all $x, y, z \in X$,

(MM1) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;
(MM2) $d(x, y) = d(y, x)$;
(MM3) $d(x, y) \leq d(x, z) \cdot d(z, y)$ (: multiplicative triangle inequality).

A set X with a multiplicative metric d is called a *multiplicative metric space*.

Example M. Let $\mathbb{R}_+^n$ be the set of all n-tuples of nonnegative real numbers. Let $d : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ be a mapping defined as follows:

$$d(x, y) = \left|\frac{x_1}{y_1}\right| \cdot \left|\frac{x_2}{y_2}\right| \cdots \left|\frac{x_n}{y_n}\right|,$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$ and $|\cdot| : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 1, \\ \frac{1}{a}, & \text{if } a < 1. \end{cases}$$

It is easy to see that d is a multiplicative metric on $\mathbb{R}^n_+$.

Remark M1. (1) It is well known that $(0, +\infty)$ is not complete according to the *usual metric* d. For example, consider a sequence $\{x_n\} = \{\frac{1}{n}\}$. Then, $\{\frac{1}{n}\}$ is a Cauchy sequence, but $0 \notin (0, +\infty)$ and so $(0, +\infty)$ is not complete. But we know that $(0, +\infty)$ is complete with respect to the multiplicative metric d.

(2) The ordinary metrics and the multiplicative metrics may be different in more general cases. In $(0, +\infty)$, the convergence in both ordinary and multiplicative metrics is equivalent.

(3) The multiplicative metrics were introduced to solve some differential and integral equations.

Remark M2. (1) In 2012, Ozavsar and Cevikel [43] introduced the concept of multiplicative contraction mappings and proved some fixed point theorems for this type of mappings.

(2) Recently, some fixed point theorems for some contractive mappings in multiplicative metric spaces have been improved and extended in many ways by some authors.

(3) In 2014, He et al. [44] proved some common fixed point theorems for weak commutative mappings in multiplicative metric spaces.

A mapping $T : X \to X$ is called:

(1) the *multiplicative contraction* if there exists $\lambda \in [0, 1)$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq [d(x, y)]^{\lambda}.$$

(2) *multiplicative Kannan's contraction* if there exists $\lambda \in [0, \frac{1}{2})$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq [d(x, Tx)d(y, Ty)]^{\lambda}.$$

(3) *multiplicative Chatterjea's contraction* if there exists $\lambda \in [0, \frac{1}{2})$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq [d(x, Ty)d(y, Tx)]^{\lambda}.$$

Recently, Tiammee et al. [45] proved the following:

Theorem TSC. *Let (X, d) be a complete multiplicative metric space. Suppose that the mappings $S, T, A, B : X \to X$ satisfy the following conditions*:

(a) $S(X) \subset B(X)$ *and* $T(X) \subset A(X)$;
(b) *the pairs A, S and B, T are compatible*;
(c) *one of the mappings S, T, A, B is continuous*;
(d) *there exist $a_1, a_2, a_3, a_4, a_5 \in [0, \infty)$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and $a_1 = a_2$* or *$a_3 = a_4$ such that, for all $x, y \in X$,*

$$d(Sx, Ty) \leq [d(Ax, Sx)]^{a_1}[d(By, Ty)]^{a_2}[d(Ax, Ty)]^{a_3}[d(By, Sx)]^{a_4}[d(Ax, By)]^{a_5}. \tag{GMC}$$

Then, S, T, A, and B have a unique common fixed point in X.

Now, we give one example to illustrate Theorem TSC as follows:

Example TSC. Let $X = [0, \infty)$ be the usual metric space. Define a mapping $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = e^{|x-y|}$ for all $x, y \in X$. Then, (X, d) is a complete multiplicative metric space. Define four mappings $S, T, A, B : X \to X$ by

$$Sx = \frac{1}{64}x, \quad Tx = \frac{1}{32}x, \quad Ax = x, \quad Bx = 2x.$$

Then, we have the following:

(a) $S(X) = T(X) = A(X) = B(X) = X$;
(b) S, T, A, and B are all continuous mappings;
(c) the pairs S, A and T, B are compatible mappings;
(d) Let $x, y \in X$ and choose $a_1 = \frac{1}{32}, a_2 = \frac{1}{32}, a_3 = \frac{1}{16}, a_4 = \frac{1}{8}, a_5 = \frac{1}{4}$. Then, we obtain

$$d(Sx, Ty) = [d(Ax, Sx)]^{\frac{1}{32}}[d(By, Ty)]^{\frac{1}{32}}[d(Ax, Ty)]^{\frac{1}{16}}[d(By, Sx)]^{\frac{1}{8}}[d(Ax, By)]^{\frac{1}{4}}.$$

Therefore, all the conditions of Theorem TSC are satisfied. Also, we see that $S(0) = T(0) = A(0) = B(0) = 0$ and so 0 is a unique common fixed point of S, T, A, and B.

Remark M3. From our generalized multiplicative contraction GMC, we have the following multiplicative contractions:

(1) If we put $a_1 = a_2 = a_3 = a_4 = 0$ and $S = T$ in (GMC), then we have the *multiplicative contraction*, that is, for some $a_5 \in [0, 1)$,

$$d(Tx, Ty) \leq [d(x, y)]^{a_5}$$

for all $x, y \in X$.

(2) If we put $a_1 = a_2$, $a_3 = a_4 = a_5 = 0$ and $S = T$ in **(GMC)**, then we have *multiplicative Kannan's contraction*, that is, for some $a_1 \in [0, \frac{1}{2})$,

$$d(Tx, Ty) \leq [d(x, Tx)d(y, Ty)]^{a_1}$$

for all $x, y \in X$.

(3) If we put $a_1 = a_2 = a_5 = 0$, $a_3 = a_4$ and $S = T$ in **(GMC)**, then we have *multiplicative Chatterjea's contraction*, that is, for some $a_3 \in [0, \frac{1}{2})$,

$$d(Tx, Ty) \leq [d(y, Tx)d(x, Ty)]^{a_3}$$

for all $x, y \in X$.

(4) Also, we can get some more kinds of multiplicative contractions from (GMC).

Remark M4. For some relations between usual metric spaces and multiplicative metric spaces, recently, in 2016, Agarwal et al. [46] pointed out the following:

Although the *multiplicative metric* was announced as a new distance notion, we note that composition of the multiplicative metric with a logarithmic function yields a *usual metric*. Hence all fixed point results in the context of multiplicative metric spaces can easily be concluded from the corresponding existing famous fixed point results *in the context of the standard metrics*.

It is clear that all topological notions, for example, convergence, Cauchy sequence, completeness, and others for multiplicative metric spaces, are the consequences of the standard topology of metric spaces.

In 2016, Agarwal et al. [46] proved the following:

Theorem AKS1. *Let X be a nonempty set and $d^* : X \times X \to [0, \infty)$ be a multiplicative metric. Then, the mapping $d : X \times X \to [0, \infty)$ defined by*

$$d(x, y) = \ln(d^*(x, y))$$

is a usual metric.

Proof. From the definition of a multiplicative metric d^*, we have the following:

$$\begin{aligned} d(x, y) &= \ln(d^*(x, y)) \\ &\leq \ln(d^*(x, y) \cdot d^*(y, z)) \\ &= \ln(d^*(x, y)) + \ln(d^*(y, z)) \\ &= d(x, y) + d(y, z). \end{aligned}$$

This completes the proof.

In 2015, Abbas et al. [47] published the following result in multiplicative metric spaces:

Theorem AAS. *Let (X, d^*) be a complete multiplicative metric space and $f : X \to X$ be a mapping. Suppose that*

$$\psi(d^*(fx, fy)) \leq \frac{\psi(M^f_{d^*}(x, y))}{\varphi(M^f_{d^*}(x, y))} \tag{D1}$$

for any $x, y \in X$, where

$$\begin{aligned} &M^f_{d^*}(x, y) \\ &= \max\{d^*(x, y), d^*(fx, x), d^*(y, fy), [d^*(fx, y) \cdot d^*(x, fy)]^{\frac{1}{2}}\}, \end{aligned} \tag{D2}$$

$\psi : [1, \infty) \to [1, \infty)$ is continuous and nondecreasing, $\psi^{-1}(\{1\}) = \{1\}$ and $\varphi : [1, \infty) \to [1, \infty)$ is lower semi-continuous and $\varphi^{-1}(\{1\}) = \{1\}$. Then, f has a unique fixed point in X.

In 2009, Dorić [48] proved the following extension of Banach's contraction principle in **metric spaces**.

Theorem D. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that, for all $x, y \in X$,*

$$\psi(d(fx, fy)) \leq \psi(M^f(x, y)) - \varphi(M^f(x, y)), \tag{E1}$$

where

$$\begin{aligned} &M^f(x, y) \\ &= \max\left\{d(x, y), d(fx, x), d(y, fy), \frac{1}{2}[d(fx, y) + d(x, fy)]\right\}, \end{aligned} \tag{E2}$$

$\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing, $\psi^{-1}(\{0\}) = \{0\}$ and $\varphi : [0, \infty) \to [0, \infty)$ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}$. Then, f has a unique fixed point in X.

Theorem AKS2. *Theorem AAS is a consequence of Theorem D.*

Proof. By using $d(x, y) = \ln(d^*(x, y))$, we easily see that the equation (D2) yields (E2). Hence, the inequality (D1) implies (E1). Consequently, Theorem D provides the existence and uniqueness of the fixed point of f.

Remark M5. (1) Some authors misuse the notion of the multiplicative calculus since they misunderstand the place and role of this calculus like other non-Newtonian calculuses.

(2) Notice that, in Newtonian calculus, the reference function is linear, whereas the reference function for multiplicative calculus is exponential. Consequently, every definition and also every theorem of Newtonian calculus have an analogue in multiplicative calculus and vice versa.

(3) Therefore, some ordinary and multiplicative fixed point theorems are applicable to the same class of functions.

(II) Fixed Point Theorems in M-Metric Spaces

In 1994, Matthews [49] extended the concept of a *metric* to a *partial metric* and obtained many results in partial metric spaces. Indeed, the motivation for introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, to give the improvement of Banach's fixed point theorem.

Afterward, many mathematicians have studied the existence and uniqueness of a fixed point for nonlinear mappings satisfying various contractive conditions in the setting of partial metric spaces.

Definition M. Let X be a nonempty set and $p : X \times X \to \mathbb{R}_+$ be a function satisfying the following condition: For all $x, y, z \in X$,

(P1) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then, p is said to be a *partial metric* or a *distance function* on X, and a pair (X, p) is called a *partial metric space*.

It is easy to see that a metric d is also a partial metric p, but the converse is not true in general case.

Example P1. Let $X = [0, \infty)$ and $p : X \times X \to \mathbb{R}_+$ be a function defined by

$$p(x, y) = \max\{x, y\}$$

for all $x, y \in X$. Then, p is a partial metric on X, but it is not a metric on X. Indeed, for any $x > 0$, we have $p(x, x) = x \neq 0$.

Example P2. Let $X = \{[a, b] : a, b \in \mathbb{R},\ a \leq b\}$ and $p : X \times X \to \mathbb{R}_+$ be a function defined by

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$$

for all $[a, b], [c, d] \in X$. Then, p is a partial metric on X, but it is not a metric on X. Indeed, $p([1, 2], [1, 2]) = 1$.

Recently, in 2014, Asadi et al. [50] extended the concept of a *partial metric* to the concept of an *m-metric* as follows:

For a nonempty set X and a function $m : X \times X \to \mathbb{R}_+$, the following notations are useful in the sequel:

(1) $m_{xy} := \min\{m(x, x), m(y, y)\}$;
(2) $M_{xy} := \max\{m(x, x), m(y, y)\}$.

Definition AKS. Let X be a nonempty set and $m : X \times X \to \mathbb{R}_+$ be a function satisfying the following condition: For all $x, y, z \in X$,

(MM1) $m(x, x) = m(y, y) = m(x, y)$ if and only if $x = y$;
(MM2) $m_{xy} \leq m(x, y)$;
(MM3) $m(x, y) = m(y, x)$;
(MM4) $m(x, y) - m_{xy} \leq [m(x, z) - m_{xz}] + [m(z, y) - m_{zy}]$.

Then, m is said to be an *m-metric*, and a pair (X, m) is called an *M-metric space*.

Also, they studied topological properties in such spaces and established some fixed point results in M-metric spaces, which are generalizations of Banach's and Kannan's fixed point theorems in the framework of partial metric spaces as follows:

Theorem ASKS1. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a mapping satisfying the following condition: There exists $k \in [0, 1)$ such that*

$$m(Tx, Ty) \leq km(x, y)$$

for all $x, y \in X$. Then, T has a unique fixed point.

Theorem ASKS2. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a mapping satisfying the following condition: There exists $k \in [0, \frac{1}{2})$ such that*

$$m(Tx, Ty) \leq k[m(x, Tx) + m(y, Ty)]$$

for all $x, y \in X$. Then, T has a unique fixed point.

Remark P1. According to the definitions of a p-metric and an m-metric,

(1) The condition (P1) in Definition M is same to the condition (MM1) in Definition AKS.
(2) The condition (P2) for $p(x, x)$ is expressed by just $p(y, y) = 0$ (we may have $p(y, y) \neq 0$) and so the condition (P2) is replaced by $\min\{p(x, x), p(y, y)\} \leq p(x, y)$, that is, the condition (MM2).
(3) The condition (P3) is same to the condition (MM3).
(4) Also, we improve the condition (P4) to the form of (MM4).

Thus, every p-metric is an m-metric, but the converse is not true as in the following examples.

Let (X, m) be an M-metric space. For all $x, y \in X$,

(1) $0 \leq M_{xy} + m_{xy} = m(x, x) + m(y, y)$;
(2) $0 \leq M_{xy} - m_{xy} = |m(x, x) - m(y, y)|$;

(3) $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

Example P3. Let $X := [0, \infty)$. Then, $m(x, y) = \frac{x+y}{2}$ on X is an m-metric.
The next examples show that m^s and m^w are metrics.

Example P4. Let (X, m) be an m-metric space and define two functions $m^w, m^s : X \times X \to \mathbb{R}_+$ by

$$m^w(x, y) := m(x, y) - 2m_{xy} + M_{xy}$$

and

$$m^s(x, y) := \begin{cases} m(x, y) - m_{xy}, & x \neq y, \\ 0, & x = y. \end{cases}$$

Then, m^w and m^s are metrics on X.

Let (X, m) be an M-metric space. For all $x, y \in X$,

(1) $m(x, y) - M_{xy} \leq m^w(x, y) \leq m(x, y) + M_{xy}$;
(2) $m(x, y) - m_{xy} \leq m^s(x, y) \leq m(x, y) + m_{xy}$;
(3) From (1) and (2), we have

$$|m^w(x, y) - m(x, y)| \leq M_{xy}$$

and

$$|m^s(x, y) - m(x, y)| \leq m_{xy}.$$

In the following example, we give an example of an m-metric which is not a p-metric:

Example P5. Let $X = \{1, 2, 3\}$ and define

$$m(1, 1) = 1, \quad m(2, 2) = 9, \quad m(3, 3) = 5,$$

$$m(1, 2) = m(2, 1) = 10, \quad m(1, 3) = m(3, 1) = 7,$$

$$m(3, 2) = m(2, 3) = 7.$$

Then, m is an m-metric, but it is not a p-metric.

Example P6. Let $X = [0, \infty)$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then, m is an m-metric, but it is not a p-metric. Indeed, $m(3, 3) = 3 > 2 = m(1, 3)$.

Example P7. Let $X = \{1, 2, 3\}$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \begin{cases} 1, & x = y = 1, \\ 9, & x = y = 2, \\ 5, & x = y = 3, \\ 10, & x, y \in \{1, 2\} \text{ and } x \neq y, \\ 7, & x, y \in \{1, 3\} \text{ and } x \neq y, \\ 8, & x, y \in \{2, 3\} \text{ and } x \neq y. \end{cases}$$

Then, m is an m-metric but it is not a p-metric. Indeed, $m(2, 2) = 9 > 8 = m(2, 3)$.

Thus, we obtain the following relation:

$$\boxed{\text{metric}} \Longrightarrow \boxed{\text{partial metric}} \Longrightarrow \boxed{\text{m-metric}}$$

Next, we show the relation between the Banach contraction in an m-metric space and the Banach contraction in a metric space (X, d).

Example P8. Let (X, d) be a metric space and $\phi : [0, \infty) \to [\phi(0), \infty)$ be a one-to-one and nondecreasing or strictly increasing mapping with $\phi(0)$ defined such that

$$\phi(x + y) \leq \phi(x) + \phi(y) - \phi(0)$$

for all $x, y \geq 0$. Then, $m(x, y) = \phi(d(x, y))$ is an m-metric.

Example P9. Let (X, d) be a metric space. Then, $m(x, y) = ad(x, y) + b$, where $a, b > 0$ is an m-metric, since we can put $\phi(t) = at + b$.

According to Example P9 and the Banach contraction (BC), since there exists $k \in [0, 1)$ such that, for all $x, y \in X$,

$$m(Tx, Ty) \leq km(x, y)$$

it follows that if $m(Tx, Ty) = ad(Tx, Ty) + b \leq kad(x, y) + kb$,

$$d(Tx, Ty) \leq kd(x, y) + \frac{b(k - 1)}{a},$$

which does not imply the ordinary Banach contraction in a metric space (X, d), that is, there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$, where $T : X \to X$ is a mapping.

Thus, this states that even if the m-metric m and the ordinary metric d have the same topology, then *the Banach contraction of the m-metric does not imply the Banach contraction of the ordinary metric d.*

Now, we give the concepts of a convergent sequence, a Cauchy sequence, and the completeness in M-metric spaces.

Let (X, m) be an m-metric space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if

$$\lim_{n\to\infty}[m(x_n, x) - m_{x_n x}] = 0;$$

(2) A sequence $\{x_n\}$ in X is called an m-*Cauchy sequence* if

$$\lim_{n,m\to\infty}[m(x_n, x_m) - m_{x_n x_m}], \quad \lim_{n,m\to\infty}[M_{x_n x_m} - m_{x_n x_m}]$$

exist (and are finite);

(3) A space X is said to be *complete* if every m-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n\to\infty}[m(x_n, x) - m_{x_n x}] = 0, \quad \lim_{n\to\infty}[M_{x_n x} - m_{x_n x}] = 0.$$

Example P10. Let $X = [0, \infty)$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then, (X, m) is a complete M-metric space since $(X, m^w) = ([0, \infty), \frac{3}{2}|\cdot|)$ is a complete metric space.

On the other hand, a basic question in the *stability* of functional equations is as follows:

When is it true that a function that approximately satisfies a functional equation must be close to an exact solution of the equation?

The stability problem of functional equations was initially studied from a question of Ulam [51] in 1940 on the stability of group homomorphisms:

Let G_1 be a group and G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, then we say that the equation of homomorphism $H(x \cdot y) = H(x) \cdot H(y)$ is *stable*.

In next year, Hyers [52] first gives some partial answer of Ulam's question for Banach spaces and then this type of stability is called the *Ulam–Hyers stability*.

This opened an avenue for further study and development of analysis in this field. Subsequently, many researchers have studied and extended the Ulam–Hyers stability in many ways (see [53, 54]).

Also, the notion of the *well-posedness* and the *limit shadowing property* of the fixed point problem has evoked much interest to many researchers, for example, De Blassi and Myjak [55], Reich and Zaslavski [56], Lahiri and Das [57], and Popa [58, 59].

Now, we show the following:

First, we define some types of the Ulam–Hyers stability, the well-posedness, and the limit shadowing property of the fixed point problem in an M-metric space which is a generalization of a metric space.

Second, we deal with the Ulam–Hyers stability, the well-posedness, and the limit shadowing property of the fixed point problem for Banach's and Kannan's contraction mappings in M-metric spaces.

Finally, we furnish two examples to illustrate our main results in this section.

The following lemma is useful to prove the main results in this paper:

Lemma AKS1. (Asadi et al. [50]) *Let* (X, m) *be an* M*-metric space. Then, we have the following*:

(1) $\{x_n\}$ *is an* m*-Cauchy sequence in*(X, m) *if and only if it is a Cauchy sequence in the metric space* (X, m^w).
(2) (X, m) *is complete if and only if the metric space* (X, m^w) *is complete. Furthermore, for a sequence* $\{x_n\}$ *in* X *and*$x \in X$, *we have*

$$\lim_{n\to\infty} m^w(x_n, x) = 0 \iff \lim_{n\to\infty} [m(x_n, x) - m_{x_n x}] = 0,$$

$$\lim_{n\to\infty} [M_{x_n x} - m_{x_n x}] = 0.$$

Moreover, two above assertions hold for m^s.

Now, we introduce the concepts of *Ulam–Hyers stability*, *well-posedness*, and the *limit shadowing property* of the fixed point problem in M-metric spaces. Also, we study the Ulam–Hyers stability, the well-posedness, and the limit shadowing property results for the fixed point problem of Banach's contractive mappings in M-metric spaces. Finally, we furnish one example to illustrate the first main result.

Definition SPL. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping.

(1) The fixed point problem

$$x = Tx \tag{FPP}$$

is said to be *Ulam–Hyers stable* if there exists $c > 0$ such that for any $\epsilon > 0$ and for each $w^* \in X$ which is an ϵ-solution of the fixed point problem (FPP), i.e., w^* satisfies the inequality

$$m(w^*, Tw^*) \leq \epsilon,$$

there exists a solution $x^* \in X$ of the problem (FPP) such that

$$m(x^*, w^*) \leq c\epsilon.$$

(2) The fixed point problem (FPP) of T is said to be *well-posed* if the following conditions hold:

(a) T has a unique fixed point x^* in X;
(b) For any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, we have $\lim_{n\to\infty} m(x_n, x^*) = 0$.

(3) The fixed point problem (FPP) of T is said to have the *limit shadowing property* in X if, for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, there exists $z \in X$ such that

$$\lim_{n\to\infty} m(T^n z, x_n) = 0.$$

Now, we give some results on Ulam–Hyers stability, well-posedness, and the limit shadowing property of the fixed point problem in M-metric spaces (see [60]):

Theorem PSCC1. *Let* (X, m) *be a complete* M*-metric space and* $T : X \to X$ *be Banach's contractive mapping satisfying the condition* (FPP). *Then, the following assertions hold*:

(1) *The fixed point problem of* T *is Ulam–Hyers stable*;
(2) *The fixed point problem of* T *is well-posed*;
(3) *The fixed point problem of* T *has the limit shadowing property in* X.

Now, we give one example to illustrate Theorem PSCC1.

Example PSCC1. Let $X = [0, \infty)$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then, (X, m) is a complete M-metric space. Define a mapping $T : X \to X$ by $Tx = \frac{x}{2}$ for all $x \in X$. For each $x, y \in X$, we obtain

$$m(Tx, Ty) = \frac{1}{2}\left(\frac{x}{2} + \frac{y}{2}\right) = \frac{1}{2} m(x, y)$$

and so T is Banach's contractive mapping.

First, we claim that the fixed point problem of T is Ulam–Hyers stable. Assume that $\epsilon > 0$ and $w^* \in X$ is an ϵ-solution of the fixed point problem of T, that is,

$$m(w^*, Tw^*) \leq \epsilon \implies \frac{1}{2}\left(w^* + \frac{w^*}{2}\right) \leq \epsilon \implies \frac{w^*}{2} \leq \frac{2}{3}\epsilon.$$

It is easy to see that $x^* = 0$ is a solution of the fixed point of T and

$$m(x^*, w^*) = m(0, w^*) = \frac{w^*}{2} \leq \frac{2}{3}\epsilon$$

and so the fixed point problem of T is Ulam–Hyers stable.

Second, we prove that the fixed point problem of T is well-posed. We can see that $x^* = 0$ is a unique fixed point of T. Now, we assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, that is,

$$\lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{x_n}{2}\right) = 0 \implies \lim_{n\to\infty} x_n = 0.$$

Then, we obtain

$$\lim_{n\to\infty} m(x_n, x^*) = \lim_{n\to\infty} m(x_n, 0) = \lim_{n\to\infty} \frac{x_n}{2} = 0$$

and so the fixed point problem of T is well-posed.

Finally, we show that the fixed point problem of T has the limit shadowing property in X. Suppose that $\{x_n\}$ is any sequence in X so that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. It follows that $\lim_{n\to\infty} x_n = 0$. We can see that there is $z = 0 \in X$ such that

$$\lim_{n\to\infty} m(T^n z, x_n) = \lim_{n\to\infty} m(0, x_n) = \lim_{n\to\infty} \frac{x_n}{2} = 0,$$

which implies that the fixed point problem of T has the limit shadowing property in X.

Next, we introduce another types of the Ulam–Hyers stability, the well-posedness, and the limit shadowing property of the fixed point problem in M-metric spaces. By using these concepts, we give the main result for the fixed point problem of *Kannan's contractive mappings* in M-metric spaces.

Definition PSCC. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping.

(1) The fixed point problem

$$x = Tx \tag{FPP}$$

is said to be *Ulam–Hyers stable type* (K) if there exists $c > 0$ such that for each $\epsilon > 0$, for each $w^* \in X$ which is an ϵ-solution of the fixed point equation (FPP), i.e., w^* satisfies the inequality

$$m(w^*, Tw^*) \leq \epsilon,$$

there exists a solution $x^* \in X$ of the equation (FPP) such that

$$m(x^*, w^*) - cm(x^*, x^*) \le c\epsilon.$$

(2) The fixed point problem of T is said to be *well-posed type* (K) if the following conditions hold:

(a) T has a unique fixed point x^* in X;
(b) There exists $c > 0$ such that for any sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, we have

$$\lim_{n\to\infty} m(x_n, x^*) = cm(x^*, x^*).$$

(3) The fixed point problem of T is said to have the *limit shadowing property type* (K) in X if there exists $c > 0$ such that for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, there exists $z \in X$ such that

$$\lim_{n\to\infty} m(T^n z, x_n) = cm(z, z).$$

Note that it is easy to see that the *Ulam–Hyers stability* of the fixed point problem implies the *Ulam–Hyers stability type* (K).

Now, we give the following result on another types of the Ulam–Hyers stability, the well-posedness, and the limit shadowing property of the fixed point problem in M-metric spaces:

Theorem PSCC2. *Let* (X, m) *be a complete* M*-metric space and* $T : X \to X$ *be Kannan's contractive mapping satisfying the condition* (FPP). *Then, the following assertions hold*:

(1) *The fixed point problem of* T *is Ulam–Hyers stable type* (K);
(2) *The fixed point problem of* T *is well-posed type* (K);
(3) *The fixed point problem of* T *has the limit shadowing property type* (K) *in* X.

Remark P2. In this survey, based on the fixed point results of Asadi [61], we have studied the Ulam–Hyers stability, the well-posedness, and the limit shadowing property for the fixed point problems of *Banach's and Kannan's contractive mappings* in M-metric spaces. We gave some examples to illustrate our results. However, several fixed point results established in M-metric spaces and other spaces have been studied by many mathematicians, for example, see Asadi's results in [50, 61].

Therefore, the author suggests to study the Ulam–Hyers stability, well-posedness, and limit shadowing of fixed point problems for *various kinds of nonlinear mappings in many distance spaces*.

4 Extensions of Banach's Fixed Point Theorem to Multi-valued Mappings

In 1969, Nadler, Jr. [62] extended Banach's fixed point theorem for a single-valued mapping in a complete metric space (X, d) to a multi-valued mapping in metric spaces. Since Nadler's theorem, many authors have improved, extended, and generalized this theorem in several ways.

In particular, in 1996, Kada et al. [63] proved the following theorem (*nonconvex minimization theorem*), which is very useful to prove *Ekeland's variational principle* and *Caristi's fixed point theorem* which generalize *Banach's fixed point theorem* for multi-valued mappings in metric spaces.

Theorem KST. *Let (X, d) be a complete metric space and $f : X \to (-\infty, +\infty)$ be a proper bounded below and lower semi-continuous function. Suppose that, for all $u \in X$ with*

$$\inf_{x \in X} f(x) < f(u),$$

there exists $v \in X$ such that $u \neq v$ and

$$f(v) + d(u, v) \leq f(u).$$

Then, there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$.

By using Theorem KST, we can prove the following *Ekeland's variational principle*, which was proved by Ekeland [64] in 1979.

Theorem E. *Let (X, d) be a complete metric space and $f : X \to (-\infty, +\infty)$ be a proper bounded below and lower semi-continuous function. Then, for any $\varepsilon > 0$ and $u \in X$ with*

$$f(u) \leq \inf_{x \in X} f(x) + \varepsilon,$$

there exists $v \in X$ such that

(a) $f(v) \leq f(u)$;
(b) $d(u, v) \leq 1$;
(c) $f(w) > f(v) - \varepsilon d(u, w)$ *for all $w \in X$ with $w \neq v$.*

By using Ekeland's variational principle, we can prove the following:

Corollary E1. *Let (X, d) be a complete metric space and $f : X \to (-\infty, +\infty)$ be a proper bounded below and lower semi-continuous function. Then, for any $\varepsilon > 0$, there exists $v \in X$ such that*

(a) $f(v) \leq \inf_{x \in X} +\varepsilon > 0$,
(b) $f(w) > f(v) - \varepsilon d(u, w)$ *for all $w \in X$.*

By using Corollary E1, we can prove *Banach's fixed point theorem*. In fact, let $f(w) = d(w, T(w))$ and choose ε with $0 < \varepsilon < 1 - L$. By Corollary E1, there exists

$v \in X$ such that

$$f(w) > f(v) - \varepsilon d(v, w)$$

for all $w \in X$. Putting $w = T(v)$, we have

$$\begin{aligned} d(v, T(v)) &\leq d(T(v), T(T(v))) + \varepsilon d(v, T(v)) \\ &\leq Ld(v, T(v)) + \varepsilon d(v, T(v)) \\ &= (L + \varepsilon)d(v, T(v)). \end{aligned}$$

If $v \neq T(v)$, then we have $1 \leq L + \varepsilon$, which contradicts $L + \varepsilon < 1$. Therefore, we have $v = T(v)$, that is, v is a fixed point of T. The uniqueness of the fixed point v follows easily.

By using Theorem KST, we can prove the following *Caristi's fixed point theorem*, which was proved by Caristi [18] in 1976:

Theorem C. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(x, T(x)) + f(T(x)) \leq f(x)$$

for all $x \in X$, where $f : X \to (-\infty, +\infty]$ be a proper bounded below and lower semi-continuous function. Then, there exists $z \in X$ such that $T(z) = z$ and $f(z) < +\infty$.

By using Caristi's fixed point theorem, we can prove a fixed point theorem for a multi-valued mapping in metric spaces.

Theorem C1. *Let (X, d) be a complete metric space and T be a mapping from X into 2^X, the power set of X, such that, for all $x \in X$, there exists $y \in T(x)$ satisfying*

$$f(y) + d(x, y) \leq f(x),$$

where $f : X \to (-\infty, +\infty]$ be a proper bounded below and lower semi-continuous function. Then, there exists $z \in X$ such that $z \in T(z)$ and $f(z) < +\infty$.

Let (X, d) be a metric space and $CB(X)$ be a family of all nonempty bounded closed subsets of X. For all $A, B \in CB(X)$, define the *Hausdorff metric* as follows:

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where

$$\delta(A, B) = \sup\{d(x, B) : x \in A\}$$

for all $A, B \in CB(X)$ and

$$d(x, B) = \inf\{d(x, y) : y \in B\}$$

for all $x \in X$ and $B \in CB(X)$.

By using Theorem KST, we can prove the following *Nadler's fixed point theorem* for a multi-valued mapping in a complete metric space (X, d):

Theorem N. *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be a* L*-contractive mapping, that is, there exists* $L \in (0, 1)$ *such that*

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$*. Then, there exists* $x_0 \in X$ *such that* $x_0 \in Tx_0$.

Now, we give one example to illustrate Nadler's fixed point theorem.

Example N1. Let $X = [0, 1]$ be a metric space with the usual metric and define a function $f : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\ -\frac{1}{2}x + 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Define a mapping $T : X \to CB(X)$ by

$$T(x) = \{0\} \cup \{f(x)\}$$

for all $x \in X$. Then, T is a multi-valued contraction and the fixed points of T are 0 and $\frac{2}{3}$.

Next, we consider to show the existence of fixed points of *multi-valued nonself-mappings* in a metric space (X, d).

In 1972, Assad and Kirk [65] first gave some sufficient conditions for a multi-valued nonself-mapping to have a fixed point.

Recall that a metric space (X, d) is called a *convex metric space* in the sense of Menger if, for all $x, y \in X$ with $x \neq y$, there exists $z \in X$, $z \neq x$ and $z \neq y$, such that

$$d(x, z) + d(z, y) = d(x, y).$$

Further, they showed that if K is a nonempty closed subset of X, $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ (∂K denotes the boundary of K) such that

$$d(x, z) + d(z, y) = d(x, y).$$

Also, they proved the following:

Theorem AK. *Let* (X, d) *be a complete and convex metric space*, K *be a closed subset of* X, *and* $T : K \to CB(X)$ *be a multi-valued mapping such that there exists* $\lambda \in (0, 1)$ *such that*

$$H(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in K$. *If* $Tx \subset K$ *for all* $x \in \partial K$, *then* T *has a fixed point in* K.

5 Existence of Common Fixed Points for Two Nonlinear Mappings

Let (X, d) be a metric space and $S, T : X \to X$ be two mappings.

(I) Two mappings S and T are said to be *commuting* on X (Jungck, [66]) if, for all $x \in X$,

$$STx = TSx.$$

Example J. Let $X = \mathbb{R}^2$ be a Euclidean *two*-dimensional space with the usual metric d. Define two mappings $S, T : X \to X$

$$S(p) = \Big(7x, \frac{y}{3} + 4\Big), \quad T(p) = \Big(11x, \frac{y}{2} + 3\Big)$$

for all $p = (x, y) \in X$. Then, we have

$$T(S(p)) = \Big(77x, \frac{y}{6} + 5\Big) = S(T(p))$$

and so S and T are commuting on X.

In 1976, Jungck [66] proved the following theorem:

Theorem J. *Let* T *be a continuous mapping from a complete metric space* (X, d) *into itself. Then,* T *has a fixed point in* X *if and only if there exist* $\alpha \in (0, 1)$ *and a mapping* $S : X \to X$ *such that*

(a) S *and* T *are commuting on* X;
(b) $S(X) \subset T(X)$;
(c) $d(S(x), S(y)) \leq \alpha d(T(x), T(y))$ *for all* $x, y \in X$.

Indeed, T *and* S *have a unique common fixed point in* X *if the conditions* (b) *and* (c) *hold.*

In 1982, Sessa [67] introduced the concept of weakly commuting mappings in a metric space (X, d) as follows:

(II) Two mappings T and S are said to be *weakly commuting* if

$$d(TSx, STx) \leq d(Tx, Sx)$$

for all $x \in X$.

Note that commuting mappings are weakly commuting, but the converse is not true.

Example J2. Let $X = [0, 1]$ with the usual metric d. Define two mappings $S, T : X \to X$ by

$$Sx = \frac{x}{2}, \quad Tx = \frac{x}{2+x}$$

for all $x, y \in X$, respectively. Then, we have

$$\begin{aligned} d(STx, TSy) &= \frac{x}{4+x} - \frac{x}{4+2x} = \frac{x^2}{(4+x)(4+2x)} \\ &\le \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx) \end{aligned}$$

for all $x \in X$ and so S and T are weakly commuting, but they are not commuting because, for all $x \in X$, we have

$$TSx = \frac{x}{x+4} \neq \frac{x}{2x+4} = STx.$$

In 1986, Jungck [68] introduced the concept of compatible mappings in a metric space (X, d) as follows:

(III) Two mappings $S, T : X \to X$ are said to be *compatible* on X if

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0$$

when there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Remark COM. (1) The weak commutativity does not imply the existence of a sequence of points satisfying the condition of compatibility.

(2) If S and T are compatible mappings, then $d(STx, TSx) = 0$ whenever $d(Sx, Tx) = 0$ for some $x \in X$.

(3) Weakly commuting mappings are compatible, but the converse is not true.

Example J3. Let $X = (-\infty, +\infty)$ be the set of real numbers with the usual metric d. Define two mappings $S, T : X \to X$ by

$$Sx = x^3, \quad Tx = 2 - x$$

for all $x, y \in X$, respectively. From

$$d(Sx_n, Tx_n) = |x_n - 1||x_n^2 + x_n + 2| \to 0 \iff x_n \to 1,$$

we have

$$\lim_{n\to\infty} d(STx_n, TSx_n) = \lim_{n\to\infty} 6|x_n - 1|^2 = 0$$

as $x_n \to 1$. Thus, S and T are compatible, but they are not weakly commuting because, if $x = 0$ in X, we have

$$d(STx, TSx) = 6 > 2 = d(Ax, Bx).$$

In 1998, Jungck and Rhoades [69] introduced the concept of weakly compatible mappings in a metric space (X, d) as follows:

(IV) Two mappings S and T are said to be *weakly compatible* if they are commuting at their coincident points, that is, if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

Note that compatible mappings are weakly compatible, but the converse is not true.

Example J4. Let $X = (-\infty, +\infty)$ be the set of real numbers with the usual metric d and $E = [0, 1]$. Define two mappings $S, T : E \to E$ by

$$Sx = \begin{cases} 0, & 0 \le x < \frac{2}{3}, \\ \frac{4}{3} - x, & \frac{2}{3} \le x \le 1, \end{cases}$$

and

$$Tx = \begin{cases} \frac{1}{2}, & 0 \le x < \frac{2}{3}, \\ 1 - \frac{1}{2}x, & \frac{2}{3} \le x \le 1. \end{cases}$$

Then, S and T are weakly compatible on E, but they are not compatible on E.

In 1993, Jungck et al. [70] introduced the concept of compatible mappings of type (A) in a metric space (X, d) as follows:

(V) Two mappings $S, T : X \to X$ are said to be *compatible of type* (A) on X if

$$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0, \quad \lim_{n\to\infty} d(STx_n, TTx_n) = 0$$

when there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Remark COM1. (1) If S and T are continuous, then S and T are compatible if and only if they are compatible of type (A).

(2) If S and T are not continuous, (1) is not true.

In 1994, Pant [71] introduced the following:

(VI) Two mappings S and T are said to be R-*weakly commuting* if there exists a real number $R > 0$ such that

$$d(STx, TSx) \le Rd(Sx, Tx)$$

for all $x \in X$.

Note that If $R = 1$, then S and T are weakly commuting.

Under the assumption of R-weak commutativity, Pant [71] proved the following common fixed point theorem:

Theorem P. *Let (X, d) be a complete metric space, S and T be R-weakly commuting self-mappings of X satisfying the condition*:

(a) $d(Sx, Sy) \leq \gamma(d(Tx, Ty))$ *for all* $x, y \in X$, *where* $\gamma : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* $\gamma(t) < t$ *for each* $t > O$;
(b) $S(X) \subset T(X)$;
(c) *either S or T is continuous*.

Then, S and T have a unique common fixed point in X.

Simple statements and elegant proofs of Theorem P reveal the fact that Theorem P does not hold if we allow both the mappings S and T to be *discontinuous* on X or the space X is not *complete*.

To this end, we have the following example:

Example PCK. Let $X = \{O, 1, \frac{1}{2}, \frac{1}{2^2}, \ldots\}$ be a metric space with the usual metric d. Define two mappings $S, T : X \to X$ by

$$S(0) = \frac{1}{2^2}, \quad S\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+2}}, \quad T(0) = \frac{1}{2}, \quad T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$$

for each $n \geq 0$, respectively. Clearly, (X, d) is complete and $S(X) \subset T(X)$. Since S and T are commuting on X, they are R-weakly commuting for $R > 0$. Define $\gamma(t) = \frac{1}{2}t$ for all $t > 0$. Then, S and T both are not continuous at 0. Hence, all the conditions of Theorem P are satisfied except the *continuity* of either S or T, but neither S nor T have a common fixed point in X.

Now, there arises a natural question:

"*How Theorem P can be improved to the setting of noncomplete metric spaces and without the continuity of S and T over the whole space X*?"

In 1997, Pathak et al. [72] gave the partial answer. It seems that Theorem P can be improved in two ways:

Either imposing certain restrictions on the space X or by replacing the notion of R-weakly commutativity of mappings with certain improved notion.

Here, we chosen the second option. In this perspective, we introduced the following definitions:

(VII) (1) Two mappings S and T are said to be R-*weakly commuting of type* (A_f) if there exists a real number $R > 0$ such that

$$d(STx, TTx) \leq Rd(Sx, Tx)$$

for all $x \in X$.

(2) Two mappings S and T are said to be *R-weakly commuting of type* (A_g) if there exists a real number $R > 0$ such that

$$d(TSx, SSx) \leq Rd(Sx, Tx)$$

for all $x \in X$.

In [72], we can find suitable examples which show that R-weakly commuting mappings are not necessarily R-weakly commuting of type (A_f) (see an example of Pant's paper [71]).

Also, we proved the following:

Theorem PCK. *Let (X, d) be a metric space, C be a subset of X, S, and T be R-weakly commuting self-mappings of type (A_g) or type (A_f) of X satisfying the following condition*:

(a) $d(Sx, Sy) \leq \gamma(d(Tx, Ty))$ *for all* $x, YinC$, *where* $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function such that* $\gamma(t) < t$ *for each* $t > 0$;
(b) $S(C) \subset T(C)$;
(c) $S(C)$ *is complete*;
(d) *either* S or T *is continuous*.
Then, S *and* T *have a unique common fixed point in* X.

In [72], we can find some examples to illustrate Theorem PCK, and Theorem PCK improves, extends, and generalizes the corresponding fixed point theorems given by some authors.

6 Improvement of Banach's Fixed Point Theorem

In 1922, since Banach's fixed point theorem, most of fixed point theorems for nonlinear mappings in a metric space (X, d) proved by many authors have required the following conditions:

(1) the *completeness* of the given space X;
(2) the *closedness* and *convexity* of a subset C of X;
(3) the *continuity* of one mapping or more mappings;
(4) the *containments of the range* of the given mappings in metric spaces.

Recently, by using the following properties, some authors have obtained some fixed point theorems without using the conditions mentioned above.

(I) Two mappings S and T are said to satisfy the (E, A)*-property* (Aamri and Moutawakil, [73]) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

(II) Two mappings S and T are said to satisfy the *common limit in the range of S* (shortly, the $(CLR-S)$-*property*) (Sintunavarat and Kumam, [74]) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Su$$

for some $u \in X$.

Now, we give one example satisfying the $(CLR-S)$-property as follows:

Example CLR. Let $X = [0, \infty)$ be the set of real numbers with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define two mappings $S, T : X \to X$ by

$$Sx = \frac{x}{2}, \quad Tx = 2x$$

for all $x \in X$. Consider a sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$ for each $n \geq 1$. Then, we have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0 = S(0).$$

Therefore, S and T satisfy the $(CLR-S)$-property.

If two mappings $S, T : X \to X$ are noncompatible, then there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or nonexistent. Thus, two noncompatible mappings S and T satisfy the (E, A)-property.

(III) Two mappings S and T are said to be *occasionally weakly compatible* (shortly, (owc)-*property*) (Bouhadjera, [75]) if there exists a point $u \in X$ such that

$$Su = Tu, \quad STu = TSu.$$

Now, we give one example of occasionally weakly compatible mappings as follows:

Example OWC. Let $X = [0, \infty)$ be the set of real numbers with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define two mappings $S, T : X \to X$ by

$$Sx = \begin{cases} 4, & 0 \leq x < 1, \\ x^4, & 1 \leq x < \infty, \end{cases}$$

and

$$Tx = \begin{cases} 3, & 0 \le x < 1, \\ \frac{1}{x^4}, & 1 \le x < \infty. \end{cases}$$

Then, $S(1) = T(1) = 1$ and $ST(1) = 1 = TS(1)$ and so the mappings S and T are occasionally weakly compatible mappings.

7 Extensions of the Picard Iterative Sequence

[A] Recently, many authors introduced the following iterations, which are generalizations of Picard iteration:

Let X be a normed linear space and $S, T : X \to X$ be two nonlinear mappings.

(1) The *Picard iteration* (Picard 1890): The sequence $\{x_n\}$ is defined by

$$x_{n+1} = Tx_n$$

for each $n \ge 0$;

(2) The *Jungck iteration* (Jungck 1976): The sequence $\{x_n\}$ is defined by

$$Sx_{n+1} = Tx_n$$

for each $n \ge 0$;

(3) The *Mann iteration* (Mann 1953): The sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n$$

for each $n \ge 0$, where $\{\lambda_n\}$ is a real sequence satisfying $0 \le \lambda_n < 1$ for each $n \ge 0$;

(4) The *Krasnoselskij iteration* (Krasnoselskij 1955): The sequence $\{x_n\}$ is defined by

$$x_{n+1} = T_{\frac{1}{2}}x_n = \frac{1}{2}x_n + \frac{1}{2}Tx_n = \frac{1}{2}(x_n + Tx_n)$$

for each $n \ge 0$;

(5) The *Schäefer iteration* (Schäefer 1957): The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n \end{cases}$$

for each $n \ge 0$, where $\lambda \in (0, 1)$.

(6) The *Halpern iteration* (Halpern 1967): For any fixed $u, x_0 \in X$, the sequence $\{x_n\}$ is defined by

$$x_n = \lambda_n u + (1 - \lambda_n)Tx_n$$

for each $n \geq 0$, where $\{\lambda_n\}$ is a real sequence satisfying $0 \leq \lambda_n \leq 1$ for each $n \geq 0$;

(7) The *Ishikawa iteration* (Ishikawa 1974): The sequence $\{x_n\}$ is defined by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \alpha_n,\ \beta_n < 1$;

(8) The *Noor iteration* (Noor 2000): The sequence $\{x_n\}$ is defined by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n,\ \beta_n < 1$;

(9) The *Moudafi viscosity iteration* (Moudafi 2000): The sequence $\{x_n\}$ is defined by

$$x_n = \lambda_n f(x_n) + (1-\lambda_n)T x_n$$

for each $n \geq 0$, where $f : X \to X$ is a contractive mapping and $\{\lambda_n\}$ is a real sequence satisfying $0 \leq \lambda_n \leq 1$ for each $n \geq 0$;

(10) The *Singh–Bhatnagar–Mishra iteration* (Singh et al. 2005): The sequence $\{x_n\}$ is defined by

$$Sx_{n+1} = (1-\lambda_n)Sx_n + \lambda_n T x_n$$

for each $n \geq 0$, where $\{\lambda_n\}$ is a real sequence satisfying $0 \leq \lambda_n < 1$ for each $n \geq 0$;

(11) The *Ishikawa-type iteration* (Agarwal et al. 2007): The sequence $\{x_n\}$ is defined by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \alpha_n,\ \beta_n < 1$;

(12) The *Jungck–Ishikawa iteration* (Olatinwo 2008): The sequence $\{x_n\}$ is defined by

$$\begin{cases} Sy_n = (1-\beta_n)Sx_n + \beta_n T x_n, \\ Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \alpha_n,\ \beta_n < 1$;

(13) The *Jungck–Noor iteration* (Olatinwo 2008): The sequence $\{x_n\}$ is defined by

$$\begin{cases} Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n, \\ Sy_n = (1-\beta_n)Sx_n + \beta_n Tz_n, \\ Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n, \ \beta_n < 1$;

(14) The *Kirk–Mann iteration* (Olatinwo 2009): The sequence $\{x_n\}$ is defined by

$$x_{n+1} = \sum_{i=1}^{k} \alpha_{n,i} T^i x_n$$

for each $n \geq 0$, where $\alpha_{n,i} \in [0, 1]$, $\alpha_{n,0} \neq 0$, $\sum_{i=1}^{k} \alpha_{n,i} = 1$, and k is a fixed number.

(15) The *Kirk–Ishikawa iteration* (Olatinwo 2009): The sequence $\{x_n\}$ is defined by

$$\begin{cases} y_n = \sum_{j=0}^{s} \beta_{n,j} T^j x_n, \\ x_{n+1} = \alpha_{n,0} x_n + \sum_{i=1}^{k} \alpha_{n,i} T^i y_n \end{cases}$$

for each $n \geq 0$, where k, s are fixed integers with $k \geq s$, $\alpha_{n,i}, \beta_{n,j} \in [0, 1]$, $\alpha_{n,0} \neq 0$, $\alpha_{n,0} \neq 0$ and

$$\sum_{i=1}^{k} \alpha_{n,i} = \sum_{j=0}^{s} \beta_{n,j} = 1;$$

(16) The *SP-iteration* (Suantai 2011): The sequence $\{x_n\}$ is defined by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n, \ \beta_n < 1$;

(17) The *Jungck- SP-iteration* (Chugh and Kumar 2011): The sequence $\{x_n\}$ is defined by

$$\begin{cases} Sz_n = (1-\gamma_n)Sy_n + \gamma_n Tx_n, \\ Sy_n = (1-\beta_n)Sz_n + \beta_n Tz_n, \\ Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n, \ \beta_n < 1$;

(18) The *Jungck–Kirk–Noor iteration* (Chugh and Kumar 2012): The sequence $\{x_n\}$ is defined by

$$\begin{cases} Sz_n = \gamma_{n,0} Sx_n + \sum_{k=1}^{t} \gamma_{n,k} T^k x_n, \\ Sy_n = \beta_{n,0} Sx_n + \sum_{j=1}^{s} \beta_{n,j} T^j z_n, \\ Sx_{n+1} = \alpha_{n,0} Sx_n + \sum_{i=1}^{k} \alpha_{n,i} T^i y_n \end{cases}$$

for each $n \geq 0$, where k, s, t are fixed integers with $k \geq s \geq t$, $\alpha_{n,i}, \beta_{n,j}, \gamma_{n,k} \in [0, 1]$, $\alpha_{n,0} \neq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,0} \neq$ and

$$\sum_{i=0}^{k} \alpha_{n,i} = \sum_{j=0}^{s} \beta_{n,j} = \sum_{k=0}^{t} \gamma_{n,k} = 1;$$

(19) The *Jungck-CR-iteration* (Hussain et al. 2013): The sequence $\{x_n\}$ is defined by

$$\begin{cases} Sz_n = (1 - \gamma_n) Sx_n + \gamma_n T x_n, \\ Sy_n = (1 - \beta_n) Sx_n + \beta_n T z_n, \\ Sx_{n+1} = (1 - \alpha_n) Sy_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n, \beta_n \leq 1$;

(20) The *Noor-type iteration* (I) (Thakur et al. 2014): The sequence $\{x_n\}$ is defined by

$$\begin{cases} z_n = (1 - \gamma_n) x_n + \gamma_n T x_n, \\ y_n = (1 - \beta_n) z_n + \beta_n T z_n, \\ x_{n+1} = (1 - \alpha_n) T y_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n, \beta_n < 1$;

(21) The *Picard S-iteration* (Thakur et al. 2014): The sequence $\{x_n\}$ is defined by

$$\begin{cases} z_n = (1 - \gamma_n) x_n + \gamma_n T x_n, \\ y_n = (1 - \beta_n) T x_n + \beta_n T z_n, \\ x_{n+1} = T y_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$ and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \beta_n < 1$;

(22) The *Noor-type iteration* (II) (Abbas et al. 2014): The sequence $\{x_n\}$ is defined by

$$\begin{cases} z_n = (1 - \gamma_n) x_n + \gamma_n T x_n, \\ y_n = (1 - \beta_n) T x_n + \beta_n T z_n, \\ x_{n+1} = (1 - \alpha_n) T y_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\{\gamma_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n, \beta_n < 1$.

Recently, many authors have proved strong and weak convergence theorems in Hilbert spaces and Banach spaces for many kinds of nonlinear mappings.

[B] Comparing the Convergence Rates of the Iterations to a Fixed Point

Recently, some authors have compared the rates of the convergence of some kinds of the iterations. First, we give the definitions of the convergence rates as follows:

Definition CR1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers which converge to a and b, respectively. Assume that there exists a real number l such that

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

(1) If $l = 0$, then we say that $\{a_n\}$ *converges faster* to a than $\{b_n\}$ to b;
(2) If $0 < l < 1$, then we say that $\{a_n\}$ and $\{b_n\}$ have the *same rate of convergence*.

Definition CR2. Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iterations converging to the same fixed point p (say) with error estimates:

$$\|u_n - p\| \leq a_n, \quad \|v_n - p\| \leq b_n$$

for each $n \geq 0$, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to 0. If $\{a_n\}$ converges faster than $\{b_n\}$, then we say that $\{u_n\}$ *converges faster* than $\{v_n\}$ to p.

For more details on the convergence rates of the iterations mentioned above, see the following papers:

(1) B. E. Rhoades and Z. Xue, Comparison of the rate of convergence among Picard, Mann, Ishikawa, and Noor iterations applied to quasi-contractive maps, Fixed Point Theory Appl. 2010, 2010:169062.
(2) N. Hussain, A. Rafiq and B. Damjanović, R. Lazović, On rate of convergence of various iterative schemes, Fixed Point Theory Appl. 2010, 2010:169062.
(3) W. Phuengrattana and S. Suantai, Strong convergence theorems and rate of convergence of multi-step iterative methods for continuous mappings on an arbitrary interval, Fixed Point Theory Appl. 2012, 2012:9.
(4) W. Phuengrattana and S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous mappings on an arbitrary interval, J. Comput. Appl. Math. 235(2011), 3006–3014.
(5) Y. Song and X. Liu, Convergence comparison of several iteration algorithms for the common fixed point problems, Fixed Point Theory Appl. 2009, 2009: 824374.
(6) A. Alotaibi, V. Kumar and N. Hussain, Convergence comparison and stability of Jungck-Kirk-type algorithms for common fixed point problems, Fixed Point Theory Appl. 2013, 2013:173.

(7) S. L. Singh, C. Bhatnagar and S. N. Mishra, Stability of Jungck-type iterative procedures, Internat. J. Math. Math. Sci. 19(2005), 3035–3043.
(8) M. O. Olatinwo and C. O. Imoru, Some convergence results for the Jungck-Mann and the Jungck-Ishikawa iteration processes in the class of generalized Zamfirescu operators, Acta Math. Univ. Comen. LXXVII(2008), 299–304.
(9) M. O. Olatinwo, A generalization of some convergence results using the Jungck-Noor three-step iteration process in an arbitrary Banach space, Fasc. Math. 40(2008), 37–43.
(10) M. O. Olatinwo, Some stability results for two hybrid fixed point iterative algorithms in normed linear space, Mat. Vesnik 61(2009), 247–256.
(11) R. Chugh and V. Kumar, Stability of hybrid fixed point iterative algorithms of Kirk-Noor type in normed linear space for self and nonself operators, Internat. J. Contemp. Math. Sci. 7(24)(2012), 1165–1184.
(12) R. Chugh and V. Kumar, Strong convergence and stability results for Jungck-SP iterative scheme, Internat. J. Comput. Appl. 36(12)(2011), 40–46.
(13) R. Chugh and V. Kumar, On the rate of convergence of some new modified iterative schemes, Amer. J. Comput. Math. 3(2013), 270–290.
(14) N. Hussain, R. Chugh, V. Kumar and A. Rafiq, On the rate of convergence of Kirk-type iterative schemes, J. Appl. Math. 2012, Article ID 526503 (2012).
(15) N. Hussain, V. Kumar and M.A. Kutbi, On the rate of convergence of Jungck-type iterative schemes, Abstr. Appl. Anal. 2013, Article ID 132626 (2013).
(16) Y. Qing and B. E. Rhoades, Comments on the rate of convergence between Mann and Ishikawa iterations applied to Zamfirescu operators, Fixed Point Theory Appl. 2008, Article ID 387504 (2008).
(17) V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, Fixed Point Theory Appl. 2(2004), 97–105.
(18) V. Berinde and M. Berinde, The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings, Carpath. J. Math. 21(2005), 13–20.
(19) V. Kumar, A. Latif, A. Rafiq and N. Hussain, S-iteration process for quasi-contractive mappings, J. Inequal. Appl. 2013, 2013:206.
(20) S. S. Chang, Y. J. Cho and J. K. Kim, The equivalence between the convergence of modified Picard, modified Mann, and modified Ishikawa iterations, Math. Comput. Model. 37(2003), 985–991.
(21) S. S. Chang, J. K. Kim and Y. J. Cho, On the equivalence for the convergence of Mann iteration and Ishikawa iteration with mixed errors for Lipschitz strongly pseudo-contractive mappings, Comm. Appl. Nonlinear Anal. 12(2005), 79–88.
(22) O. Popescu, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, Math. Commun. 12(2007), 195–202.
(23) G. V. R. Babu, K. N. V. V Vara Prasad and V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, Fixed Point Theory Appl. 2004 2004:716359.
(24) G. V. R. Babu and K. N. V. V Vara Prasad, Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators, Fixed Point Theory Appl. 2006, 2006: 49619.

(25) S. Fathollahi, A. Ghiura, M. Postolache and S. Rezapour, A comparative study on the convergence rate of some iteration methods involving contractive mappings, Fixed Point Theory Appl. 2015, 2015:234.

(26) S. Akbulut and M. Ozdemir, Picard iteration converges faster than Noor iteration for a class of quasi-contractive operators, Chiang Mai J. Sci. 39(2012), 688–692.

(27) M. Abbas and T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesnik 66(2014), 223–234.

(28) O. Olaleru, On the convergence rates of Picard, Mann, and Ishikawa iterations of generalized contractive operators, Studia Univ., "Babes Bolyai", Math. LIV(2009), 103–114.

(29) Q. L. Dong, S. He and X. C. Liu, Rate of convergence of Mann, Ishikawa and Noor iterations for continuous functions on an arbitrary interval, J. Inequal. Appl. 2013, 2013:269.

(30) S. M. Soltuz, The equivalence of Picard, Mann and Ishikawa iterations dealing with quasi-contractive operators, Math. Commun. 10(2005), 81–88.

(31) V. Kumar, Comments on convergence rates of Mann and Ishikawa schemes for generalized contractive operators, Internat. J. Math. Anal. 7(2013), 1317–1321.

(32) O. Celikler, Convergence analysis for a modified SP-iterative Method, Sci. World J. 2014, Article ID 84054 (2014).

[C] Some Results on Comparisons of the Convergence Rates of the Iterations mentioned above

(1) We know that, in general, the speed of some kinds of iterations depends on the control conditions by numerical examples (see A. Alotaibi, V. Kumar and N. Hussain, Fixed Point Theory Appl., 2013);

(2) Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators (see V. Brinde, Fixed Point Theory Appl. 2004);

(3) Picard iteration converges faster than Noor iteration for a class of quasi-contractive operators (see S. Akbulut and M. Ozdemir, Chiang Mai J. Sci., 2012);

(4) Some examples to show that Ishikawa iteration is faster than Mann iteration for a certain class of quasi-contractive operators (see Y. Qing and B. E. Rhoades, Fixed Point Theory Appl., 2008);

(5) The SP-iterative scheme with error terms converges faster than Ishikawa and Noor iterative schemes for accretive type mappings (see R. Chugh and V. Kumar, Internat. J. Comput. Math., 2013);

(6) The SP-iteration converges faster than the Mann, Ishikawa and Noor iterations (see W. Phuengrattana and S. Suantai, J. Comput. Appl. Math., 2011);

(7) The Jungck-Kirk iteration converge faster than the corresponding Jungck iteration to the common fixed point of T and S (see A. Alotaibi, V. Kumar and N. Hussain, Fixed Point Theory Appl., 2013);

(8) We can see the convergence rates of some iterations mentioned above in the paper by A. Alotaibi, V. Kumar and N. Hussain, Fixed Point Theory Appl., 2013;

(9) The Krasnoselskij iteration converges faster than the Mann, Ishikawa and Noor iterations (see B. E. Rhoades and Z. Xue, Fixed Point Theory Appl., 2010);
(10) A new faster iteration process applied to constrained minimization and feasibility problems (see M. Abbas and T. Nazir, Mat. Vesnik, 2014);
(11) In some papers, we can show some *Open Problems and Nice Examples* on comparisons of the convergence rates.

In particular, in the following papers, we can see very nice examples and some open problems on comparisons of the convergence rates of some kinds of iterations:

(1) A. Alotaibi, V. Kumar and N. Hussain, Convergence comparison and stability of Jungck-Kirk-type algorithms for common fixed point problems, Fixed Point Theory Appl. 2013, 2013:173;
(2) W. Phuengrattana and S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous mappings on an arbitrary interval, J. Comput. Appl. Math. 235(2011), 3006–3014.

8 Applications of Banach's Fixed Point Theorem

[A] Consider an usual equation $F(x) = 0$.

Let $F(a) < 0$, $F(b) > 0$, and $0 < C_1 \leq F'(x) \leq C_2$ for all $x \in [a, b]$. Define a function $f : [a, b] \to [a, b]$ by

$$f(x) = x - \alpha F(x)$$

for all $x \in [a, b]$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, the equations $f(x) = x$ and $F(x) = 0$ are equivalent. Since $f'(x) = 1 - \alpha F'(x)$, it follows that

$$1 - \alpha C_2 \leq f'(x) \leq 1 - \alpha C_1$$

for all $x \in [a, b]$. So, we can choose α such that $|f'(x)| \leq \lambda$ for some $\lambda < 1$ and $f(x) \in [a, b]$ for all $x \in [a, b]$. Then, we have

$$|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|$$

for all $x_1, x_2 \in [a, b]$. Hence, it follows that f is a Banach contraction and so, by Banach's fixed point theorem, a sequence $\{x_n\}$ defined by, for any $x_0 \in [a, b]$,

$$x_1 = f(x_0),\ x_2 = f(x_1),\ \ldots,\ x_{n+1} = f(x_n),\ \ldots$$

converges to a unique solution of the equation $f(x) = x$, that is, to a solution of the equation $F(x) = 0$.

[B] Consider the $m \times n$ system $Ax = b$ for all $m, n \geq 1$, that is,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

From the $m \times n$ system $Ax = b$ for all $m, n \geq 1$, we can consider the following cases:

(1) If $m < n$, that is, in the system $Ax = b$, the unknowns are more than the equations, then the system $Ax = b$ has many solutions;
(2) If $m = n$, that is, in the system $Ax = b$, the unknowns and the equations are same, then the system has a unique solution;
(3) If $m > n$, that is, in the system $Ax = b$, the equations are more than the unknowns, then the system is usually inconsistent.

Thus, in general, we cannot expect to find a vector (solution) $x \in \mathbb{R}^n$ for which $Ax = b$. But we can look for a vector x for which Ax is "closest" to b by using the *linear least squares*.

For the case (3), consider the $m \times n$ system of the equations $Ax = b$. For each $x \in \mathbb{R}^n$, we can form a residual

$$r(x) = b - Ax.$$

The distance between b and ax is given by

$$\|b - Ax\| = \|r(x)\|.$$

Now, we wish to find a vector $x \in \mathbb{R}^n$ for which $\|r(x)\|$ will be a *minimum*. In fact,

$$\text{Minimizing } \|r(x)\| \text{ is equivalent to Minimizing } \|r(x)\|^2$$

A vector x that accomplishes this is called a *least squares solution* to the system $Ax = b$.

For the case (3), we will discuss some *minimum-norm problems* again.

Now, we consider the case (2) by applying Banach's fixed point theorem.

The $m \times n$ system of the equations $Ax = b$ can be written as follows:

$$\begin{cases} x_1 = (1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n + b_1 \\ x_2 = -a_{21}x_1 + (1 - a_{22})x_2 - a_{23}x_3 - \cdots - a_{2n}x_n + b_2 \\ x_3 = -a_{31}x_1 - a_{32}x_2 + (1 - a_{33})x_3 - \cdots - a_{3n}x_n + b_3 \\ \quad \cdots \\ x_n = -a_{n1}x_1 - a_{n2}x_2 - a_{n3}x_3 - \cdots + (1 - a_{nn})x_n + b_n. \end{cases}$$

Put $\alpha_{ij} = -a_{ij} + \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then, the above system is equivalent to the following:

$$x_i = \sum_{j=1}^{n} \alpha_{ij} x_j + b_i \quad (i = 1, 2, 3, \ldots, n).$$

Thus, if $x = (x_1, x_2, \ldots, x_n),\ b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, then the above system is equivalent to the following:

$$x = Ax + b.$$

In other words, the problem $x = Ax + b$ is to find the fixed point of the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(x) = Ax + b$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

If T is a contractive mapping, then we can use Banach's fixed point theorem and obtain the unique solution of $T(x) = x$ by the method of successive approximation.

The conditions under which T is a contractive mapping depend on the choice of the metric on $X = \mathbb{R}^n$.

Theorem SE. *Let $X = \mathbb{R}^n$ be a metric space with the metric $d_\infty(x, y) = \max_{i \leq i \leq n} |x_i - y_i|$. If*

$$\sum_{j=1}^{n} |\alpha_{ij}| \leq \alpha < 1 \quad (i = 1, 2, 3, \ldots, n),$$

then the $n \times n$ system of the equations $Ax = b$ has a unique solution.

[C] Consider the following (ordinary) differential equation:

$$\begin{cases} \frac{dy}{dx} = f(x, y), \\ y(x_0) = y_0. \end{cases} \tag{DE}$$

By using Banach's fixed point theorem, we can show the following:

Theorem DE. *Let $f(x, y)$ be a continuous function on $A = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$ satisfies the following*:

$$|f(x, y) - f(x, y')| \leq \alpha |y - y'|$$

for all $y, y' \in [c, d]$. *Further, let* (x_0, y_0) *be an interior point of* A. *Then, the differential equation* **(DE)** *with the given initial condition has a unique solution.*

[D] Let $K(x, y)$ be a continuous function on $[a, b] \times [a, b]$ and $\phi(x)$ be a continuous function on $[a, b]$. Consider the following integral equation:

$$f(x) = \phi(x) + \lambda \int_a^x K(x, y) f(y) dy \tag{VE}$$

for all $x \in [a, b]$, where λ is a parameter, which is called the **Volterra equation**.

By using Banach's fixed point theorem, we can show the following:

Theorem IE. *For each* $\lambda \in \mathbb{R}$, *the Volterra equation* **(VE)** *has a unique solution* f *which is continuous on* $[a, b]$.

Proof. Let $X = C[a, b]$ be the set of all continuous functions on $[a, b]$ with the uniform metric. Since K is continuous, there exists $k > 0$ such that

$$|K(x, y)| \le k$$

for all $x, y \in [a, b]$. Define a mapping $T : f \longmapsto T(f)$ on X by

$$T(f(x)) = \phi(x) + \lambda \int_a^x K(x, y) f(y) dy.$$

For all $f, g \in X$, we have

$$\begin{aligned} |T(f(x)) - T(g(x))| &= \left| \lambda \int_a^x K(x, y) |f(y) - g(y)| dy \right| \\ &\le |\lambda| k (x - a) d(f, g) \end{aligned}$$

for all $x \in [a, b]$. Since $T^2(f) - T^2(g) = T(T(f) - T(g))$, we have

$$\begin{aligned} |T^2(f(x)) - T^2(g(x))| &= \left| \lambda \int_a^x K(x, y) |T(f(y)) - T(g(y))| dy \right| \\ &\le |\lambda| \int_a^x |K(x, y)| |\lambda| k (y - a) d(f, g) dy \\ &\le |\lambda|^2 k^2 \int_a^x (y - a) dy d(f, g) \\ &\le \frac{|\lambda|^2 k^2 (x - a)^2}{2} d(f, g). \end{aligned}$$

Continuing this iterative process, we have

$$|T^n(f(x)) - T^n(g(x))| \leq \frac{|\lambda|^n k^n (x-a)^n}{n!} d(f,g)$$

for all $x \in [a,b]$. Therefore, we have

$$|T^n(f) - T^n(g)| \leq \frac{|\lambda| k(x-a)|^n}{n!} d(f,g).$$

Since $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any $r \in \mathbb{R}$, we conclude that there exists a positive integer n such that T^n is a Banach contraction. For sufficiently large n, we have

$$\frac{|\lambda| k(x-a)|^n}{n!} < 1.$$

Therefore, by Banach's fixed point theorem, there exists a unique solution $f \in X$ such that $T(f) = f$. Obviously, if $T(f) = f$, then f solves the integral equation (VE). This completes the proof.

9 Converses of Banach's Fixed Point Theorem

Now, we consider the converse problem of Banach's fixed point theorem. The most elegant result in this diction is due to Bessaga [76] in 1959.

Theorem BE. *Suppose that M is an arbitrary nonempty set and $T : M \to M$ has the property that T and each of its iterates T^n have a unique fixed point in M. Then, for each $\lambda \in (0,1)$, there exists a metric d_λ on M such that (M, d_λ) is complete and*

$$d_\lambda(Tx, Ty) \leq \lambda d_\lambda(x, y)$$

for all $x, y \in M$.

In 1975, Subrahmanyam [77] proved that Kannan's contraction (KC) characterizes the metric completeness, that is,

Theorem SU. *A metric space (X, d) is complete if and only if every Kannan's contraction on X has a fixed point.*

Recall that a metric space (M, d) is *ultrametric* if and only if, for all $x, y, z \in M$,

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

An ultrametric space (M, d) is said to be *spherically complete* if every descending sequence of closed balls in M has the nonempty intersection. Thus, a spherically complete ultrametric space (M, d) is complete.

In 1990, Priess-Crampe [78] proved an analogue to Banach's fixed point theorem in ultrametric spaces.

Theorem PC. *An ultrametric space* (M, d) *is spherically complete if and only if every strictly contractive mapping* $T : M \to M$ *has a unique fixed point in* M*, where a mapping* $T : M \to M$ *is strictly contractive if*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in M$.

In 1993, Park and Kang [79] gave some characterizations of metric completeness by using Caristi's contraction (CC).

Theorem PK. *A metric space* (X, d) *is complete if and only if, for every self-mapping* T *of* X *with a uniformly continuous function* $\phi : X \to [0, \infty)$ *such that*

$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$

for all $x \in X$*,* T *has a fixed point in* X.

In 1996, Suzuki and Takahashi [80] gave some characterizations of metric completeness by using weakly contractive mapping in metric spaces.

Let X be a metric space with a metric d. Then, a function $p : X \times X \to [0, \infty)$ is called a w*-distance* on X if the following are satisfied:

(a) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(b) For any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi-continuous;
(c) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Note that the metric d is a w-distance on X. Some other examples of w-distances are given in [63].

A mapping $T : X \to X$ is said to be *weakly contractive* or p*-contractive* if there exists a w-distance p on X and $\alpha \in [0, 1)$ such that

$$p(Tx, Ty) \leq \alpha p(x, y)$$

for all $x, y \in X$. In the case of $p = d$, the mapping T is said to be *contractive*.

They proved the following:

Theorem ST1. *A metric space* $X, d)$ *is complete if and only if every weakly contractive mapping* $T : X \to X$ *has a fixed point in* X.

Theorem ST2. *Let X be a normed linear space and D be a convex subset of X. Then, D is complete if and only if every contractive mapping $T : D \to D$ has a fixed point in D.*

From Theorem ST2, we have the following:

Corollary ST. *Let X be a normed linear space. Then, X is a Banach space if and only if every contractive mapping $T : X \to X$ has a fixed point in X.*

On the other hand, in 2014, Ansari [81] gave some characterizations of metric completeness (the converse of *Ekeland's variational principle*) in metric spaces.

Theorem A. *A metric space (X, d) is complete if, for every functional $f : X \to \mathbb{R} \cup \{+\infty\}$ which is proper, bounded below, and low semi-continuous on X and, for any $\epsilon > 0$, there exists $\bar{x} \in X$ such that, for all $x \in X$,*

$$f(\bar{x}) \le \inf_{x \in X} f(x) + \epsilon, \quad f(\bar{x}) \le f(x) + \epsilon d(x, \bar{x}).$$

10 Conjectures of Banach's Fixed Point Theorem

The following theorem was one of the interesting conjecture connected with Banach's fixed point theorem, which was suggested by Jachymski et al. [82] in 1999:

Theorem GBC. (Generalized Banach's fixed point theorem Conjecture) *Let (E, d) be a complete metric space and $T : E \to E$ be a mapping. Suppose that there exist an integer p and a number $L \in [0, 1)$ such that*

$$\min\{d(T^i x, T^i y) : 1 \le i \le p\} \le L d(x, y) \tag{GCM}$$

for all $x, y \in E$. Then, T has exactly one fixed point $z \in E$?

$$\min\{d(T^i x, T^i y) : 1 \le i \le p\} \le L d(x, y) \tag{GCM}$$

Remark BGC. (1) The condition (GCM) does not imply the continuity of T.

(2) If $p = 1$ in the condition (GCM), then T is a contractive mapping on E.

(3) If T^k is a contractive mapping, then the condition (GCM) holds for all $p \ge k$.

(4) In 1999, if $p = 2$, then, without any additional assumption on T, Jachymski et al. [82] showed that Theorem GBC is true. Moreover, Theorem GBC is true if $p = 3$ with the additional assumption that T is continuous on E.

(5) In 1999, Jachymski and Stein [83] showed that Theorem GBC is true for any p if T is uniformly continuous.

(6) In 2000, Stein [84] showed that Theorem GBC is true for any p if T is strongly continuous, where we say that a mapping $T : E \to E$ is strongly continuous if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} d(x_k, y_k) < \delta \implies \sum_{k=1}^{n} d(Tx_k, Ty_k) < \varepsilon.$$

(7) In 2001, Merryfield et al. [85] showed that Theorem GBC is true if T is continuous. Moreover, Theorem GBC is true for $p = 3$ without any additional assumption on T.

(8) In fact, Theorem GBC is not true for all $p \in \mathbb{N}$ (see [84]).

Example S. Let $E = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all x, $y \in E$. Define a mapping $T : E \to E$ by

$$Tx = \sqrt{x^2 + 1}$$

for all $x \in E$. It is easy to show that $T^n x = \sqrt{x^2 + n}$ for all $x \in E$ and, for all $x, y \in E$ with $x < y$,

$$\min\{d(T^i x, T^i y) : i \in \mathbb{N}\} \leq L d(x, y).$$

However, it is clear that T has no fixed point in E.

11 Relations Between Banach's Fixed Point Theorem and Best Proximity Point Theorems

Banach's fixed point theorem plays an important role in showing the existence of solutions of various equations of the form $Tx = x$ for a *self-mapping* $T : A \to A$ defined on a subset A of a metric space E.

Now, if $T : A \to B$ is a *nonself-mapping*, where A and B are subsets of E, then the equation $Tx = x$ does not necessarily have a solution, which is known as a fixed point of the mapping T. Thus, in such circumstance, it may be considered to determine an element x for which the error $d(x, Tx)$ is minimum, in which case x and Tx are in close proximity to each other.

In this perspective, *best approximation theorems* and *best proximity point theorems* are very relevant.

One of the most interesting results, *best approximation theorem*, in this direction is due to Ky Fan [86]:

Theorem KF. *Let K be a nonempty compact convex subset of a normed space E and $T : K \to E$ be a continuous nonself-mapping. Then, there exists $x \in K$ such that*

$$\|x - Tx\| = d(K, Tx) = \inf\{\|Tx - u\| : u \in K\}.$$

Let A and B be nonempty subsets of a metric space E. Then, we recall the following:

$$d(A, B) = \inf\{d(x, y) : x \in A, \quad y \in B\}.$$

It is clear that if $T : A \to B$ is a nonself-mapping, then, for all $x \in A$,

$$d(x, Tx) \geq d(A, B).$$

When a nonself-mapping $T : A \to B$ has no fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum. The best proximity point theorems assure the existence of an element x^* such that

$$d(x^*, Tx^*) = d(A, B).$$

This element x^* is called the *best proximity point* of T.

Moreover, if the mapping T under discussion is a *self-mapping*, then a best proximity point theorem becomes to a fixed point result. In fact, the best proximity point evolves as a generalization of the idea of the best approximation.

The best approximation results provide an approximate solution to the fixed point equation $Tx = x$, when the nonself-mapping T has no fixed point, that is, the best approximation theorem assures the existence of an approximate solution (see Theorem KF). But such solution need not yield an optimal solution. But the best proximity point theorem is considered for solving the problem to find an approximate solution which is optimal.

Indeed, if there is no exact solution of the fixed point equation $Tx = x$ for a nonself-mapping $T : A \to B$, then a best proximity theorem offers sufficient conditions for the existence of an optimal approximate solution x, which is called a *best proximity point* of the mapping T, satisfying the following condition:

$$d(x, Tx) = d(A, B).$$

Let (E, d) be a metric space and A, B be subsets of E. Let $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subset B$ and $T(B) \subset A$, where the mapping T is said to be *cyclic*.

(1) A point $x \in A \cup B$ is called a *best proximity point* of T if

$$d(x, Tx) = d(A, B);$$

(2) T is called a *cyclic proximity contraction* if there exists $\alpha \in (0, 1)$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B);$$

Let A and B be nonempty subsets of a complete metric space (E, d) and $T : A \cup B \to A \cup B$ be a cyclic proximity contraction. If $A \cap B \neq \emptyset$, then $d(A, B) = 0$

and so T is the Banach contraction on a complete metric space $(A \cap B, d)$. Thus, applying Banach's fixed point theorem, T has a unique fixed point in $A \cap B$.

In 2013, Yadav et al. [87] introduced the following:

(3) T is called a *generalized cyclic proximity contraction* if there exists $0 \leq \alpha_1, \alpha_2, \alpha_3$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that, for all $x, y \in X$,

$$\begin{aligned} d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) \\ + [1 - (\alpha_1 + \alpha_2 + \alpha_3)] d(A, B). \end{aligned}$$

Example YTS. Let $E = \mathbb{R}$ be a complete metric space with the usual metric and let $A = [0, \frac{1}{2}]$, $B = [1, \frac{1}{2}]$. Define a mapping $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } y \in B. \end{cases}$$

If $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{3}$ and $\alpha_3 = \frac{1}{9}$, then T is a generalized cyclic proximity contraction.

(4) Let $S, T : A \cup B \to A \cup B$ be two mappings such that $T(A) \subset B$ and $T(B) \subset A$. Then, a pair of mappings S, T is called a *TS-cyclic proximity contraction* if there exists $0 \leq \alpha_1, \alpha_2, \alpha_3$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that

$$\begin{aligned} d(Tx, Sy) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Sy) \\ + [1 - (\alpha_1 + \alpha_2 + \alpha_3)] d(A, B) \end{aligned}$$

for all $x, y \in X$.

In 2003, Kirk et al. [88] proved the following:

Theorem KSV. *Let A and B be two nonempty closed subsets of a complete metric space (E, d). Suppose that a mapping $T : A \cup B \to A \cup B$ satisfies the following*:

(a) $T(A) \subset B$ and $T(B) \subset A$;
(b) *There exists $\alpha \in (0,1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x \in A$ and $y \in B$.
Then, T has a unique fixed point in $A \cap B$.

Note that, if $A = B$ in Theorem KSV, we have Banach's fixed point theorem.
Recently, in 2013, Yadav et al. proved the following [87]:

Theorem YTS1. *Let A, B be two nonempty closed subsets of a complete metric space (E, d) and $T : A \cup B \to A \cup B$ be a generalized cyclic proximity contraction. For any $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence to $x^* \in A$, then x^* is a best proximity point of T.*

Theorem YTS2. *Let A, B be two nonempty closed subsets of a complete metric space (E, d) and $S, T : A \cup B \to A \cup B$ be a TS-cyclic proximity contraction. For any $x_0 \in A$, define $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$ for each $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence to $x^* \in A \cup B$, then x^* is a best proximity point of T and S.*

Recently, some authors have considered the following problems:

(1) How to find more generalized cyclic proximity contractions than a generalized cyclic proximity contraction?
(2) How to show the existence of a common best proximity point of two mappings?
(3) How to extend best proximity point theorems in metric spaces to the classes of probabilistic metric spaces, fuzzy metric spaces, ordered metric spaces, and other spaces?

12 Some Better Nonlinear Mappings than Banach's Contraction

In particular, Banach's fixed point theorem is a widely applied tool for an iterative approximation of fixed points, but, unfortunately, its application is restricted to *contractive mappings*. Thus, we need appropriate, nice nonlinear mappings for some iterative approximations of fixed points, for example, *nonexpansive mappings*, *firmly nonexpansive mappings*, and other nonlinear mappings.

Let C be a nonempty subset of a normed linear space E. A mapping $T : C \to C$ is said to be *nonexpansive* if, for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Note that if a nonexpansive mapping $T : E \to E$ has a fixed point in E, then it need not be unique (for example, the identity mapping) and the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

for all $n \geq 0$ may fail to converge to a fixed point of T. For example, define a mapping $T : R \to \mathbb{R}$ by

$$Tx = 1 - x$$

for all $x \in \mathbb{R}$. Then, for $x_0 = 1$,

$$x_1 = Tx_0 = 0, \quad x_2 = Tx_1 = 1, \quad x_3 = Tx_2 = 0, \quad \ldots,$$
$$x_{2n} = Tx_{2n-1} = 1, \quad x_{2n+1} = Tx_{2n} = 0, \quad \ldots.$$

Then, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

for all $n \geq 0$ does not converge to the fixed point $\frac{1}{2}$ even if the mapping T has a unique fixed point $\frac{1}{2} \in \mathbb{R}$.

Further, if T is a nonexpansive mapping on $\mathbb{R}$, then T need not have a fixed point as the example

$$Tx = x + 1$$

for all $x \in \mathbb{R}$.

In 1973, Bruck [89] introduced a class of nonexpansive mappings which he called firmly nonexpansive mappings as follows:

Let C be a nonempty closed convex subset of a Banach space X. A mapping $T : C \to X$ is said to be *firmly nonexpansive* if, for all $x, y \in C$ and for $t \geq 0$,

$$\|Tx - Ty\| \leq \|t(x - y) + (1 - t)(Tx - Ty)\|.$$

Remark B1. (1) Firmly nonexpansive mappings are nonexpansive;

(2) The resolvent of an accretive mapping is firmly nonexpansive;

(3) In some sense, the class of firmly nonexpansive mappings is quite restrictive. For example, the identity mapping $I_X : B_X \to B_X$ is trivially firmly nonexpansive, but $-I_X$ fails to be firmly nonexpansive, where B_X the closed unit ball of a Banach space X.

Recently, some authors introduced the concept of a firmly nonexpansive mapping has been widely studied and generalized in several ways as follows:

Definition FNM1. A mapping $T : c \to X$ is said to be:

(1) *λ-firmly nonexpansive* [90] if there exists $\lambda \in (0, 1)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\|.$$

(2) *nonspreading* [91] if, for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$

(3) *hybrid* [92] if, for all $x, y \in C$,

$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|x - y\|^2.$$

(4) T *J_1-mapping* [93] if, for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2.$$

(5) T *J_2-mapping* [93] if, for all $x, y \in C$,

$$3\|Tx - Ty\|^2 \leq 2\|Tx - y\|^2 + \|Ty - x\|^2.$$

Remark B2. (1) The class of λ-firmly nonexpansive mappings is wider than the class of firmly nonexpansive mappings;

(2) Every λ-firmly nonexpansive mapping is nonexpansive.

Definition FNM2. Let C be a nonempty closed convex subset of a Hilbert space H and let $\lambda \in \mathbb{R}$. A mapping $T : C \to X$ is said to be λ-*hybrid* [94] if, for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle.$$

Definition FNM3. A mapping $T : C \to X$ is said to be α-*nonexpansive* [95] if, for all $x, y \in C$ and $\alpha < 1$,

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

Remark B3. (1) Every firmly nonexpansive mapping is α-nonexpansive for all $\alpha \in [0, \frac{1}{2}]$;

(2) For any $\lambda \in [0, 1)$, if a mapping $T : C \to X$ is λ-firmly nonexpansive, then T is α-nonexpansive with $\alpha = \frac{\lambda}{1+\lambda}$;

(3) For all $\lambda \in [0, \frac{1}{2}]$, if a mapping $T : C \to X$ is λ-firmly nonexpansive, then T is α-nonexpansive with $\alpha = \lambda$.

The following properties between α-nonexpansive mappings and other nonlinear mappings can be found in Ariza-Ruiz et al. [96]:

Remark B4. (1) The identity mapping I_X is α-nonexpansive for all $\alpha < 1$;

(2) A mapping $T : C \to X$ is 0-nonexpansive if and only if T is nonexpansive;

(3) $\frac{1}{2}$-nonexpansive mappings are nonspreading;

(4) $\frac{1}{3}$-nonexpansive mappings are hybrid;

(5) For all $\alpha < 0$, the unique α-nonexpansive mapping is the identity mapping $I_C : C \to C$. In fact, taking $y = x$ in the definition of the α-nonexpansive mapping, we have the following: For all $x \in C$,

$$0 \leq 2\alpha\|Tx - x\|^2.$$

So, since $\alpha < 0$, it follows that $Tx = x$ for all $x \in C$.

In [96], we can see that there exists a constant mapping failing to be α-nonexpansive for $\alpha > \frac{2}{3}$.

(6) For all $0 \leq \alpha \leq \frac{2}{3}$, every constant mapping $T : C \to C$ is α-nonexpansive.

(7) Every TJ_1-mapping is $\frac{1}{4}$-nonexpansive;
(8) Every TJ_2-mapping is nonspreading and so $\frac{1}{2}$-nonexpansive.

Now, we give some relations between *Banach's contraction* and *α-nonexpansive mappings* as follows:

(1) Let C be a nonempty subset of a Banach space X. If $T : C \to X$ is k-contractive for some $k \in (\frac{1}{3}, 1)$, then T is α-nonexpansive for all $\alpha \in [0, \frac{1-k}{1+k}]$;
(2) If $T : C \to X$ is k-contractive for some $k \in [0, \frac{1}{3}]$, then T is α-nonexpansive for all $\alpha \in [0, \frac{1}{2}]$.

Recall that a mapping $T : C \to X$ is said to be *generalized nonexpansive* if there exist nonnegative constants $a_1, a_2, \ldots, a_5$ with $a_1 + a_2 + \cdots + a_5 \leq 1$ such that, for all $x, y \in C$,

$$\begin{aligned}&\|Tx - Ty\| \\ &\leq a_1\|x-y\| + a_2\|x-Tx\| + a_3\|y-Ty\| + a_4\|x-Ty\| + a_5\|y-Tx\|.\end{aligned} \tag{GNM1}$$

Since the distance function is symmetric, we can replace a_2, a_3 with $\frac{a_2+a_3}{2}$ and a_4, a_5 with $\frac{a_4+a_5}{2}$ and so the generalized nonexpansive mapping (GNM1) is equivalent to the following: There exist nonnegative constants a, b, c with $a + 2b + 2c \leq 1$ such that, for all $x, y \in C$,

$$\begin{aligned}&\|Tx - Ty\| \\ &\leq a\|x-y\| + b(\|x-Tx\| + \|y-Ty\|) + c(\|x-Ty\| + \|y-Tx\|).\end{aligned} \tag{GNM2}$$

Remark B5. Every generalized nonexpansive mapping $T : C \to X$ with $b = 0$ is c-nonexpansive.

In 2007, Pineda and Goebel [97] introduced the new class of mappings called α-mean nonexpansive mappings, which is wider the class of nonexpansive mappings, as follows:

Definition PG1. A mapping $T : C \to C$ is said to be *α-mean nonexpansive* if, for all $x, y \in C$,

$$\sum_{i=1}^{n} a_i\|T^i x - T^i y\| \leq \|x - y\|,$$

where $a_i \geq 0$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} a_i = 1$.

Shortly, we consider the following α-mean nonexpansive mapping:

Definition PG2. Let $\alpha \in (0, 1]$ and C be a nonempty subset of a normed space X. A mapping $T : C \to C$ is said to be *α-mean nonexpansive* if, for all $x, y \in C$,

$$a\|Tx - Ty\| + (1-a)\|T^2x - T^2y\| \leq \|x - y\|.$$

Remark B6. (1) Every α-mean nonexpansive mapping T is continuous since $a > 0$;

(2) For any $\alpha \in (0, 1)$, there exists an α-nonexpansive mapping which is not an α-mean nonexpansive mapping;

(3) In fact, none of the classes of *α-mean nonexpansive mappings* and *α-nonexpansive mappings* is included in the other one.

Now, we can consider the following problems:

(1) How to study the structures of the fixed point sets of the mappings introduced above?
(2) How to find approximating fixed point sequences $\{x_n\}$ for the mappings introduced above?
(3) How to prove some classic fixed point theorems, demiclosedness principles, Pazy's fixed point theorems, ergodic theorems for the mappings introduced above?

Next, we introduce the concept of asymptotically nonexpansive mappings on a normed linear space E.

Let C be a nonempty subset of a normed linear space E. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* [98] if there exists a sequence $\{k_i\}$ of real numbers with $k_i \to 1$ as $i \to \infty$ such that

$$\|T^i x - T^i y\| \le k_i \|x - y\|$$

for all $x, y \in C$.

Note that every nonexpansive mapping is an asymptotically nonexpansive mapping.

Example GK. Let $B_X = \{x \in X : \|x\| \le 1\}$ be the closed unit ball in a Hilbert space $X = l_2$, where

$$l_2 = \left\{x : x = (x_1, x_2, \ldots, x_i, \ldots), \sum_{i=1}^{\infty} |x_i|^2 < \infty\right\},$$

and $T : B_H \to B_H$ be a mapping defined by

$$T(x_1, x_2, \ldots) = (0, x_1^2, a_2 x_2, a_3 x_3, \ldots),$$

where $\{a_i\}$ is a sequence of real numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$. Then, we have

$$\|Tx - Ty\| \le 2\|x - y\|$$

for all $x, y \in B_H$, that is, T is a Lipschitz mapping, but not a nonexpansive mapping. Note that

$$\|T^n x - T^n y\| \le 2\prod_{i=2}^{\infty} a_i \|x - y\|$$

for all $x, y \in B_H$ and $n \ge 2$, where $2\prod_{i=2}^{\infty} a_i \to 1$ as $n \to \infty$. Therefore, T is an asymptotically nonexpansive mapping, but not a nonexpansive mapping.

Now, we consider the class of multi-valued asymptotically nonexpansive mappings.

Let C be a nonempty subset of a metric space E and $CB(C)$ denote the family of nonempty bounded closed subsets of C. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}$$

for all $A, B \in CB(C)$, where $d(u, A) = \inf_{v \in A} d(u, v)$.

Definition GK. (1) A multi-valued mapping $T : C \to CB(C)$ is said *nonexpansive* if

$$H(Tx, Ty) \le \|x - y\|$$

for all $x, y \in E$.
(2) A multi-valued mapping $T : K \to CB(C)$ is said to be *asymptotically quasi-nonexpansive* if $F(T)$ is nonempty and there exists a sequence $\{k_n\}$ with $\lim_{n \to \infty} k_n = 0$ such that

$$\phi(x_n, z) \le (k_n + 1)\phi(x, z)$$

for all $x_n \in T^n x$, $z \in F(T)$, $x \in C$ and $n \ge 1$.

Here, we have one question: *What means* $x_n \in T^n x$?

Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T : K \to CB(C)$ be a multi-valued mapping. For any $z \in C$, define the following:

$$Tz = \{z_1 : z_1 \in Tz\},$$

$$T^2 z = TTz = \cup_{w_1 \in Tz} Tw_1,$$

$$T^3 z = TT^2 z = \cup_{w_2 \in T^2 z} Tw_2,$$

$$\cdots$$

$$T^{n+1} z = TT^n z = \cup_{w_n \in T^n z} Tw_n,$$

$$\cdots$$

Then, by using this new definition,

Can we extend the corresponding results for the class of single-valued asymptotically nonexpansive mappings to the class of multi-valued asymptotically nonexpansive mappings?

Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^* and $\langle\cdot,\cdot\rangle$ be the paring between E and E^*. Let $A : E \to 2^{E^*}$ be a multi-valued mapping and the *graph* of A, $G(A)$ is defined by

$$G(A) = \{(x, x^*) : x \in E,\ x^* \in E^*\}.$$

In 2007, Bartz et al. [99] introduced the following:

Definition BB1. (1) A mapping $A : E \to 2^{E^*}$ is said to be *n-cyclically monotone* if, for all $n \in \{2, 3, \ldots\}$,

$$\begin{cases} (a_1, a_1^*) \in G(A) \\ (a_2, a_2^*) \in G(A) \\ \quad \cdots \\ (a_n, a_n^*) \in G(A) \\ a_{n+1} = a_1 \end{cases} \implies \sum_{i=1}^{n} \langle a_{i+1} - a_i, a_i^* \rangle \le 0.$$

(2) A mapping $A : E \to 2^{E^*}$ is said to be *monotone* if T is 2-cyclically monotone, equivalently,

$$\begin{cases} (x, x^*) \in G(A) \\ (y, y^*) \in G(A) \end{cases} \implies \langle x - y, x^* - y^* \rangle \ge 0.$$

(3) A mapping $A : E \to 2^{E^*}$ is said to be *cyclically monotone* if, for all $n \in \{2, 3, \ldots\}$, A is n-cyclically monotone.
(4) A mapping $A : E \to 2^{E^*}$ is said to be *maximal n-cyclically monotone* if, for all $n \in \{2, 3, \ldots\}$, A is monotone and no proper extension of A is n-cyclically monotone.
(5) A mapping $A : E \to 2^{E^*}$ is said to be *maximal cyclically monotone* if A is cyclically monotone and no proper extension of A is cyclically monotone.
(6) A mapping $A : E \to 2^{E^*}$ is said to be *maximal monotone* if A is maximal 2-cyclically monotone.

Remark BB. (1) There exists a maximal 3-cyclically monotone mappings on $\mathbb{R}^2$ which is not maximal monotone.

(2) We can find some examples of mappings which are n-cyclically monotone, but not $(n+1)$-cyclically monotone.

(3) For some more details on *n-cyclically monotone* and *maximal n-cyclically monotone mappings*, see the paper [99].

In 2007, Bartz et al. [99] also introduced the following in a Hilbert space H:

Definition BB3. A mapping $A : H \to H$ is said to be *cyclically firmly nonexpansive* if, for all $n \in \{2, 3, \ldots\}$,

$$\sum_{i=1}^{n} \langle x_i - Tx_i, Tx_i - Tx_{i+1} \rangle \geq 0$$

for all set $\{x_1, x_2, \ldots, x_n\}$ of $x_1, x_2, \ldots, x_n \in H$ with $x_{n+1} = x_1$.

Now, we can consider the following problems:

How to solve some nonlinear equation, fixed point theorems, equilibrium problems, variational inequality problems, optimization problems, and some other nonlinear problems in Banach spaces, and Hilbert spaces by using n-cyclically monotone, *maximal n-cyclically monotone* and *cyclically firmly nonexpansive mappings*?

References

1. L. Brouwer, Uber abbildungen von mannigfaltigkeiten. Math. Ann. **71**, 97–115 (1912)
2. J. Schauder, Der fixpunktsatz in funktionalrumen. Studia Math. **2**, 171–180 (1930)
3. A. Tychonoff, Ein Fixpunktsatz. Math. Ann. **111**,767–776 (1935)
4. S. Kakutani, A gneralization of Tychonoffs fixed point theorem. Duck Math. J. **8**, 457–459 (1968)
5. A. Tarski, A lattice theoretical fixpoint theorem and its applications. Pacific J. Math. **5**, 285–309 (1955)
6. S. Hayashi, Self-similar sets as Tarski's fixed points. Publ. RIMS Kyoto Univ. **21**, 1059–1066 (1985)
7. S. Heikkila, On fixed points through a generalized iteration method with applications to differential and integral equations involving discontinuities. Nonlinear Anal. **14**, 413–426 (1990)
8. B. S. W. Schröder, Algorithms for the fixed point property. Theoret. Comput. Sci. **217**, 301–358 (1999)
9. J. Jachymski, L. Gajek and K. Pokarowski, The Tarski-Kantorovitch principle and the theory of iterated function systems. Bull. Austral. Math. Soc. **61**, 247–261 (2000)
10. R. Uhl, Smallest and greatest fixed points of quasimonotone increasing mappings. Math. Nachr. **248–249**, 204–210 (2003)
11. E. A. Ok, Fixed set theory for closed correspondences with applications to self-similarity and games. Nonlinear Anal. **56**, 309–330 (2004)
12. A. C. Davis, A characterization of complete lattices. Pacific J. Math. **5**, 311–319 (1955)
13. E. Picard, Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. J. Math. Pures et Appl. **6**, 145–210 (1890)
14. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. Fund. Math. **3**, 133–181 (1922)
15. A. Meir, E. Keeler, A theorem on contraction mappings. J. Math. Anal. Appl. **28**, 326–329 (1969)
16. S. Park, B.E. Rhoades, Meir-Keeler type contractive conditions. Math. Japon. **26**, 13–20 (1981)

17. B.E. Rhoades, S. Park, On generalizations of the Meir-Keeler type contraction maps. J. Math. Anal. Appl. **146**, 482–494 (1990)
18. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions. Trans. Amer. Math. Soc. **215**, 241–251 (1976)
19. R. Kannan, Some results on fixed points. Bull. Cal. Math. Soc. **62**, 71–76 (1968)
20. S.K. Chatterjea, Fixed point theorems. C.R. Acad. Bulgare Sci. **25**, 727–730 (1972)
21. B.E. Rhoades, A comparison of various definitions of contractive mappings. Trans. Amer. Math. Soc **226**, 257–290 (1977)
22. V. Berinde, Approximating fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum **9**, 45–53 (2004)
23. S. Reich, Some remarks concerning contraction mappings. Canad. Math. Bull. **14**, 121–124 (1971)
24. Lj.B. Ćirić, Generalized contractions and fixed-point theorems. Publ. Inst. Math. (Belgr.) **12**(26), 19–26 (1971)
25. T. Zamfirescu, Fixed point theorems in metric spaces. Arch. Math. (Basel) **23**, 292–298 (1972)
26. G.E. Hary, T.D. Rogers, A generalization of a fixed point theorem of Reich. Canad. Math. Bull. **16**, 201–206 (1973)
27. Lj.B. Ćirić, A generalization of Banach's contraction principle. Proc. Amer. Math. Soc. **45**, 267–273 (1974)
28. M. Edelstein, On fixed and periodic points under contraction mappings. J. London Math. Soc. **37**, 74–79 (1962)
29. S. Prešić, Sur la convergence des suites. C.R. Acad. Paris **260**, 3828–3830 (1965)
30. H. Fukhar-Ud-Din, V. Berinde, A.R. Khan, Fixed point approximation of Prešić nonexpansive mappings in product of CAT(0) spaces. Carpathian J. Math. **32**, 315–322 (2016)
31. P. Boriwan, N. Petrot, S. Suantai, Fixed point theorems for Prešić almost contraction mappings in orbitally complete metric spaces endowed with directed graphs. Carpathian J. Math. **32**, 303–313 (2016)
32. M.S. Khan, M. Berzig, B. Samet, Some convergence results for iterative sequences of Presic type and applications. Advan. Differ. Equ. **2012**(38), 12 pp (2012)
33. M. Păcurar, Approximating common fixed points of Prešić-Kannan type operators by a multi-step iterative method. An. St. Univ. Ovidius Constanta Ser. Mat. **17**, 153–168 (2009)
34. M. Păcurar, A multi-step iterative method for approximating common fixed points of Prešić-Rus type operators on metric spaces. Studia Univ. Babes-Bolyai Math. **55**, 149–162 (2010)
35. M. Păcurar, Common fixed points for almost Prešić type operators. Carpathian J. Math. **28**, 117–126 (2012)
36. S. Shukla, B. Fisher, A generalization of Prešić type mappings in metric-like spaces. J. Oper. **2013**, Article ID 368501, 5 pp (2013)
37. R. George, K.P. Reshma, R. Rajagopalan, A generalized fixed point theorem of Prešić type in cone metric spaces and application to morkov process. Fixed Point Theory Appl. **2011**(85), 8 pp (2011)
38. M.S. Khan, M. Samanipour, Prešić type extension in cone metric space. Int. J. Math. Anal. **7**(36), 1795–1802 (2013)
39. S.K. Malhotra, S. Shukla, R. Sen, A generalization of Banach contraction principle in ordered cone metric spaces. J. Adv. Math. Stud. **5**, 59–67 (2012)
40. M.S. Khan, S. Shukla, S.M. Kang, Fixed point theorems of weakly monotone Prešić type mappings in ordered cone metric spaces. Bull. Korean Math. Soc. **52**, 881–893 (2015)
41. J. Zhu, Y.J. Cho, S.M. Kang, Equivalent contraction conditions in symmetric spaces. Comput. Math. Appl. **50**, 1621–1628 (2005)
42. A.E. Bashirov, E.M. Kurplnara, A. Ozyaplcl, Multiplicative calculus and its applications. J. Math. Anal. Appl. **337**, 36–48 (2008)
43. M. Ozavsar, A.C. Cevikel, Fixed point of multiplicative contractions on multiplicative metric space, arXiv:1205.5131v1 [mathGN]
44. X. He, M. Song, D. Chen, Common fixed points for weak commutative mappings on a multiplicative metric space. Fixed Point Theory Appl. **2014**, 48 (2014)

45. J. Tiammee, S. Suantai, Y.J. Cho, Common fixed point theorems for generalized multiplicative contractions in multiplicative metric spaces, submitted (2016)
46. R.P. Agarwal, E. Karapinar, B. Samet, An essential remark on fixed point results on multiplicative metric spaces. Fixed Point Theory Appl. **2016**, 21 (2016)
47. M. Abbas, B. Ali, Y.I. Suleiman, Common fixed points of locally contractive mappings in multiplicative metric spaces with applications, Int. J. Math. Math. Sci. **2015**, Article ID 218683 (2015)
48. D. Dorić, Common fixed point for generalized (ψ, ϕ)-weak contractions. Appl. Math. Lett. **22**, 1896–1900 (2009)
49. S. Matthews, Partial metric topology. Ann. N.Y. Acad. Sci. **728**, 183–197 (1994)
50. M. Asadi, E. Karapinar, P. Salimi, New extension of p-metric spaces with some fixed-point results on M-metric spaces. J. Inequal. Appl. **2014**, 18 (2014)
51. S.M. Ulam, *Problems in Modern Mathematics* (Wiley, New York, 1964)
52. D.H. Hyers, On the stability of the linear functional equation. Proceedings of National Academy of Sciences, USA **27**, 222–224 (1941)
53. Y.J. Cho, ThM Rassias, R. Saadati, *Stability of Functional Equations in Random Normed Spaces* (Springer, New York, 2013)
54. Y.J. Cho, C. Park, ThM Rassias, R. Saadati, *Stability of Functional Equations in Banach Algebras* (Springer, New York, 2015)
55. F.S. de Blassi, J. Myjak, Sur la porosite des contractions sans point fixe. Comptes Rendus de l'Academie des Sciences Paris **308**, 51–54 (1989)
56. S. Reich, A.J. Zaslavski, Well-posedness of fixed point problems. Far East J. Math. Sci. **3**, 393–401 (2001)
57. B.K. Lahiri, P. Das, Well-posednes and porosity of certain classes of operators. Demonstratio Math. **38**, 170–176 (2005)
58. V. Popa, Well posedness of fixed point problem in orbitally complete metric spaces, Studii si Cercetari Stiintifice, Seria Matematic, Department of Mathematics and Informatics, Faculty of Sciences, University of Bacau, Romania, 16(2006), 209–214, 2006, Proceedings ICMI 45, Bacau, September 18–20 (2006)
59. V. Popa, Well posedness of fixed point problem in compact metric spaces, Buletinul Universitatii Petrol-Gaze din Ploiesti. Seria Matematica–Informatica–Fizica **60**, 1–4 (2008)
60. A. Pansuwan, W. Sintunavarat, J.Y. Choi, Y.J. Cho, Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problem in M-metric spaces. J. Nonlinear Sci. Appl. **9**, 4489–4499 (2016)
61. M. Asadi, Fixed point theorems for Meir-Keeler type mappings in M-metric spaces with applications. Fixed Point Theory Appl. **2015**, 210 (2015)
62. S.B. Nadler Jr., Multi-valued contraction mappings. Pacific J. Math. **30**, 474–487 (1969)
63. O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Math. Japon. **44**, 381–391 (1996)
64. I. Ekeland, Nonconvex minimization problems. Bull. Amer. Math. Soc. **1**, 443–474 (1979)
65. N.A. Assad, W.A. Kirk, Fixed point theorems for set-valued mappings of contractive type. Pacific J. Math. **43**, 553–562 (1972)
66. G. Jungck, Commuting mappings and fixed points. Amer. Math. Monthly **83**, 261–263 (1976)
67. S. Sessa, On a weak commutativity condition of mappings in fixed point consideration. Publ. Inst. Math. (Belg.) **32**, 149–153 (1982)
68. G. Jungck, Compatible mappings and common fixed points. Int. J. Math. Math. Sci. **9**, 771–779 (1986)
69. G. Jungck, B.E. Rhoades, Fixed points for set-valued functions without continuity. Indian J. Pure Appl. Math. **29**, 227–238 (1998)
70. G. Jungck, P.P. Murthy, Y.J. Cho, Compatible mappings of type (A) and common fixed points. Math. Japon. **38**, 381–390 (1993)
71. R.P. Pant, Common fixed points of noncommuting mappings. J. Math. Anal. Appl. **188**, 436–440 (1994)

72. H.K. Pathak, Y.J. Cho, S.M. Kang, Remarks on R-weakly commuting mappings and common fixed point theorems. Bull. Korean Math. Soc. **34**, 247–257 (1997)
73. M. Aamri, D. El Moutawakil, Some new common fixed points theorems under strict contractive conditions. J. Math. Anal. Appl. **270**, 181–188 (2002)
74. W. Sintunavarat, P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces. J. Appl. Math. **2011**, Art. ID 637958, 14 pp (2011)
75. H. Bouhadjera, On common fixed point theorems for three and four self-mappings satisfying contractive conditions. Acta Univ. Palscki. Olomuc., Fac. rer. nat., Math. **49**, 25–31 (2010)
76. C. Bessaga, On the converse of Banach fixed-point principle. Colloq. Math. **7**, 41–43 (1959)
77. V. Subrahmanyam, Completeness and fixed-points. Monatsh. Math. **80**, 325–330 (1975)
78. S. Priess-Crampe, Der Banachsche Fixpunkstaz fur ultrametrische Raume. Results Math. **18**, 178–186 (1990)
79. S. Park, G. Kang, Generalizations of the Ekeland type variational principle. Chinese J. Math. **21**, 313–325 (1993)
80. T. Suzuki, W. Takahashi, Fixed point theorems and characterizations of metric completeness. Top. Methods in Nonlinear Anal. **8**, 371–382 (1996)
81. Q.H. Ansari, Ekeland's variational principle and its extensions with applications, in *Topics in Fixed Point Theory*, ed. by S. Almezel, Q.H. Ansari, M.A. Khamsi (Springer, Berlin, 2014)
82. J.R. Jachymski, B. Schroder, J.D. Stein Jr., A connection between fixed point theorems and tiling problems. J. Combin. Theory Ser. A **87**, 273–286 (1999)
83. J.R. Jachymski, J.D. Stein Jr., A minimum condition and some related fixed-point theorems. J. Austral. Math. Soc. Ser. A **66**, 224–243 (1999)
84. J.D. Stein Jr., A systematic generalization procedure for fixed-point theorems. Rocky Mountain J. Math. **30**, 735–754 (2000)
85. J. Merryfield, B. Rothschild, J.D. Stein Jr., An application of Ramsey's Theorem to the Banach contraction principle. Proc. Am. Math. Soc. **130**, 927–933 (2001)
86. Ky Fan, Extensions of two fixed point theorems of F E. Browder. Math. Z. **112**, 234–240 (1969)
87. M.R. Yadav, B.S. Thakur, A.K. Sharma, Best proximity points for generalized proximity contraction in complete metric spaces. Adv. Fixed Point Theory **3**, 393–405 (2013)
88. W.A. Kirk, P.S. Srinvasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory **4**, 79–89 (2003)
89. R.E. Bruck, Nonexpansive projections on subsets of Banach spaces. Pacific J. Math. **47**, 341–355 (1973)
90. Y.M. Hong, Y.Y. Huang, On λ-firmly nonexpansive mappings in nonconvex sets. Bull. Inst. Math. Acad. Sinica **21**, 35–42 (1993)
91. F. Kohsaka, W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces. Arch. Math. (Basel) **91**, 166–177 (2008)
92. W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space. J. Nonlinear Convex Anal. **11**, 79–88 (2010)
93. W. Takahashi, J.C. Yao, Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces. Taiwan. J. Math. **15**, 457–472 (2011)
94. K. Aoyama, S. Iemoto, F. Kohsaka, W. Takahashi, Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces. J. Nonlinear Convex Anal. **11**, 335–343 (2010)
95. K. Aoyama, F. Kohsaka, Fixed point theorem for α-nonlinear mappings in Banach spaces. Nonlinear Anal. **74**, 4387–4391 (2011)
96. D. Ariza-Ruiz, C.H. Linares, E. Llorens-Fuster, E. Moreno-Galvez, On α-nonexpansive mappings in Banach spaces. Carpathian J. Math. **32**, 13–28 (2016)
97. M.A. Japon-Pineda, K. Goebel, On a type of generalized nonexpansiveness, in *Proceedings of the 8th International Conference of Fixed Point Theory and its Application* (Yokohama Publishers, 2008), pp. 71–82
98. K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. **35**, 171–174 (1972)
99. S. Bartz, H.H. Bauschke, J.M. Borwein, S. Reich, X.F. Wang, Fitzpatrick functions, cyclic monotonicity and Rockafellar's antiderivative. Nonlinear Anal. **66**, 1198–1223 (2007)

Sums of Finite Products of Euler Functions

Taekyun Kim, Dae San Kim, Gwan Woo Jang and Jongkyum Kwon

Abstract In this paper, we consider three types of functions given by sums of finite products of Euler functions and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

Keywords Fourier series · Sums of finite products of Euler functions

2010 Mathematics Subject Classification 11B68 · 42A16

1 Introduction

Let $E_m(x)$ be the Euler polynomials given by the generating function

$$\frac{2}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!}, \quad \text{(see [1-3,7,12,13,15,19])}. \tag{1}$$

T. Kim (✉) · G.W. Jang
Department of Mathematics, Kwangwoon University, Seoul 139-701,
Republic of Korea
e-mail: tkkim@kw.ac.kr

G.W. Jang
e-mail: gwjang@kw.ac.kr

D.S. Kim
Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
e-mail: dskim@sogang.ac.kr

J. Kwon
Department of Mathematics Education and RINS, Gyeongsang National University,
Jinju, Gyeongsangnamdo 52828, Republic of Korea
e-mail: mathkjk26@gnu.ac.kr

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_10

When $x = 0$, $E_m = E_m(0)$ are called Euler numbers. For any real number x, we let

$$< x >= x - [x] \in [0, 1) \tag{2}$$

denote the fractional part of x.

Fourier series expansion of higher-order Bernoulli functions was treated in the recent paper [14]. Here we will consider the following three types of functions given by sums of finite products of Euler functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

(1) $\alpha_m(\langle x\rangle) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 0} E_{c_1}(\langle x\rangle)E_{c_2}(\langle x\rangle)\cdots E_{c_r}(\langle x\rangle)$, $(m \geq 1)$;
(2) $\beta_m(\langle x\rangle) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 0} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(\langle x\rangle)E_{c_2}(\langle x\rangle)\cdots E_{c_r}(\langle x\rangle)$, $(m \geq 1)$;
(3) $\gamma_{r,m}(\langle x\rangle) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1} \frac{1}{c_1c_2\cdots c_r} E_{c_1}(\langle x\rangle)E_{c_2}(\langle x\rangle)\cdots E_{c_r}(\langle x\rangle)$, $(m \geq r)$.

For elementary facts about Fourier analysis, the reader may refer to any book (e.g., see [16, 20]).

As to $\beta_m(< x >)$, we note that the next polynomial identity follows immediately from Theorems 3.1 and 3.2, which is in turn derived from the Fourier series expansion of $\beta_m(< x >)$:

$$\begin{aligned}
&\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(x)\cdots E_{c_r}(x) \\
&= \frac{1}{r}\Omega_{m+1} + \sum_{j=1}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j(x),
\end{aligned}$$

where

$$\begin{aligned}
\Omega_l = &\sum_{0<a\leq r} \binom{r}{a}(-1)^a 2^{r-a} \sum_{c_1+c_2+\cdots+c_a=l} \frac{1}{c_1!c_2!\cdots c_a!} E_{c_1}E_{c_2}\cdots E_{c_a} \\
&- \sum_{c_1+c_2+\cdots+c_r=l} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}E_{c_2}\cdots E_{c_r}.
\end{aligned} \tag{3}$$

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\gamma_m(< x >)$ from Theorems 2.1 and 2.2, and Remark 4.1 and Theorem 4.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(< x >)B_{m-k}(< x >)$, we can derive the Faber–Pandharipande–Zagier identity (see [5, 8–11]) and the Miki's identity (see [4, 6, 8–11, 17, 18]).

2 The Function $\alpha_m(< x >)$

Let $\alpha_m(x) = \sum_{c_1+c_2+\cdots+c_r=m} E_{c_1}(x)E_{c_2}(x)\cdots E_{c_r}(x)$, $(m \geq 1)$. Here the sum runs over all nonnegative integers $c_1, c_2, \ldots, c_r$ with $c_1 + c_2 + \cdots + c_r = m$, $(r \geq 1)$. Then, we will consider the function

$$\alpha_m(< x >) = \sum_{c_1+c_2+\cdots+c_r=m} E_{c_1}(< x >)E_{c_2}(< x >)\cdots E_{c_r}(< x >), \tag{4}$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{5}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >)e^{-2\pi i n x}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx. \end{aligned} \tag{6}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{c_1+c_2+\cdots+c_r=m} \Big(c_1 E_{c_1-1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\ &\quad + \cdots + c_r E_{c_1}(x)E_{c_2}(x)\cdots E_{c_r-1}(x)\Big) \\ &= \sum_{c_1+c_2+\cdots+c_r=m, c_1\geq 1} c_1 E_{c_1-1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\ &\quad + \cdots + \sum_{c_1+c_2+\cdots+c_r=m, c_r\geq 1} c_r E_{c_1-1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\ &= (m+r-1) \sum_{c_1+c_2+\cdots+c_r=m-1} E_{c_1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\ &= (m+r-1)\alpha_{m-1}(x). \end{aligned} \tag{7}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+r}\right)' = \alpha_m(x). \tag{8}$$

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+r}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{9}$$

For $m \geq 1$, we put

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \Big(E_{c_1}(1)E_{c_2}(1)\cdots E_{c_r}(1) - E_{c_1}E_{c_2}\cdots E_{c_r} \Big) \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \Big((-E_{c_1} + 2\delta_{0,c_1})\cdots(-E_{c_r} + 2\delta_{0,c_r}) - E_{c_1}E_{c_2}\cdots E_{c_r} \Big) \\
&= \sum_{0<a\leq r} \binom{r}{a}(-1)^a 2^{r-a} \sum_{c_1+c_2+\cdots+c_a=m} E_{c_1}E_{c_2}\cdots E_{c_a} - \sum_{c_1+c_2+\cdots+c_r=m} E_{c_1}E_{c_2}\cdots E_{c_r}.
\end{aligned} \tag{10}$$

Observe here that the sum over all $c_1 + c_2 + \cdots + c_r = m$ of any term with a of $-E_{c_e}$ and b of $2\delta_{0,c_f}$, $(1 \leq e, f \leq r, a + b = r)$, all give the same sum

$$\begin{aligned}
&\sum_{c_1+c_2+\cdots+c_r=m} (-E_{c_1})\cdots(-E_{c_a})(2\delta_{0,c_{a+1}})\cdots(2\delta_{0,c_{a+b}}) \\
&= \sum_{c_1+c_2+\cdots+c_a=m} (-1)^a 2^{r-a} E_{c_1}E_{c_2}\cdots E_{c_a}.
\end{aligned} \tag{11}$$

Note here that, for $a = 0$, the sum in (11) is $2^r\delta_{0,m} = 0$.

Also,

$$\alpha_m(1) = \alpha_m(0) \iff \Delta_m = 0, \tag{12}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+r}\Delta_{m+1}. \tag{13}$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.

$Case 1 : n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big[\alpha_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+r-1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m+r-1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m \\
&= \frac{m+r-1}{2\pi in}\left(\frac{m+r-2}{2\pi in}A_n^{(m-2)} - \frac{1}{2\pi in}\Delta_{m-1}\right) - \frac{1}{2\pi in}\Delta_m
\end{aligned} \tag{14}$$

$$
\begin{aligned}
&= \frac{(m+r-1)_2}{(2\pi in)^2} A_n^{(m-2)} - \sum_{j=0}^{2} \frac{(m+r-1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1} \\
&= \cdots \\
&= \frac{(m+r-1)_m}{(2\pi in)^m} A_n^{(0)} - \sum_{j=1}^{m} \frac{(m+r-1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1} \\
&= -\frac{1}{m+r} \sum_{j=1}^{m} \frac{(m+r)_j}{(2\pi in)^j} \Delta_{m-j+1},
\end{aligned}
$$

where $A_n^{(0)} = \int_0^1 e^{-2\pi inx} dx = 0$.

$Case 2 : n = 0$.

$$
A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+r} \Delta_{m+1}. \tag{15}
$$

As to Bernoulli functions $B_m(< x >)$, we recall the following facts:

(a) for $m \geq 2$,

$$
B_m(< x >) = -m! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{16}
$$

(b) for $m = 1$,

$$
-\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{17}
$$

$\alpha_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers m with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m \neq 0$.

Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(1) = \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise C^∞ and continuous. Thus, the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$
\begin{aligned}
\alpha_m(< x >) &= \frac{1}{m+r} \Delta_{m+1} \\
&\quad + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\frac{1}{m+r} \sum_{j=1}^{m} \frac{(m+r)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx} \\
&= \frac{1}{m+r} \Delta_{m+1} + \frac{1}{m+r} \sum_{j=1}^{m} \binom{m+r}{j} \Delta_{m-j+1}
\end{aligned}
$$

$$\times \left(-j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \tag{18}$$

$$= \frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=2}^{m}\binom{m+r}{j}\Delta_{m-j+1}B_j(<x>)$$

$$+\Delta_m \times \begin{cases} B_1(<x>), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

We are now ready to state our first result.

Theorem 2.1 *For each positive integer l, let*

$$\Delta_l = \sum_{0<a\leq r}\binom{r}{a}(-1)^a 2^{r-a} \sum_{c_1+c_2+\cdots+c_a=l} E_{c_1}E_{c_2}\cdots E_{c_a} - \sum_{c_1+c_2+\cdots+c_r=l} E_{c_1}E_{c_2}\cdots E_{c_r}.$$

Assume that $\Delta_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{c_1+c_2+\cdots+c_r=m} E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>)$ *has the Fourier series expansion*

$$\sum_{c_1+c_2+\cdots+c_r=m} E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>)$$

$$= \frac{1}{m+r}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r}\sum_{j=1}^{m}\frac{(m+r)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{c_1+c_2+\cdots+c_r=m} E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>)$$

$$= \frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=2}^{m}\binom{m+r}{j}\Delta_{m-j+1}B_j(<x>),$$

for all $x \in \mathbb{R}$, where $B_j(<x>)$ is the Bernoulli function.

Assume next that $\Delta_m \neq 0$, for a positive integer m. Then $\alpha_m(1) \neq \alpha_m(0)$. Hence $\alpha_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right)=\alpha_m(0)+\frac{1}{2}\Delta_m, \tag{19}$$

for $x \in \mathbb{Z}$.

Now, we are going to state our second result.

Theorem 2.2 *For each positive integer l, let*

$$\Delta_l=\sum_{0<a\le r}\binom{r}{a}(-1)^a 2^{r-a}\sum_{c_1+c_2+\cdots+c_a=l}E_{c_1}E_{c_2}\cdots E_{c_a}-\sum_{c_1+c_2+\cdots+c_r=l}E_{c_1}E_{c_2}\cdots E_{c_r}.$$

Assume that $\Delta_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}
&(\text{a})\frac{1}{m+r}\Delta_{m+1}\\
&+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r}\sum_{j=1}^{m}\frac{(m+r)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}\\
&=\begin{cases}\sum_{c_1+c_2+\cdots+c_r=m}E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>), & \textit{for } x\notin\mathbb{Z},\\ \sum_{c_1+c_2+\cdots+c_r=m}E_{c_1}E_{c_2}\cdots E_{c_r}+\frac{1}{2}\Delta_m, & \textit{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}$$

$$\begin{aligned}
&(\text{b})\frac{1}{m+r}\Delta_{m+1}+\frac{1}{m+r}\sum_{j=1}^{m}\binom{m+r}{j}\Delta_{m-j+1}B_j(<x>)\\
&=\sum_{c_1+c_2+\cdots+c_r=m}E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>),\ \ for\ x\notin\mathbb{Z};
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{m+r}\Delta_{m+1}+\frac{1}{m+r}\sum_{j=2}^{m}\binom{m+r}{j}\Delta_{m-j+1}B_j(<x>)\\
&=\sum_{c_1+c_2+\cdots+c_r=m}E_{c_1}E_{c_2}\cdots E_{c_r}+\frac{1}{2}\Delta_m,\ \ for\ x\in\mathbb{Z}.
\end{aligned}$$

3 The Function $\beta_m(<x>)$

Let $\beta_m(x)=\sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}E_{c_1}(x)E_{c_2}(x)\cdots E_{c_r}(x)$, $(m\ge 1)$. Here the sum runs over all nonnegative integers $c_1, c_2, \ldots, c_r$ with $c_1+c_2+\cdots+c_r=m$, $(r\ge 1)$. Then we will investigate the function

$$\beta_m(<x>)=\sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>), \tag{20}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{21}$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(<x>) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \tag{22}$$

$$\begin{aligned}
\beta_m'(x) &= \sum_{c_1+c_2+\cdots+c_r=m} \Big(\frac{c_1}{c_1!c_2!\cdots c_r!} E_{c_1-1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\
&+ \cdots + \frac{c_r}{c_1!c_2!\cdots c_r!} E_{c_1}(x)\cdots E_{c_{r-1}}(x)E_{c_r-1}(x) \Big) \\
&= \sum_{c_1+c_2+\cdots+c_r=m, c_1\geq 1} \frac{1}{(c_1-1)!c_2!\cdots c_r!} E_{c_1-1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\
&+ \cdots + \sum_{c_1+c_2+\cdots+c_r=m, c_r\geq 1} \frac{1}{c_1!c_2!\cdots (c_r-1)!} E_{c_1}(x)\cdots E_{c_{r-1}}(x)E_{c_r-1}(x) \\
&= \sum_{c_1+c_2+\cdots+c_r=m-1} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\
&+ \cdots + \sum_{c_1+c_2+\cdots+c_r=m-1} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(x)E_{c_2}(x)\cdots E_{c_r}(x) \\
&= r\beta_{m-1}(x).
\end{aligned} \tag{23}$$

From (23), we obtain $\left(\frac{\beta_{m+1}(x)}{r}\right)' = \beta_m(x)$,
and

$$\int_0^1 \beta_m(x)dx = \frac{1}{r}\Big(\beta_{m+1}(1) - \beta_{m+1}(0)\Big). \tag{24}$$

Let

$$\begin{aligned}
\Omega_m &= \beta_m(1) - \beta_m(0) \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(1)E_{c_2}(1)\cdots E_{c_r}(1) \\
&\quad - \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}E_{c_2}\cdots E_{c_r} \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} (-E_{c_1}+2\delta_{0,c_1})\cdots(-E_{c_r}+2\delta_{0,c_r})
\end{aligned} \tag{25}$$

$$- \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1} E_{c_2} \cdots E_{c_r}.$$

Observe here that the sum over all $c_1 + c_2 + \cdots + c_r = m$ of any term with a of $-E_{c_e}$ and b of $2\delta_{0,c_f}$, $(1 \le e, f \le r, a+b=r)$, all give the same sum

$$\begin{aligned} &\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} (-E_{c_1}) \cdots (-E_{c_a})(2\delta_{0,c_{a+1}}) \cdots (2\delta_{0,c_{a+b}}) \\ &= \sum_{c_1+c_2+\cdots+c_a=m} \frac{1}{c_1!c_2!\cdots c_r!} (-1)^a 2^{r-a} E_{c_1} E_{c_2} \cdots E_{c_a}. \end{aligned} \tag{26}$$

Thus

$$\begin{aligned} \Omega_m = &\sum_{0<a\le r} \binom{r}{a} (-1)^a 2^{r-a} \sum_{c_1+c_2+\cdots+c_a=m} \frac{1}{c_1!c_2!\cdots c_a!} E_{c_1} E_{c_2} \cdots E_{c_a} \\ &- \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1} E_{c_2} \cdots E_{c_r}. \end{aligned} \tag{27}$$

In addition,

$$\begin{aligned} &\beta_m(1) = \beta_m(0) \Leftrightarrow \Omega_m = 0, \\ &\int_0^1 \beta_m(x)dx = \frac{1}{r}\Omega_{m+1}. \end{aligned} \tag{28}$$

Next, we would like to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} \Big[\beta_m(x) e^{-2\pi i n x} \Big]_0^1 + \frac{1}{2\pi i n} \int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} \Big(\beta_m(1) - \beta_m(0) \Big) + \frac{r}{2\pi i n} \int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\ &= \frac{r}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n} \Omega_m \\ &= \frac{r}{2\pi i n} \left(\frac{r}{2\pi i n} B_n^{(m-2)} - \frac{1}{2\pi i n} \Omega_{m-1} \right) - \frac{1}{2\pi i n} \Omega_m \\ &= \left(\frac{r}{2\pi i n} \right)^2 B_n^{(m-2)} - \sum_{j=1}^{2} \frac{r^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \\ &= \cdots \end{aligned} \tag{29}$$

$$= \left(\frac{r}{2\pi in}\right)^m B_n^{(0)} - \sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j} \Omega_{m-j+1}$$
$$= -\sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j} \Omega_{m-j+1},$$

where $B_n^{(0)} = \int_0^1 e^{-2\pi inx} dx = 0$.

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) = \frac{1}{r}\Omega_{m+1}. \tag{30}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those positive integers m with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. Hence $\beta_m(< x >)$ is piecewise C^∞ and continuous. Thus, the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} &\beta_m(< x >) \\ &= \frac{1}{r}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-\sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right) e^{2\pi inx} \\ &= \frac{1}{r}\Omega_{m+1} + \sum_{j=1}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1} \times \left(-j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) \\ &+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{31}$$

Now, we can state our first result.

Theorem 3.1 *For each positive integer l, let*

$$\begin{aligned} \Omega_l = &\sum_{0<a\leq r} \binom{r}{a} (-1)^a 2^{r-a} \sum_{c_1+c_2+\cdots+c_a=l} \frac{1}{c_1!c_2!\cdots c_a!} E_{c_1}E_{c_2}\cdots E_{c_a} \\ &- \sum_{c_1+c_2+\cdots+c_r=l} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}E_{c_2}\cdots E_{c_r}. \end{aligned} \tag{32}$$

Assume that $\Omega_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(< x >)E_{c_2}(< x >)\cdots E_{c_r}(< x >)$ *has the Fourier series expansion*

$$\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(< x >)E_{c_2}(< x >)\cdots E_{c_r}(< x >) = \frac{1}{r}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(-\sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big) e^{2\pi inx}, \tag{33}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)
$$\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}(< x >)E_{c_2}(< x >)\cdots E_{c_r}(< x >) = \frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) \tag{34}$$

for all $x \in \mathbb{R}$, where $B_j(< x >)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Omega_m \neq 0$. Then, $\beta_m(1) \neq \beta_m(0)$. Hence $\beta_m(< x >)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned} \frac{1}{2}(\beta_m(0) + \beta_m(1)) &= \beta_m(0) + \frac{1}{2}\Omega_m \\ &= \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}E_{c_2}\cdots E_{c_r} + \frac{1}{2}\Omega_m, \end{aligned} \tag{35}$$

for $x \in \mathbb{Z}$.

Now, we can state our second result.

Theorem 3.2 *For each positive integer l, let*

$$\begin{aligned} \Omega_l = &\sum_{0<a\leq r} \binom{r}{a} (-1)^a 2^{r-a} \sum_{c_1+c_2+\cdots+c_a=l} \frac{1}{c_1!c_2!\cdots c_a!} E_{c_1}E_{c_2}\cdots E_{c_a} \\ &- \sum_{c_1+c_2+\cdots+c_r=l} \frac{1}{c_1!c_2!\cdots c_r!} E_{c_1}E_{c_2}\cdots E_{c_r}. \end{aligned} \tag{36}$$

Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

$$(a)\frac{1}{r}\Omega_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Big(-\sum_{j=1}^{m}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx}$$
$$=\begin{cases}\sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>), & \textit{for } x\notin\mathbb{Z},\\ \sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}E_{c_1}E_{c_2}\cdots E_{c_r}+\frac{1}{2}\Omega_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\frac{1}{r}\Omega_{m+1}+\sum_{j=1}^{m}\frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)$$
$$=\sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}E_{c_1}(<x>)\cdots E_{c_r}(<x>),\qquad \textit{for } x\notin\mathbb{Z};$$

$$\frac{1}{r}\Omega_{m+1}+\sum_{j=2}^{m}\frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)$$
$$=\sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}E_{c_1}E_{c_2}\cdots E_{c_r}+\frac{1}{2}\Omega_m,\qquad \textit{for } x\in\mathbb{Z}.$$

4 The Function $\gamma_{r,m}(<x>)$

Let $\gamma_{r,m}(x)=\sum_{c_1+c_2+\cdots+c_r=m,c_1,\cdots,c_r\geq 1}\frac{1}{c_1c_2\cdots c_r}E_{c_1}(x)E_{c_2}(x)\cdots E_{c_r}(x)$, $(m\geq r\geq 1)$, where the sum runs over all positive integers $c_1,c_2,\cdots,c_r$ with $c_1+c_2+\cdots+c_r=m$.

$$\begin{aligned}\gamma'_{r,m}(x)=&\sum_{c_1+c_2+\cdots+c_r=m,c_1,\cdots,c_r\geq 1}\frac{1}{c_2\cdots c_r}E_{c_1-1}(x)E_{c_2}(x)\cdots E_{c_r}(x)\\ &+\sum_{c_1+c_2+\cdots+c_r=m,c_1,\cdots,c_r\geq 1}\frac{1}{c_1c_3\cdots c_r}E_{c_1}(x)E_{c_2-1}(x)E_{c_3}(x)\cdots E_{c_r}(x)\\ &+\cdots+\sum_{c_1+c_2+\cdots+c_r=m,c_1,\cdots,c_r\geq 1}\frac{1}{c_1c_2\cdots c_{r-1}}E_{c_1}(x)\cdots E_{c_{r-1}}(x)E_{c_r-1}(x)\\ =&\sum_{c_2+\cdots+c_r=m-1,c_2,\cdots,c_r\geq 1}\frac{1}{c_2\cdots c_r}E_{c_2}(x)\cdots E_{c_r}(x)\\ &+\sum_{c_1+\cdots+c_r=m-1,c_1,\cdots,c_r\geq 1}\frac{1}{c_2\cdots c_r}E_{c_1}(x)\cdots E_{c_r}(x)\\ &+\cdots+\sum_{c_1+c_2+\cdots+c_{r-1}=m-1,c_1,\cdots,c_{r-1}\geq 1}\frac{1}{c_1c_2\cdots c_{r-1}}E_{c_1}(x)\cdots E_{c_{r-1}}(x)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{c_1+c_2+\cdots+c_r=m-1, c_1,\cdots,c_r\geq 1} \frac{1}{c_1 c_2 \cdots c_{r-1}} E_{c_1}(x)\cdots E_{c_r}(x) \\
&= r\gamma_{r-1,m-1}(x) + (m-1) \sum_{c_1+c_2+\cdots+c_r=m-1, c_1,\cdots,c_r\geq 1} \frac{1}{c_1\cdots c_r} E_{c_1}(x)E_{c_2}(x) \\
&\quad \cdots E_{c_r}(x) \\
&= r\gamma_{r-1,m-1}(x) + (m-1)\gamma_{r,m-1}(x). \qquad (37)
\end{aligned}
$$

Thus,

$$
\gamma'_{r,m}(x) = r\gamma_{r-1,m-1}(x) + (m-1)\gamma_{r,m-1}(x), \ (m \geq r), \qquad (38)
$$

with $\gamma_{r,r-1}(x) = 0$.

From this, we obtain.

$$
\gamma_{r,m}(x) = -\frac{r}{m}\gamma_{r-1,m}(x) + \frac{1}{m}\gamma'_{r,m+1}(x).
$$

Let $\Lambda_{r,m} = \gamma_{r,m}(1) - \gamma_{r,m}(0)$. Denoting $\int_0^1 \gamma_{r,m}(x)dx$ by $a_{r,m}$, we have

$$
a_{r,m} = -\frac{r}{m}a_{r-1,m} + \frac{1}{m}\Lambda_{r,m+1}. \qquad (39)
$$

From (39), we can easily obtain

$$
\int_0^1 \gamma_{r,m}(x)dx = \sum_{j=1}^{r} \frac{(-1)^{j-1}(r)_{j-1}}{m^j}\Lambda_{r-j+1,m+1}. \qquad (40)
$$

Here we note that

$$
a_{1,m} = \int_0^1 \gamma_{1,m}(x)dx = \frac{1}{m}\int_0^1 E_m(x)dx, \qquad (41)
$$

$$
\Lambda_{1,m+1} = \frac{1}{m+1}\left(E_{m+1}(1) - E_{m+1}(0)\right) = \int_0^1 E_m(x)dx. \qquad (42)
$$

$$
\begin{aligned}
\Lambda_{r,m} &= \gamma_{r,m}(1) - \gamma_{r,m}(0) \\
&= \sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1} \frac{1}{c_1\cdots c_r}\left(E_{c_1}(1)\cdots E_{c_r}(1) - E_{c_1}\cdots E_{c_r}\right) \\
&= \sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1} \frac{1}{c_1\cdots c_r}(-E_{c_1} + 2\delta_{0,c_1})\cdots(-E_{c_r} + 2\delta_{0,c_r})
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1}\frac{1}{c_1\cdots c_r}E_{c_1}\cdots E_{c_r} \\
&=\left((-1)^r-1\right)\sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1}\frac{1}{c_1\cdots c_r}E_{c_1}\cdots E_{c_r} \\
&=\begin{cases}-2\sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1}\frac{1}{c_1\cdots c_r}E_{c_1}\cdots E_{c_r}, & \text{for } r \ \ odd, \\ 0, & \text{for } r \ \ even.\end{cases}
\end{aligned} \tag{43}
$$

Remark 4.1 (a) We note here that $\Lambda_{r,m}=0$, and hence $\gamma_{r,m}(1)=\gamma_{r,m}(0)$, for any even positive integer r and any integer m with $m\geq r$.

(b) For r even,

$$
\int_0^1 \gamma_{r,m}(x)dx=\sum_{j=1}^{\frac{r}{2}}\frac{(-1)^{2j-1}(r)_{2j-1}}{m^{2j}}\Lambda_{r-2j+1,m+1}; \tag{44}
$$

for r odd,

$$
\int_0^1 \gamma_{r,m}(x)dx=\sum_{j=1}^{\frac{r+1}{2}}\frac{(r)_{2j-2}}{m^{2j-1}}\Lambda_{r-2j+2,m+1}. \tag{45}
$$

Also, $\gamma_{r,m}(1)=\gamma_{r,m}(0)\Leftrightarrow\Lambda_{r,m}=0$.

Now, we are going to consider the function

$$
\gamma_{r,m}(<x>)=\sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1}\frac{1}{c_1c_2\cdots c_r}E_{c_1}(<x>)E_{c_2}(<x>)\cdots E_{c_r}(<x>), \tag{46}
$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_{r,m}(<x>)$ is

$$
\sum_{n=-\infty}^{\infty}C_n^{(r,m)}e^{2\pi inx}, \tag{47}
$$

where

$$
C_n^{(r,m)}=\int_0^1\gamma_{r,m}(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_{r,m}(x)e^{-2\pi inx}dx. \tag{48}
$$

Now, we would like to determine the Fourier coefficients $C_n^{(r,m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(r,m)} &= \int_0^1 \gamma_{r,m}(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\gamma_{r,m}(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma'_{r,m}(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Lambda_{r,m} + \frac{1}{2\pi i n}\int_0^1 \Big(r\gamma_{r-1,m-1}(x) + (m-1)\gamma_{r,m-1}(x)\Big) e^{-2\pi i n x} dx \\
&= \frac{m-1}{2\pi i n} C_n^{(r,m-1)} + \frac{r}{2\pi i n} C_n^{(r-1,m-1)} - \frac{1}{2\pi i n}\Lambda_{r,m} \\
&= \frac{m-1}{2\pi i n}\Big(\frac{m-2}{2\pi i n} C_n^{(r,m-2)} + \frac{r}{2\pi i n} C_n^{(r-1,m-2)} - \frac{1}{2\pi i n}\Lambda_{r,m-1}\Big) \\
&+ \frac{r}{2\pi i n} C_n^{(r-1,m-1)} - \frac{1}{2\pi i n}\Lambda_{r,m} \\
&= \frac{(m-1)_2}{(2\pi i n)^2} C_n^{(r,m-2)} + \sum_{j=1}^{2} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,m-j)} - \sum_{j=1}^{2} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{r,m-j+1} \\
&= \cdots \\
&= \frac{(m-1)_{m-r}}{(2\pi i n)^{m-r}} C_n^{(r,r)} + \sum_{j=1}^{m-r} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,m-j)} - \sum_{j=1}^{m-r} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{r,m-j+1} \\
&= \sum_{j=1}^{m-r+1} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,m-j)} - \sum_{j=1}^{m-r+1} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{r,m-j+1}.
\end{aligned} \tag{49}$$

Here we need to note that

$$\begin{aligned}
C_n^{(r,r)} &= \int_0^1 \left(x - \frac{1}{2}\right)^r e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\left(\frac{1}{2}\right)^r - \left(-\frac{1}{2}\right)^r\right) + \frac{r}{2\pi i n} C_n^{(r-1,r-1)},
\end{aligned} \tag{50}$$

$$\Lambda_{r,r} = \gamma_{r,r}(1) - \gamma_{r,r}(0) = (\frac{1}{2})^r - (-\frac{1}{2})^r. \tag{51}$$

In addition, we can show that, for $n \neq 0$,

$$\begin{aligned}
C_n^{(1,m)} &= \frac{1}{m}\int_0^1 E_m(x) e^{-2\pi i n x} dx \\
&= \frac{2}{m}\sum_{j=1}^{m} \frac{(m)_{j-1}}{(2\pi i n)^j} E_{m-j+1},
\end{aligned} \tag{52}$$

For $n \neq 0$, (49) together with (52) determines all $C_n^{(r,m)}$ recursively.

Case 2: $n = 0$.

$$C_0^{(r,m)} = \int_0^1 \gamma_{r,m}(x)dx = \sum_{j=1}^{r} \frac{(-1)^{j-1}(r)_{j-1}}{m^j} \Lambda_{r-j+1,m+1}. \tag{53}$$

$\gamma_{r,m}(< x >)$, $(m \geq r \geq 1)$ is piecewise C^{∞}. In addition, $\gamma_{r,m}(< x >)$ is continuous for those integers r, m with $\Lambda_{r,m} = 0$ and discontinuous with jump discontinuities at integers for those integers r, m with $\Lambda_{r,m} \neq 0$. We recall here that $\Lambda_{r,m} = 0$, and hence $\gamma_{r,m}(1) = \gamma_{r,m}(0)$, for any even positive integer r and any integer m with $m \geq r$. Assume first that $\Lambda_{r,m} = 0$, for some integers r, m with $m \geq r \geq 1$. Then $\gamma_{r,m}(1) = \gamma_{r,m}(0)$.Hence $\gamma_{r,m}(< x >)$ is piecewise C^{∞} and continuous. Thus, the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\gamma_m(< x >) = C_0^{(r,m)} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n^{(r,m)} e^{2\pi i n x},$$

where $C_0^{(r,m)}$ is given by (53), and $C_n^{(r,m)}$, for each $n \neq 0$, are determined recursively from the relations (49) and (52).

Now, we are ready to state our first theorem.

Theorem 4.2 *For all integers s, l with $l \geq s \geq 1$, we let*

$$\Lambda_{s,l} = ((-1)^s - 1) \sum_{c_1+c_2+\cdots+c_s=l, c_1,\cdots,c_s \geq 1} \frac{1}{c_1 \cdots c_s} E_{c_1} \cdots E_{c_s}.$$

Assume that $\Lambda_{r,m} = 0$, for some integers r, m with $m \geq r \geq 1$. Then we have the following.

$\sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r \geq 1} \frac{1}{c_1 \cdots c_r} E_{c_1}(< x >) \cdots E_{c_r}(< x >)$ has the Fourier series expansion

$$\begin{aligned} &\sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r \geq 1} \frac{1}{c_1 \cdots c_r} E_{c_1}(< x >) \cdots E_{c_r}(< x >) \\ &= C_0^{(r,m)} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n^{(r,m)} e^{2\pi i n x}, \end{aligned}$$

where $C_0^{(r,m)} = \sum_{j=1}^{r} \frac{(-1)^{j-1}(r)_{j-1}}{m^j} \Lambda_{r-j+1,m+1}$, and $C_n^{(r,m)}$, for each $n \neq 0$, are determined recursively from

$$C_n^{(r,m)} = \sum_{j=1}^{m-r+1} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,m-j)} - \sum_{j=1}^{m-r+1} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{r,m-j+1}, \tag{54}$$

and

$$C_n^{(1,m)} = \frac{2}{m}\sum_{j=1}^{m}\frac{(m)_{j-1}}{(2\pi in)^j}E_{m-j+1}. \tag{55}$$

Here the convergence is uniform.

Next, assume that $\Lambda_{r,m} \neq 0$, for some integers r, m with $m \geq r \geq 1$. Then $\gamma_{r,m}(1) \neq \gamma_{r,m}(0)$. Hence $\gamma_{r,m}(< x >)$ is piecewise C^{∞} and discontinuous with jump discontinuities at integers. Then the Fourier series of $\gamma_{r,m}(< x >)$ converges pointwise to $\gamma_{r,m}(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}&\frac{1}{2}(\gamma_{r,m}(0) + \gamma_{r,m}(1)) = \gamma_{r,m}(0) + \frac{1}{2}\Lambda_{r,m}\\ &= \sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1}\frac{1}{c_1\cdots c_r}E_{c_1}\cdots E_{c_r} + \frac{1}{2}\Lambda_{r,m},\end{aligned} \tag{56}$$

for $x \in \mathbb{Z}$.

Hence we can now state our second theorem.

Theorem 4.3 *For all integers s, l with $l \geq s \geq 1$, we get*

$$\Lambda_{s,l} = ((-1)^s - 1)\sum_{c_1+c_2+\cdots+c_s=l, c_1,\cdots,c_s\geq 1}\frac{1}{c_1\cdots c_s}E_{c_1}\cdots E_{c_s}.$$

Assume that $\Lambda_{r,m} \neq 0$, for some integers r, m with $m \geq r \geq 1$. Then we have the following.

Let $C_0^{(r,m)}$, $C_n^{(r,m)}(n \neq 0)$ be as in Theorem 4.2. Then we have the following.

$$\begin{aligned}&C_0^{(r,m)} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} C_n^{(r,m)} e^{2\pi inx}\\ &= \begin{cases}\sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1}\frac{1}{c_1\cdots c_r}E_{c_1}(< x >)\cdots E_{c_r}(< x >), & \text{for } x \notin \mathbb{Z},\\ \sum_{c_1+c_2+\cdots+c_r=m, c_1,\cdots,c_r\geq 1}\frac{1}{c_1\cdots c_r}E_{c_1}\cdots E_{c_r} + +\frac{1}{2}\Lambda_{r,m}, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{57}$$

References

1. A. Bayad, T. Kim, Higher recurrences for Apostol-Bernoulli-Euler numbers. Russ. J. Math. Phys. **16**(1), 1–10 (2012)
2. L. Carlitz, Some formulas for the Bernoulli and Euler polynomials. Math. Nachr. **25**, 223–231 (1963)
3. D. Ding, J. Yang, Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **20**(1), 7–21 (2010)

4. G.V. Dunne, C. Schubert, Bernoulli number identities from quantum field theory and topological string theory. Commun. Number Theory Phys. **7**(2), 225–249 (2013)
5. C. Faber, R. Pandharipande, Hodge integrals and Gromov-Witten theory. Invent. Math. **139**(1), 173–199 (2000)
6. I.M. Gessel, On Miki's identities for Bernoulli numbers. J. Number Theory **110**(1), 75–82 (2005)
7. Y. He, W. Zhang, A note on the twisted Lerch type Euler zeta functions. Bull. Korean. Math. Soc. **50**(2), 659–665 (2013)
8. D.S. Kim, T. Kim, Bernoulli basis and the product of several Bernoulli polynomials. Int. J. Math. Math. Sci. (2012). Art. ID 463659
9. D.S. Kim, T. Kim, Euler basis, identities, and their applications. Int. J. Math. Math. Sci. (2012). Art. ID 343981
10. D.S. Kim, T. Kim, Some identities of higher order Euler polynomials arising from Euler basis. Integral Transform Spec. Funct. **24**(9), 734–738 (2013)
11. D.S. Kim, T. Kim, Identities arising from higher-order Daehee polynomial bases. Open Math. **13**, 196–208 (2015)
12. D.S. Kim, T. Kim, Y.H. Lee, Some arithemetic properties of Bernoulli and Euler numbers. Adv. Stud. Contemp. Math. **22**(4), 467–480 (2010)
13. T. Kim, Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials. Adv. Stud. Contemp. Math. **20**(1), 23–28 (2015)
14. T. Kim, D.S. Kim, S.-H. Rim, D.-V. Dolgy, Fourier series of higher-order Bernoulli functions and their applications. J. Inequal. Appl. 2017, 8 (2017). 7pp
15. H. Liu, W. Wang, Some identities on the the Bernoulli, Euler and Genocchi poloynomials via power sums and alternate power sums. Discret. Math. **309**, 3346–3363 (2009)
16. J.E. Marsden, *Elementary Classical Analysis* (W. H Freeman and Company, New York, 1974)
17. H. Miki, A relation between Bernoulli numbers. J. Number Theory **10**(3), 297–302 (1978)
18. K. Shiratani, S. Yokoyama, An application of p-adic convolutions. Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**, 73–83 (1982)
19. H.M. Srivastava, Some generalizations and basic extensions of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Inf. Sci. **5**(3), 390–414 (2011)
20. D.G. Zill, M.R. Cullen, *Advanced Engineering Mathematics* (Jones and Bartlett Publishers, Massachusetts, 2006)

On a New Extension of Caputo Fractional Derivative Operator

İ.O. Kıymaz, P. Agarwal, S. Jain and A. Çetinkaya

Abstract In this paper, by using a generalization of beta function we introduced a new extension of Caputo fractional derivative operator and obtained some of its properties. With the help of this extended fractional derivative operator, we also defined new extensions of some hypergeometric functions and determined their integral representations, linear and bilinear generating relations.

Keywords Caputo fractional derivative · Hypergeometric functions · Generating functions · Integral representations

2010 MSC 26A33 · 33C05 · 33C20 · 33C65

1 Introduction

In [2], Chaudhry et al. presented the following extension of Euler's beta function

$$B_p(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} e^{\left(\frac{-p}{t(1-t)}\right)} dt, \tag{1}$$

where $\Re(p) > 0$. Then in [4], Chaudhry et al. used $B_p(x, y)$ to extend the hypergeometric functions as

İ.O. Kıymaz · A. Çetinkaya
Department of Mathematics, Ahi Evran University, Kirşehir, Turkey
e-mail: shilpi.jain@poornima.org

P. Agarwal (✉)
Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India
e-mail: goyal.praveen2011@gmail.com

S. Jain
Department of Mathematics, Poornima College of Engineering, Jaipur, India
e-mail: shilpi.jain@poornima.org

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_11

$$F_p(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{B_p(b+n,c-b)}{B(b,c-b)} z^n,$$

where $p \geq 0$, $\Re(c) > \Re(b) > 0$, and $| z |< 1$. Here, the symbol $(a)_n$ denotes the Pochhammer's symbol which defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 := 1.$$

Afterward, in [8] Özarslan and Özergin obtained linear and bilinear generating relations for extended hypergeometric functions by defining the extension of the Riemann–Liouville fractional derivative (RLFD) operator by using the similar parameter with (1) as

$$\begin{aligned}\mathbf{D}_z^{\mu,p} f(z) &:= \frac{d^m}{dz^m} \mathbf{D}_z^{\mu-m} f(z)\\ &= \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int_0^z (z-t)^{-\mu+m-1} e^{\left(\frac{-pz^2}{t(z-t)}\right)} f(t)dt \right\},\end{aligned}$$

where $\Re(p) > 0$ and $m-1 < \Re(\mu) < m$.

Very recently, in [6] Kıymaz et al. used the same parameter to define the extended Caputo fractional derivative (ECFD) operator as

$$D_z^{\mu,p} f(z) := \frac{1}{\Gamma(m-\mu)} \int_0^z (z-t)^{m-\mu-1} e^{\left(\frac{-pz^2}{t(z-t)}\right)} \frac{d^m}{dt^m} f(t)dt, \tag{2}$$

where $\Re(p) > 0$ and $m-1 < \Re(\mu) < m$. In the case $p = 0$, ECFD reduces to classical Caputo fractional derivative (CFD), and also when $\mu = m \in \mathbb{N}_0$ and $p = 0$, $D_z^{m,0} f(z) := f^{(m)}(z)$. It is obvious that these extensions given above coincide with original ones when $p = 0$.

Another extension of Beta function which is given by Choi et al. in [5] is

$$B_{p,q}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} e^{\left(\frac{-p}{t} - \frac{q}{1-t}\right)} dt, \tag{3}$$

where $\min\{\Re(p), \Re(q)\} > 0$. Note that when $p = q$ and $p = q = 0$, the extension of Beta function (3) reduces to the extended Beta function (1) and the classical Beta function, respectively.

Finally in [1], Beleanu et al. defined a new extension of RLFD operator, by using the similar parameter in the definition of generalized Beta function (2) as

$$\begin{aligned}\mathbf{D}_z^{\mu}\{f(z);\,p,q\} &:= \frac{d^m}{dz^m}\mathbf{D}_z^{\mu-m}[f(z);\,p,q]\\ &= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^z (z-t)^{-\mu+m-1}e^{\left(\frac{-pz}{t}-\frac{qz}{z-t}\right)}f(t)dt\right\},\end{aligned}$$

where $\min\{\Re(p),\Re(q)\}>0$ and $m-1\le\Re(\mu)<m$. They also studied their properties in a same way with [8].

Motivated by the above works, in this paper we give a new extension of CFD operator, by using the similar parameter in the definition of generalized Beta function (3) and calculate the extended fractional derivatives of some elementary functions. Furthermore, we present extensions of some hypergeometric functions and their integral representations, and obtained linear and bilinear generating relations for extended hypergeometric functions.

2 New Extensions of Hypergeometric Functions

In this section, we introduce the new extensions of Gauss hypergeometric function ${}_2F_1$, the Appell hypergeometric functions F_1, F_2, and the Lauricella hypergeometric function F_D^3. Throughout this paper, we assume that $m\in\mathbb{N}$ and $\min\{\Re(p),\Re(q)\}>0$. The reader also should note that

(i) when $p=q$, the following definitions (4)–(7) and (8) reduce to the corresponding definitions (2.1), (2.2), (2.5), (2.6) and (3.1) which is given in [6], respectively,
(ii) when $p=q=0$, the following definitions (4)–(7) and (8) reduce to well-known Gauss hypergeometric function ${}_2F_1$, Appell functions F_1, F_2, Lauricella function F_D^3 and CFD operator, respectively.

Definition 1 The extended Gauss hypergeometric function is defined for all $|z|<1$ as

$$ {}_2F_1(a,b;c;z;\,p,q) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(b-m)_n}\frac{B_{p,q}(b-m+n,c-b+m)}{B(b-m,c-b+m)}\frac{z^n}{n!}, \tag{4}$$

where $m<\Re(b)<\Re(c)$.

Definition 2 The extended Appell hypergeometric function is defined for all $|x|<1$, $|y|<1$ as

$$\begin{aligned}&F_1(a,b,c;d;x,y;\,p,q) :=\\ &\sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_n(c)_k}{(a-m)_{n+k}}\frac{B_{p,q}(a-m+n+k,d-a+m)}{B(a-m,d-a+m)}\frac{x^n}{n!}\frac{y^k}{k!},\end{aligned} \tag{5}$$

where $m<\Re(a)<\Re(d)$.

Definition 3 The extended Appell hypergeometric function is defined for all $|\ x\ |+|\ y\ |<1$ as

$$F_2(a,b,c;d,e;x,y;p,q):=\\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(a)_{n+k}(b)_n(c)_k}{(b-m)_n(c-m)_k}\frac{B_{p,q}(b-m+n,d-b+m)}{B(b-m,d-b+m)}\right.\\ \left.\frac{B_{p,q}(c-m+k,e-c+m)}{B(c-m,e-c+m)}\frac{x^n y^k}{n!k!}\right], \tag{6}$$

where $m<\Re(b)<\Re(d)$, and $m<\Re(c)<\Re(e)$.

Definition 4 The extended Lauricella hypergeometric function is defined for all $|x|<1, |y|<1, |z|<1$ as

$$F_D^3(a,b,c,d;e;x,y,z;p,q):=\sum_{n,k,r=0}^{\infty}\left[\frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(a-m)_{n+k+r}}\right.\\ \left.\frac{B_{p,q}(a-m+n+k+r,e-a+m)}{B(a-m,e-a+m)}\frac{x^n}{n!}\frac{y^k}{k!}\frac{z^r}{r!}\right], \tag{7}$$

where $m<\Re(a)<\Re(e)$.

3 A New Extension of CFD Operator

The classical CFD operator is defined in [7] as

$$D^{\mu}f(z):=\frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1}\frac{d^m}{dt^m}f(t)dt,$$

where $m-1<\Re(\mu)<m$.

Inspired by the same idea in [1, 6], we introduce a new extension of Caputo fractional derivative (NECFD) operator as

$$D_z^{\mu}\{f(z);p,q\}:=\frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1}e^{\left(\frac{-pz}{t}-\frac{-qz}{z-t}\right)}\frac{d^m}{dt^m}f(t)dt, \tag{8}$$

where $m-1<\Re(\mu)<m$.

Now, we begin our investigation by calculating the NECFD's of some elementary functions.

Theorem 1 *Let $m-1<\Re(\mu)<m$, and $\Re(\mu)<\Re(\lambda)$ then*

$$D_z^{\mu}\left\{z^{\lambda};p,q\right\}=\frac{\Gamma(\lambda+1)B_{p,q}(\lambda-m+1,m-\mu)}{\Gamma(\lambda-\mu+1)B(\lambda-m+1,m-\mu)}z^{\lambda-\mu}.$$

Proof With direct calculation, we get

$$\begin{aligned}
D_z^{\mu}\left\{z^{\lambda};p,q\right\}&=\frac{1}{\Gamma(m-\mu)}\int_0^z(z-t)^{m-\mu-1}e^{\left(\frac{-pz}{t}-\frac{qz}{z-t}\right)}\frac{d^m}{dt^m}t^{\lambda}dt\\
&=\frac{1}{\Gamma(m-\mu)}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)}\int_0^z(z-t)^{m-\mu-1}t^{\lambda-m}e^{\left(\frac{-pz}{t}-\frac{qz}{z-t}\right)}dt\\
&=\frac{z^{\lambda-\mu}}{\Gamma(m-\mu)}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)}\int_0^1(1-u)^{m-\mu-1}u^{\lambda-m}e^{\left(\frac{-p}{u}-\frac{q}{1-u}\right)}du\\
&=\frac{\Gamma(\lambda+1)B_{p,q}(\lambda-m+1,m-\mu)}{\Gamma(\lambda-\mu+1)B(\lambda-m+1,m-\mu)}z^{\lambda-\mu}.
\end{aligned}$$

□

Remark 1 Note that $D_z^{\mu}\left\{z^{\lambda};p,q\right\}=0$ for $\lambda=0,1,\ldots,m-1$.

The next theorem expresses the NECFD of an analytic function.

Theorem 2 *If $f(z)$ is an analytic function on the disk $|z|<\rho$ with its power series expansion $f(z)=\sum_{n=0}^{\infty}a_nz^n$, then*

$$D_z^{\mu}\{f(z);p,q\}=\sum_{n=0}^{\infty}a_nD_z^{\mu}\left\{z^n;p,q\right\}$$

where $m-1<\Re(\mu)<m$.

Proof Using the power series expansion of f, we get

$$D_z^{\mu}\left\{f(z);p,q\right\}=\frac{1}{\Gamma(m-\mu)}\int_0^z(z-t)^{m-\mu-1}e^{\left(\frac{-pz}{t}-\frac{qz}{z-t}\right)}\sum_{n=0}^{\infty}a_n\frac{d^m}{dt^m}t^ndt.$$

Since the power series converges uniformly and the integral converges absolutely, then the order of the integration and the summation can be changed. So we get,

$$D_z^{\mu}\left\{f(z);\,p,q\right\}=\sum_{n=0}^{\infty}a_n\left(\frac{1}{\Gamma(m-\mu)}\int_0^z(z-t)^{m-\mu-1}e^{\left(\frac{-pz}{t}-\frac{qz}{z-t}\right)}\frac{d^m}{dt^m}t^n dt\right)$$
$$=\sum_{n=0}^{\infty}a_n D_z^{\mu}\left\{z^n;\,p,q\right\}.$$

□

The proof of the following theorem is obvious from Theorems 1 and 2.

Theorem 3 *If $f(z)$ is an analytic function on the disk $\mid z\mid<\rho$ with its power series expansion $f(z)=\sum_{n=0}^{\infty}a_n z^n$, then*

$$D_z^{\mu}\left\{z^{\lambda-1}f(z);\,p,q\right\}=\sum_{n=0}^{\infty}a_n D_z^{\mu}\left\{z^{\lambda+n-1};\,p,q\right\}$$
$$=\frac{\Gamma(\lambda)z^{\lambda-\mu-1}}{\Gamma(\lambda-\mu)}\sum_{n=0}^{\infty}a_n\frac{(\lambda)_n}{(\lambda-\mu)_n}\frac{B_{p,q}(\lambda-m+n,m-\mu)}{B(\lambda-m+n,m-\mu)}z^n$$
$$=\frac{\Gamma(\lambda)z^{\lambda-\mu-1}}{\Gamma(\lambda-\mu)}\sum_{n=0}^{\infty}a_n\frac{(\lambda)_n}{(\lambda-m)_n}\frac{B_{p,q}(\lambda-m+n,m-\mu)}{B(\lambda-m,m-\mu)}z^n$$

where $m-1<\Re(\mu)<m<\Re(\lambda)$.

The following theorems will be useful for finding the generating function relations.

Theorem 4 *Let $m-1<\Re(\lambda-\mu)<m<\Re(\lambda)$, then*

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha};\,p,q\right\}=\frac{\Gamma(\lambda)z^{\mu-1}}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\lambda)_n}{(\lambda-m)_n}\frac{B_{p,q}(\lambda-m+n,\mu-\lambda+m)}{B(\lambda-m,\mu-\lambda+m)}\frac{z^n}{n!}$$
$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}{}_2F_1(\alpha,\lambda;\mu;z;p,q) \tag{9}$$

for $\mid z\mid<1$.

Proof If we use the power series expansion of $(1-z)^{-\alpha}$ and (4), we get

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha};\,p,q\right\}=D_z^{\lambda-\mu}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!};\,p,q\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}D_z^{\lambda-\mu,p}\left\{z^{\lambda+n-1};\,p,q\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(\mu+n)}\frac{B_{p,q}(\lambda-m+n,m-\lambda+\mu)}{B(\lambda-m+n,m-\lambda+\mu)}z^{\mu+n-1}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\lambda)_n}{(\mu)_n} \frac{B_{p,q}(\lambda - m + n, m - \lambda + \mu)}{B(\lambda - m + n, m - \lambda + \mu)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\lambda)_n}{(\lambda - m)_n} \frac{B_{p,q}(\lambda - m + n, \mu - \lambda + m)}{B(\lambda - m, \mu - \lambda + m)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} {}_2F_1(\alpha, \lambda; \mu; z; p, q).$$

□

Theorem 5 *Let $m - 1 < \Re(\lambda - \mu) < m < \Re(\lambda)$, then*

$$D_z^{\lambda-\mu} \left\{ z^{\lambda-1} (1 - az)^{-\alpha} (1 - bz)^{-\beta}; p, q \right\}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k=0}^{\infty} \frac{(\lambda)_{n+k} (\alpha)_n (\beta)_k}{(\lambda - m)_{n+k}} \frac{B_{p,q}(\lambda - m + n + k, \mu - \lambda + m)}{B(\lambda - m, \mu - \lambda + m)} \frac{(az)^n}{n!} \frac{(bz)^k}{k!}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1(\lambda, \alpha, \beta; \mu; az; bz; p, q) \tag{10}$$

for $| az |< 1$ *and* $| bz |< 1$.

Proof Using the power series expansion of $(1 - az)^{-\alpha}$, $(1 - bz)^{-\beta}$, and (5), we get

$$D_z^{\lambda-\mu} \left\{ z^{\lambda-1} (1 - az)^{-\alpha} (1 - bz)^{-\beta}; p, q \right\}$$
$$= D_z^{\lambda-\mu} \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{(\beta)_k}{k!} a^n b^k z^{\lambda+n+k-1}; p, q \right\}$$
$$= \sum_{n,k=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{(\beta)_k}{k!} a^n b^k D_z^{\lambda-\mu} \left\{ z^{\lambda+n+k-1}; p, q \right\}$$
$$= \sum_{n,k=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{(\beta)_k}{k!} a^n b^k \frac{\Gamma(\lambda + n + k) B_{p,q}(\lambda - m + n + k, m - \lambda + \mu)}{\Gamma(\lambda - m + n + k) \Gamma(m - \lambda + \mu)} z^{\mu+n+k-1}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k=0}^{\infty} \frac{(\lambda)_{n+k} (\alpha)_n (\beta)_k}{(\lambda - m)_{n+k}} \frac{B_{p,q}(\lambda - m + n + k, m - \lambda + \mu)}{B(\lambda - m, m - \lambda + \mu)} \frac{(az)^n}{n!} \frac{(bz)^k}{k!}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1(\lambda, \alpha, \beta; \mu; az; bz; p, q).$$

□

Theorem 6 *Let* $m-1 < \Re(\lambda-\mu) < m < \Re(\lambda)$*, then*

$$
\begin{aligned}
&D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma};\, p,q\right\}\\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k,r=0}^{\infty} \frac{(\lambda)_{n+k+r}(\alpha)_n(\beta)_k(\gamma)_r}{(\lambda-m)_{n+k+r}} \frac{B_{p,q}(\lambda-m+n+k+r,\mu-\lambda+m)}{B(\lambda-m,\mu-\lambda+m)}\\
&\times \frac{(az)^n}{n!}\frac{(bz)^k}{k!}\frac{(cz)^r}{r!}\\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_D^3(\lambda,\alpha,\beta,\gamma;\mu;az;bz;cz;\,p,q)
\end{aligned} \tag{11}
$$

for $|\, az \,| < 1, |\, bz \,| < 1$ *and* $|\, cz \,| < 1$.

Proof Using the power series expansion of $(1-az)^{-\alpha}$, $(1-bz)^{-\beta}$, $(1-cz)^{-\gamma}$, and (7), we get

$$
\begin{aligned}
&D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma};\, p,q\right\}\\
&= D_z^{\lambda-\mu}\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha)_n}{n!}\frac{(\beta)_k}{k!}\frac{(\gamma)_r}{r!} a^n b^k c^r z^{\lambda+n+k+r-1};\, p,q\right\}\\
&= \sum_{n,k,r=0}^{\infty} \frac{(\alpha)_n}{n!}\frac{(\beta)_k}{k!}\frac{(\gamma)_r}{r!} a^n b^k c^r D_z^{\lambda-\mu}\left\{z^{\lambda+n+k+r-1};\, p,q\right\}\\
&= \sum_{n,k,r=0}^{\infty} \frac{(\alpha)_n}{n!}\frac{(\beta)_k}{k!}\frac{(\gamma)_r}{r!} a^n b^k c^r\\
&\quad \times \frac{\Gamma(\lambda+n+k+r)B_{p,q}(\lambda-m+n+k+r,m-\lambda+\mu)}{\Gamma(\lambda-m+n+k+r)\Gamma(m-\lambda+\mu)} z^{\mu+n+k+r-1}\\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k,r=0}^{\infty} \frac{(\lambda)_{n+k+r}(\alpha)_n(\beta)_k(\gamma)_r}{(\lambda-m)_{n+k+r}} \frac{B_{p,q}(\lambda-m+n+k+r,m-\lambda+\mu)}{B(\lambda-m,m-\lambda+\mu)}\\
&\quad \times \frac{(az)^n}{n!}\frac{(bz)^k}{k!}\frac{(cz)^r}{r!}\\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_D^3(\lambda,\alpha,\beta,\gamma;\mu;az;bz;cz;\,p,q).
\end{aligned}
$$

□

Theorem 7 *Let* $m-1 < \Re(\lambda-\mu) < m < \Re(\lambda)$ *and* $m < \Re(\beta) < \Re(\gamma)$*, then*

$$
D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha}\,{}_2F_1\left(\alpha,\beta;\gamma;\frac{x}{1-z};\,p,q\right);\,p,q\right\}
$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[\frac{(\alpha)_{n+k} (\beta)_n (\lambda)_k}{(\beta - m)_n (\lambda - m)_k} \frac{B_{p,q}(\beta - m + n, \gamma - \beta + m)}{B(\beta - m, \gamma - \beta + m)} \right.$$
$$\left. \cdot \frac{B_{p,q}(\lambda - m + k, \mu - \lambda + m)}{B(\lambda - m, \mu - \lambda + m)} \frac{x^n z^k}{n! k!} \right]$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2(\alpha, \beta, \lambda; \gamma, \mu; x, z; p, q). \tag{12}$$

for $| x | + | z |< 1$.

Proof Using the power series expansion of $(1 - z)^{-\alpha}$ and Eqs. (4) and (6), we get

$$D_z^{\lambda-\mu} \left\{ z^{\lambda-1} (1 - z)^{-\alpha} {}_2F_1 \left(\alpha, \beta; \gamma; \frac{x}{1 - z}; p, q \right); p, q \right\}$$
$$= D_z^{\lambda-\mu} \left\{ z^{\lambda-1} (1 - z)^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\beta - m)_n n!} \frac{B_{p,q}(\beta - m + n, \gamma - \beta + m)}{B(\beta - m, \gamma - \beta + m)} \left(\frac{x}{1 - z} \right)^n; p, q \right\}$$
$$= D_z^{\lambda-\mu} \left\{ z^{\lambda-1} (1 - z)^{-\alpha-n} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\beta - m)_n} \frac{B_{p,q}(\beta - m + n, \gamma - \beta + m)}{B(\beta - m, \gamma - \beta + m)} \frac{x^n}{n!}; p, q \right\}$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\beta - m)_n} \frac{B_{p,q}(\beta - m + n, \gamma - \beta + m)}{B(\beta - m, \gamma - \beta + m)} \frac{x^n}{n!} D_z^{\lambda-\mu} \left\{ z^{\lambda-1} (1 - z)^{-\alpha-n}; p, q \right\}$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[\frac{(\alpha)_{n+k} (\beta)_n (\lambda)_k}{(\beta - m)_n (\lambda - m)_k} \frac{B_{p,q}(\beta - m + n, \gamma - \beta + m)}{B(\beta - m, \gamma - \beta + m)} \right.$$
$$\left. \cdot \frac{B_{p,q}(\lambda - m + k, \mu - \lambda + m)}{B(\lambda - m, \mu - \lambda + m)} \frac{x^n z^k}{n! k!} \right]$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2(\alpha, \beta, \lambda; \gamma, \mu; x, z; p, q).$$

□

4 Generating Function Relations

In this section, we use the equalities (9), (10), and (12) for obtaining linear and bilinear generating relations for the extension of hypergeometric function ${}_2F_1$.

Theorem 8 *Let* $m - 1 < \Re(\lambda - \mu) < m < \Re(\lambda)$, *then*

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha + n, \lambda; \mu; z; p, q) t^n = (1 - t)^{-\alpha} {}_2F_1 \left(\alpha, \lambda; \mu; \frac{z}{1 - t}; p, q \right) \tag{13}$$

where $|z| < min\{1, |1 - t|\}$.

Proof Taking the identity

$$[(1-z)-t]^{-\alpha} = (1-t)^{-\alpha}\left(1-\frac{z}{1-t}\right)^{-\alpha}$$

in [12] and expanding the left-hand side, we get

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}(1-z)^{-\alpha}\left(\frac{t}{1-z}\right)^{n} = (1-t)^{-\alpha}\left(1-\frac{z}{1-t}\right)^{-\alpha}$$

when $|t| < |1-z|$. If we multiply both sides with $z^{\lambda-1}$ and apply the NECFD operator, we get

$$\begin{aligned} D_z^{\lambda-\mu}&\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n t^n}{n!}z^{\lambda-1}(1-z)^{-\alpha-n};\, p,q\right\} \\ &= D_z^{\lambda-\mu}\left\{(1-t)^{-\alpha}z^{\lambda-1}\left(1-\frac{z}{1-t}\right)^{-\alpha};\, p,q\right\}. \end{aligned}$$

Since $|t| < |1-z|$ and $\Re(\lambda) > \Re(\mu) > 0$, it is possible to change the order of the summation and the derivative as

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha-n};\, p,q\right\}t^n &= (1-t)^{-\alpha} \\ D_z^{\lambda-\mu}&\left\{z^{\lambda-1}\left(1-\frac{z}{1-t}\right)^{-\alpha};\, p,q\right\}. \end{aligned}$$

So we get the result after using Theorem 4 on both sides. □

Theorem 9 *Let $m-1 < \Re(\lambda-\mu) < m < \Re(\lambda)$, then*

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}{}_2F_1(\beta-n,\lambda;\mu;z;p,q)t^n = (1-t)^{-\alpha}F_1\left(\lambda,\beta,\alpha;\mu;z;\frac{-zt}{1-t};p,q\right)$$

where $|t| < \frac{1}{1+|z|}$.

Proof Taking the identity

$$[1-(1-z)t]^{-\alpha} = (1-t)^{-\alpha}\left(1+\frac{zt}{1-t}\right)^{-\alpha}$$

in [12] and expanding the left hand side, we get

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (1-z)^n t^n = (1-t)^{-\alpha} \left(1 - \frac{-zt}{1-t}\right)^{-\alpha}$$

when $|t| < |1-z|$ over minus 1. If we multiply both sides with $z^{\lambda-1}(1-z)^{-\beta}$ and apply the NECFD operator, we get

$$D_z^{\lambda-\mu} \left\{ \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^{\lambda-1}(1-z)^{-\beta}(1-z)^n t^n; p, q \right\}$$
$$= D_z^{\lambda-\mu} \left\{ (1-t)^{-\alpha} z^{\lambda-1}(1-z)^{-\beta} \left(1 - \frac{-zt}{1-t}\right)^{-\alpha}; p, q \right\}.$$

Since $|zt| < |1-t|$ and $\Re(\lambda) > \Re(\mu) > 0$, it is possible to change the order of the summation and the derivative as

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} D_z^{\lambda-\mu} \left\{ z^{\lambda-1}(1-z)^{-\beta+n}; p, q \right\} t^n$$
$$= (1-t)^{-\alpha} D_z^{\lambda-\mu} \left\{ z^{\lambda-1}(1-z)^{-\beta} \left(1 - \frac{-zt}{1-t}\right)^{-\alpha}; p, q \right\}.$$

So we get the result after using Theorems 4 and 5. □

Theorem 10 *Let $m-1 < \Re(\beta-\gamma) < m < \Re(\beta)$ and $m < \Re(\lambda) < \Re(\mu)$, then*

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha+n, \lambda; \mu; z; p, q) {}_2F_1(-n, \beta; \gamma; u; p, q)$$
$$= 1 - t^{-\alpha} F_2 \left(\alpha, \lambda, \beta; \mu, \gamma; \frac{z}{1-t}, \frac{-ut}{1-t}; p, q \right).$$

Proof If we take $t \to (1-u)t$ in (13) and then multiply both sides with $u^{\beta-1}$, we get

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha+n, \lambda; \mu; z; p, q) u^{\beta-1}(1-u)^n t^n$$
$$= u^{\beta-1}[1-(1-u)t]^{-\alpha} {}_2F_1 \left(\alpha, \lambda; \mu; \frac{z}{1-(1-u)t}; p, q \right).$$

Applying the NECFD $D_u^{\beta-\gamma}$ to both sides and changing the order, we find

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha+n, \lambda; \mu; z; p, q) D_u^{\beta-\gamma} \left\{ u^{\beta-1}(1-u)^n; p, q \right\} t^n$$
$$= D_u^{\beta-\gamma} \left\{ u^{\beta-1}[1-(1-u)t]^{-\alpha} {}_2F_1\left(\alpha, \lambda; \mu; \frac{z}{1-(1-u)t}; p, q\right); p, q \right\}$$

when $|z| < 1$, $\left|\frac{1-u}{1-z} t\right| < 1$ and $\left|\frac{z}{1-t}\right| + \left|\frac{ut}{1-t}\right| < 1$. If we write the equality like

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha+n, \lambda; \mu; z; p, q) D_u^{\beta-\gamma} \left\{ u^{\beta-1}(1-u)^n; p, q \right\} t^n$$
$$= D_u^{\beta-\gamma} \left\{ u^{\beta-1} \left[1 - \frac{-ut}{1-t}\right]^{-\alpha} {}_2F_1\left(\alpha, \lambda; \mu; \frac{\frac{z}{1-t}}{1-\frac{-ut}{1-t}}; p, q\right); p, q \right\}$$

and using Theorems 4 and 7, we get the desired result. □

5 Further Results and Observations

In this section, we apply the NECFD operator (8) to familiar functions e^z and ${}_2F_1(a, b; c; z)$. We also obtain the Mellin transforms of some NECFD, and we give the integral representations of extended hypergeometric functions.

Theorem 11 *The NECFD of $f(z) = e^z$ is*

$$D_z^{\mu}\{e^z; p, q\} = \frac{z^{m-\mu}}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{z^n}{n!} B_{p,q}(n+1, m-\mu)$$

for all z.

Proof Using the power series expansion of e^z and Theorem 2, we get

$$D_z^{\mu}\{e^z; p, q\} = \sum_{n=0}^{\infty} \frac{1}{n!} D_z^{\mu}\{z^n; p, q\}$$
$$= \sum_{n=m}^{\infty} \frac{\Gamma(n+1) B_{p,q}(n-m+1, m-\mu)}{\Gamma(n-\mu+1) B(n-m+1, m-\mu)} \frac{z^{n-\mu}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+m+1)B_{p,q}(n+1,m-\mu)}{\Gamma(n+m-\mu+1)B(n+1,m-\mu)} \frac{z^{n+m-\mu}}{(n+m)!}$$

$$= \frac{z^{m-\mu}}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{z^n}{n!} B_{p,q}(n+1,m-\mu).$$

□

Theorem 12 *The NECFD of* ${}_2F_1(a,b;c;z)$ *is*

$$D_z^{\mu}\left\{{}_2F_1(a,b;c;z);\,p,q\right\} = \frac{(a)_m(b)_m}{(c)_m}\frac{z^{m-\mu}}{\Gamma(1-\mu+m)}$$
$$\cdot \sum_{n=0}^{\infty} \frac{(a+m)_n(b+m)_n}{(c+m)_n(1-\mu+m)_n}\frac{B_{p,q}(n+1,m-\mu)\,z^n}{B(m-\mu,n+1)}$$

for $|\,z\,| < 1$.

Proof Using the power series expansion of ${}_2F_1(a,b;c;z)$ and making similar calculations, we get

$$D_z^{\mu}\{{}_2F_1(a,b;c;z);\,p,q\} = D_z^{\mu}\left\{\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!};\,p,q\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}D_z^{\mu}\left\{z^n;\,p,q\right\}$$
$$= \sum_{n=m}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\frac{\Gamma(n+1)B_{p,q}(n-m+1,m-\mu)}{\Gamma(n-\mu+1)B(m-\mu,n-m+1)}z^{n-\mu}$$
$$= \sum_{n=0}^{\infty}\frac{(a)_{n+m}(b)_{n+m}}{(c)_{n+m}(n+m)!}\frac{\Gamma(n+m+1)B_{p,q}(n+1,m-\mu)}{\Gamma(n+m-\mu+1)B(m-\mu,n+1)}z^{n+m-\mu}$$
$$= \frac{(a)_m(b)_m}{(c)_m}\frac{z^{m-\mu}}{\Gamma(1-\mu+m)}\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{(c+m)_n(1-\mu+m)_n}\frac{B_{p,q}(n+1,m-\mu)\,z^n}{B(m-\mu,n+1)}.$$

□

The following theorem is about the integral representations of new extensions of hypergeometric functions.

Theorem 13 *The following integral representations are valid*

$$ {}_2F_1(a,b;c;z;\,p,q) = \frac{1}{B(b-m,c-b+m)}\int_0^1 \left\{t^{b-m-1}(1-t)^{c-b+m-1}\right.$$
$$\left. e^{\left(\frac{-p}{t}-\frac{q}{1-t}\right)}{}_2F_1(a,b;b-m;zt)\right\}dt, \tag{14}$$

$$F_1(a,b,c;d;x,y;p,q)=\frac{1}{B(a-m,d-a+m)}\int_0^1\Big\{t^{a-m-1}(1-t)^{d-a+m-1} \\ e^{\left(\frac{-p}{t}-\frac{q}{1-t}\right)}F_1(a,b,c;a-m;xt,yt)\Big\}dt, \tag{15}$$

$$F_2(a,b,c;d,e;x,y;p,q)=\frac{1}{B(b-m,d-b+m)B(c-m,e-c+m)} \\ \int_0^1\int_0^1\Big\{t^{b-m-1}u^{c-m-1}(1-t)^{d-b+m-1}(1-u)^{e-c+m-1} \\ e^{\left(\frac{-p}{t}-\frac{q}{1-t}-\frac{p}{u}-\frac{q}{1-u}\right)}F_2(a,b,c;b-m,c-m;xt,yu)\Big\}dtdu. \tag{16}$$

Proof The integral representations (14)–(16) can be obtained directly by replacing the function $B_{p,q}$ with its integral representation in (4)–(6), respectively. □

Acknowledgements This work was supported by the Ahi Evran University Scientific Research Projects Coordination Unit. Project Number: FEF.D1.16.001.

References

1. D. Baleanu, P. Agarwal, R.K. Parmar, M. Al. Qurashi, S. Salahshour, Extension of the fractional derivative operator of the Riemann–Liouville. J. Nonlinear Sci. Appl. **10**, 2914–2924 (2017)
2. M.A. Chaudhry, A. Qadir, M. Rafique, S.M. Zubair, Extension of Euler's beta function. J. Comput. Appl. Math. **78**, 19–32 (1997)
3. M.A. Chaudhry, S.M. Zubair, Generalized incomplete gamma functions with applications. J. Comput. Appl. Math. **55**, 99–124 (1994)
4. M.A. Chaudhry, A. Qadir, H.M. Srivastava, R.B. Paris, Extended hypergeometric and confluent hypergeometric functions. Appl. Math. Comput. **159**(2), 589–602 (2004)
5. J. Choi, A.K. Rathie, R.K. Parmar, Extension of extended beta, hypergeometric and confluent hypergeometric functions. Honam Math. J. **36**(2), 357–385 (2014)
6. İ.O. Kıymaz, A. Çetinkaya, P. Agarwal, An extension of Caputo fractional derivative operator and its applications. J. Nonlinear Sci. Appl. **9**, 3611–3621 (2016)
7. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006)
8. M.A. Özarslan, E. Özergin, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator. Math. Comput. Model. **52**, 1825–1833 (2010)
9. E. Özergin, Some Properties of Hypergeometric Functions, Ph.D. Thesis, Eastern Mediterranean University, North Cyprus, Turkey (2011)
10. E. Özergin, M.A. Özarslan, A. Altın, Extension of gamma, beta and hypergeometric functions. J. Comput. Appl. Math. **235**, 4601–4610 (2011)
11. R.K. Parmar, Some Generating Relations For Generalized Extended Hypergeometric Functions Involving Generalized Fractional Derivative Operator, J. Concrete Appl. Math. 217 (2014)
12. H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions* (Ellis Horwood Limited, Chichester, 1984)

13. H.M. Srivastava, P. Agarwal, S. Jain, Generating functions for the generalized Gauss hypergeometric functions. Appl. Math. Comput. **247**, 348–352 (2014)
14. H.M. Srivastava, A. Çetinkaya, İ.O. Kıymaz, A certain generalized Pochhammer symbol and its applications to hypergeometric functions. Appl. Math. Comput. **226**, 484–491 (2014)

An Extension of the Shannon Wavelets for Numerical Solution of Integro-Differential Equations

Maryam Attary

Abstract In this work, an extension of the algebraic formulation of the Shannon wavelets for the numerical solution of a class of Volterra integro-differential equation is proposed. Our approach is based on the connection coefficients of the Shannon wavelet and collocation method for constructing the algebraic equivalent representation of the problem. Also, the Shannon approximation is applied to solve one type of nonlinear integral equation arising from chemical phenomenon. An analysis of error for the problem is given. The obtained numerical results show the accuracy of the presented method.

Keywords Integro-differential equations · Shannon wavelet · Numerical approximation of solutions

2010 AMS Math. Subject Classification Primary 45J05 · Secondary 34K28

1 Introduction

Integral, integro-differential, ordinary and fractional differential equations are used in modelling problems of engineering and science fields, including mathematical biology, electromagnetic theory, potential theory and chemical engineering, see [1, 2, 5, 7, 8, 11, 14] and references therein.

The main purpose in this article is to develop and to provide a numerical algorithm based on the coefficients of the Shannon wavelets for the following form of integro-differential equation

$$\sum_{i=0}^{1} \Gamma_i u^{(i)}(x) = f(x) + \int_a^x k(x,t)u(t)dt, \tag{1}$$

M. Attary (✉)
Department of Mathematics, Islamic Azad University,
Karaj Branch, Karaj, Iran
e-mail: maryam.attari@kiau.ac.ir

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_12

$$u(a) = a_0,$$

where Γ_i are constants, k and f are given functions and $u(x)$ is a solution to be determined. Noting that for $\Gamma_1 = 0$, (1) be transformed to integral equation.

Over the past few decades, the numerical solvability of these type of equations has been studied intensively by many authors, such as Chebyshev spectral solution [6], rationalized Haar functions [12] and Sinc-Legendre collocation method [13].

Wavelets are very powerful and useful tool in data compression, signal and operator analysis. The real part of the harmonic wavelets is Shannon wavelets. These wavelets can be used to study frequency changes as well as oscillations in a small range time interval [4].

This paper is organized as follows: Sect. 2 introduces some basic definitions and preliminaries of the Shannon wavelets. We derive formulas for a class of IDEs and give a numerical scheme based on proposed method in Sect. 3. Error analysis of our method is considered in Sect. 4. Finally, in Sect. 5, we report several numerical experiments to clarify the efficiency and accuracy of the proposed method.

2 Preliminary Definitions

Here, we give some basic definitions of the Shannon wavelets family [4, 9]. The *Sinc* function is defined on the whole real line by:

$$Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

The Shannon scaling functions and mother wavelets can be defined as:

$$\begin{cases} \varphi_{j,k}(x) = 2^{j/2} Sinc(2^j x - k) = 2^{j/2} \dfrac{\sin \pi(2^j x - k)}{\pi(2^j x - k)}, & j,k \in Z, \\ \psi_{j,k}(x) = 2^{j/2} \dfrac{\sin \pi(2^j x - k - \frac{1}{2}) - \sin 2\pi(2^j x - k - \frac{1}{2})}{\pi(2^j x - k - \frac{1}{2})}, & j,k \in Z, \end{cases}$$

we recall the following theorem from [3]:

Theorem 2.1 *If $u(x) \in L_2(\mathbb{R})$, then*

$$u(x) = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_{0,k}(x) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_{j,k} \psi_{j,k}(x), \tag{2}$$

with

$$\alpha_k =< u, \varphi_{0,k} >= \int_{-\infty}^{\infty} u(x) \varphi_{0,k}(x) dx, \tag{3}$$

$$\beta_{j,k} =< u, \psi_{j,k} >= \int_{-\infty}^{\infty} u(x)\psi_{j,k}(x)dx. \tag{4}$$

Using a finite truncated series of the above theorem, we can define an approximation function of the exact solution $u(x)$ as follows:

$$u(x) \simeq \sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k}\psi_{j,k}(x). \tag{5}$$

The nth derivatives of $u(x)$ in terms of the Shannon wavelets can be written as (see e.g. [9] for further details):

$$u^{(n)}(x) \simeq \sum_{k=-M}^{M} \alpha_k \varphi_{0,k}^{(n)}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k}\psi_{j,k}^{(n)}(x), \tag{6}$$

on the other hand, we have the following relations [4]:

$$\varphi_{0,k}^{(n)}(x) = \sum_{h=-M}^{M} \lambda_{kh}^{(n)} \varphi_{0,h}(x) \tag{7}$$

$$\psi_{j,k}^{(n)}(x) = \sum_{h=-M}^{M} \gamma_{kh}^{(n)jj} \psi_{j,h}(x). \tag{8}$$

Therefore, (6) rewritten as:

$$u^{(n)}(x) \simeq \sum_{k=-M}^{M} \alpha_k \sum_{h=-M}^{M} \lambda_{kh}^{(n)} \varphi_{0,h}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \sum_{h=-M}^{M} \gamma_{kh}^{(n)jj} \psi_{j,h}(x), \tag{9}$$

where

$$\lambda_{kh}^{(n)} = \begin{cases} (-1)^{k-h} \frac{i^n}{2\pi} \sum_{s=1}^{n} \dfrac{l!\pi^s}{s![i(k-h)]^{n-s+1}}[(-1)^s - 1], & k \neq h, \\ \dfrac{i^n \pi^{n+1}}{2\pi(n+1)}[1 + (-1)^n], & k = h, \end{cases} \tag{10}$$

$$\gamma_{kh}^{(n)jj} = \begin{cases} \dfrac{i^n 2^{jn}}{2\pi} \Sigma_{m=1}^{n} (-1)^n \dfrac{n!\pi^m(2^m-1)}{m![i(h-k)]^{n-m+1}}[(-1)^m - 1], & k \neq h, \\ \dfrac{i^n 2^{jn} \pi^{n+1}}{2\pi(n+1)}[1 + (-1)^n][2^{n+1} - 1], & k = h, \end{cases} \tag{11}$$

which $\lambda_{kh}^{(n)}$ and $\gamma_{kh}^{(n)jj}$ are known as the connection coefficients.

Moreover, it is

$$\gamma_{kh}^{(n)jj} = 2^{n(j-1)}\gamma_{kh}^{(n)11}. \tag{12}$$

3 Numerical Treatment of the Problem

In this section, we will obtain formulas for numerical solvability of (1), based on the previous results. We define an approximation function $u^{'}(x)$ as follows:

$$u^{'}(x) \simeq \sum_{k=-M}^{M} \alpha_k \sum_{h=-M}^{M} \lambda_{kh}^{(1)} \varphi_{0,h}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \sum_{h=-M}^{M} \gamma_{kh}^{(1)jj} \psi_{j,h}(x). \tag{13}$$

By taking $n = 1$ in (10), (11) and using simple computations, we obtain the following relations for $\lambda_{kh}^{(1)}$ and $\gamma_{kh}^{(1)jj}$:

$$\lambda_{kh}^{(1)} = \begin{cases} -\frac{(-1)^{k-h}}{k-h}, & k \neq h, \\ 0, & k = h, \end{cases} \quad \gamma_{kh}^{(1)jj} = \begin{cases} \frac{2^j}{(h-k)}, & k \neq h, \\ 0, & k = h, \end{cases} \tag{14}$$

and due to (12), we can write $\gamma_{kh}^{(1)jj} = 2^{(j-1)}\gamma_{kh}^{(1)11}$, for $j > 1$.

Now, we are ready to apply the obtained results for constructing the algebraic equivalent presentation of (1). Equation (1) can be rewritten as:

$$\Gamma_0 u(x) + \Gamma_1 u^{'}(x) = f(x) + \int_a^x k(x,t)u(t)dt,$$

by substituting (5), (13) and (14) in the above equation, we have:

$$\begin{aligned} &\Gamma_0 \left[\sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \psi_{j,k}(x) \right] \\ &+ \Gamma_1 \left[\sum_{k=-M}^{M} \alpha_k \sum_{h=-M}^{M} \lambda_{kh}^{(1)} \varphi_{0,h}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \sum_{h=-M}^{M} \gamma_{kh}^{(1)jj} \psi_{j,h}(x) \right] \\ &\quad = f(x) + \int_a^x k(x,t) \left[\sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(t) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \psi_{j,k}(t) \right] dt, \end{aligned}$$

and by rearranging the above equation based on unknowns α_k and $\beta_{j,k}$, we get

$$\sum_{k=-M}^{M}\alpha_k\left[\Gamma_0\varphi_{0,k}(x)+\Gamma_1\sum_{h=-M}^{M}\lambda_{kh}^{(1)}\varphi_{0,h}(x)-\int_a^x k(x,t)\varphi_{0,k}(t)dt\right] \tag{15}$$
$$+\sum_{j=0}^{N}\sum_{k=-M}^{M}\beta_{j,k}\left[\Gamma_0\psi_{j,k}(x)+\Gamma_1\sum_{h=-M}^{M}\gamma_{kh}^{(1)jj}\psi_{j,h}(x)-\int_a^x k(x,t)\psi_{j,k}(t)dt\right]=f(x).$$

We may set

$$\Phi_k(x)=\Gamma_0\varphi_{0,k}(x)+\Gamma_1\sum_{h=-M}^{M}\lambda_{kh}^{(1)}\varphi_{0,h}(x)-\int_a^x k(x,t)\varphi_{0,k}(t)dt,$$
$$\Psi_{j,k}(x)=\Gamma_0\psi_{j,k}(x)+\Gamma_1\sum_{h=-M}^{M}\gamma_{kh}^{(1)jj}\psi_{j,h}(x)-\int_a^x k(x,t)\psi_{j,k}(t)dt,$$

therefore, we can write (15) as:

$$\sum_{k=-M}^{M}\alpha_k\Phi_k(x)+\sum_{j=0}^{N}\sum_{k=-M}^{M}\beta_{j,k}\Psi_{j,k}(x)=f(x). \tag{16}$$

For obtaining $(2N+1)(2M+2)$ unknowns α_k and $\beta_{j,k}$, we take $x=x_i$ for $i=1,\ldots,(2N+1)(2M+2)-1$, where x_i be collocation points. So, we have

$$\sum_{k=-M}^{M}\alpha_k\Phi_k(x_i)+\sum_{j=0}^{N}\sum_{k=-M}^{M}\beta_{j,k}\Psi_{j,k}(x_i)=f(x_i). \tag{17}$$

On the other hand, $u(a)=a_0$ can be written as

$$\sum_{k=-M}^{M}\alpha_k\varphi_{0,k}(a)+\sum_{j=0}^{N}\sum_{k=-M}^{M}\beta_{j,k}\psi_{j,k}(a)=a_0. \tag{18}$$

According to above equations, a system of $(2N+1)(2M+2)$ linear equations is obtained. By solving the resulting system, unknowns α_k and $\beta_{j,k}$ can be determined and so the approximate solution $u(x)$ will be obtained.

The following algorithm summarizes our proposed method:

Algorithm 1. The construction of Shannon method for a class of IDEs

---

Step 1. Input:

$$\Gamma_0, \Gamma_1, f(x), k(x,t), \varphi_{0,h}(x), \psi_{j,h}(x), a, a_0.$$

Step 2. Choose N, M;

Step 3. Compute:

$$\lambda_{kh}^{(1)} = \begin{cases} -\frac{(-1)^{k-h}}{k-h}, & k \neq h, \\ 0, & k = h, \end{cases} \qquad \gamma_{kh}^{(1)jj} = \begin{cases} \frac{2^j}{(h-k)}, & k \neq h, \\ 0, & k = h. \end{cases}$$

Step 4. Compute $\Phi_k(x_i)$, $\Psi_{j,k}(x_i)$, $f(x_i)$; for $i = 1, \ldots, (2N+1)(2M+2)-1$;

Step 5. Compute α_k and $\beta_{j,k}$ from (17) and (18);

Step 6. Set: $u(x) \simeq \sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \psi_{j,k}(x)$.

---

4 Error Analysis

In this section, we will provided a convergence analysis of the numerical algorithm for a class of integro-differential equation (1).

Theorem 4.1 *Assume that $\widetilde{u}(x)$ be the approximate solution of Eq. (1). If $u^{(1)}(x) \in L_2(\mathbb{R})$, then the obtained approximation solution of the proposed method converges to the exact solution, where α_k and $\beta_{j,k}$ are given in Theorem 2.1.*

Proof Note that

$$\begin{aligned} \widetilde{u}(x) &= \sum_{k=-\infty}^{\infty} < u, \varphi_{0,k} > \varphi_{0,k}(x) + \sum_{j=0}^{N-1} \sum_{k=-\infty}^{\infty} < u, \psi_{j,k} > \psi_{j,k}(x) \\ &= \sum_{j=-\infty}^{N-1} \sum_{k=-\infty}^{\infty} < u, \psi_{j,k} > \psi_{j,k}(x). \end{aligned} \tag{19}$$

Due to [9], the following relation holds

$$\| D^{(n)} \left[\sum_{j=-\infty}^{N-1} \sum_{k=-\infty}^{\infty} < u, \psi_{j,k} > \psi_{j,k}(x) - u(x) \right] \|_2 \to 0, \quad as\ N \to \infty,$$

or

$$\|\Bigg[\sum_{k=-\infty}^{\infty} < u, \varphi_{0,k} > \varphi_{0,k}^{(n)}(x) + \sum_{j=0}^{N-1}\sum_{k=-\infty}^{\infty} < u, \psi_{j,k} > \psi_{j,k}^{(n)}(x) - u^{(n)}(x)\Bigg]\|_2 \to 0,$$
$$as\ N \to \infty,$$

according to definitions of α_k and $\beta_{j,k}$ in Theorem 2.1 and Eqs. (7) and (8), for $n = 1$ above relation can be written as

$$\lim_{N\to\infty}\left[\sum_{k=-\infty}^{\infty}\alpha_k\sum_{h=-\infty}^{\infty}\lambda_{kh}^{(1)}\varphi_{0,h}(x) + \sum_{j=0}^{N-1}\sum_{k=-\infty}^{\infty}\beta_{j,k}\sum_{h=-\infty}^{\infty}\gamma_{kh}^{(1)jj}\psi_{j,h}(x)\right] = u^{(1)}(x),$$

which proves the theorem. □

Theorem 4.2 *Let $u_M^{(1)}(x)$ be the first-order derivative of the approximate solution of Eq. (1), then there exist constants C_1 and C_2 independent of N and M, such that*

$$\begin{aligned}\left|u^{(1)}(x) - \widetilde{u}_M^{(1)}(x)\right| \le |&C_1(u(-M-1) + u(M+1)) \\ &-C_2\left[\tfrac{3\sqrt{3}}{\pi}[u(2^{-N-1}(-M-\tfrac{1}{2})) + u(2^{-N-1}(M+\tfrac{3}{2}))]\right]|,\end{aligned}$$

where $C_1 = Max\{|\sum_k\sum_h\lambda_{kh}^{(1)}|\}$, $C_2 = Max\{|\sum_k\sum_h\gamma_{kh}^{(1)jj}|\}$ and M, N refer to the given values of j and k.

Proof See [10].

Detailed analysis of the proof of this theorem can be found in [9, 10], so we refrain from going into details.

5 Numerical Results

In this section, several test problems are considered to demonstrate the accuracy of the proposed method.

Example 5.1 Consider the following equation

$$\begin{cases} u^{'}(x) - 2u(x) = f(x) + \int_0^x k(x,t)u(t)dt, \\ f(x) = 1 - 2x - \dfrac{x^4}{2} - \dfrac{x^3}{3}, \\ k(x,t) = x^2 + t, \\ u(0) = 0, \end{cases} \tag{20}$$

with the exact solution $u(x) = x$.

Example 5.2 Consider the following equation

$$\begin{cases} u^{'}(x) - 3u(x) = f(x) + \int_0^x k(x,t)u(t)dt, \\ f(x) = -1 + x - 2xe^x - e^x, \\ k(x,t) = x + t, \\ u(0) = 1, \end{cases} \tag{21}$$

with the exact solution $u(x) = e^x$.

The computational results of Examples 5.1 and 5.2 have been reported in Tables 1 and 2, to show the accurate solution of mentioned algorithm. The exact and approximate solution of these examples for different values of M and N are compared in Figs. 1 and 2.

Example 5.3 Consider the following equation with the exact solution

$$u(x) = 1 - \sinh(x).$$

$$\begin{cases} u(x) = f(x) + \int_0^x k(x,t)u(t)dt, \\ f(x) = 1 - x - \dfrac{x^2}{2}, \\ k(x,t) = x - t. \end{cases} \tag{22}$$

Table 1 Numerical results of Example 5.1 using Shannon approximation

x	Absolute errors	
	M = 1, N = 3	M = 2, N = 4
0	1.11×10^{-16}	0
0.2	3.50×10^{-2}	3.43×10^{-2}
0.4	9.33×10^{-3}	2.57×10^{-2}
0.6	1.26×10^{-1}	3.65×10^{-3}
0.8	2.92×10^{-1}	2.05×10^{-2}
1	4.08×10^{-1}	4.65×10^{-4}

Table 2 Numerical results of Example 5.2 using Shannon approximation

x	Absolute errors	
	M = 1, N = 3	M = 2, N = 4
0	2.20×10^{-16}	1.11×10^{-16}
0.2	6.50×10^{-2}	9.62×10^{-2}
0.4	2.09×10^{-1}	6.13×10^{-2}
0.6	4.16×10^{-1}	4.73×10^{-3}
0.8	6.45×10^{-1}	1.48×10^{-3}
1	8.40×10^{-1}	1.80×10^{-1}

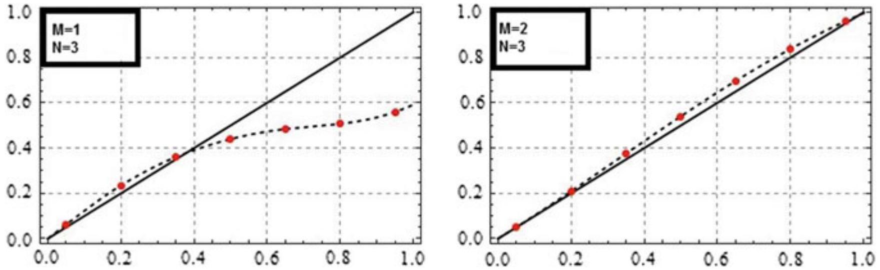

Fig. 1 Exact and approximate solution of Example 5.1 for different values of M and N using presented method

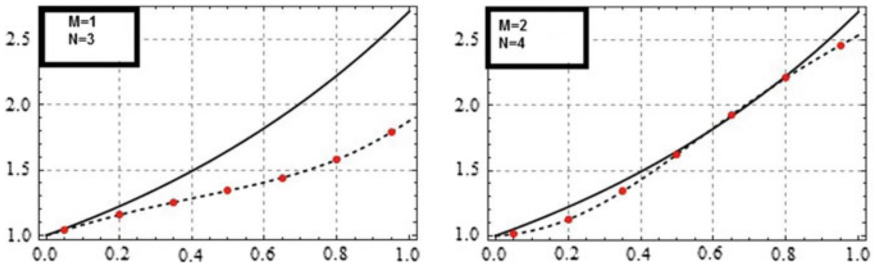

Fig. 2 Exact and approximate solution of Example 5.2 for different values of M and N using presented method

Table 3 Numerical results of Examples 5.3 and 5.4 using Shannon approximation

x	$M = 1, N = 1$		$M = 2, N = 3$	
	Example 5.3	Example 5.4	Example 5.3	Example 5.4
0	6.98×10^{-4}	1.25×10^{-4}	2.32×10^{-10}	3.51×10^{-11}
0.2	3.62×10^{-5}	4.79×10^{-6}	2.46×10^{-13}	4.20×10^{-14}
0.4	1.73×10^{-5}	1.10×10^{-6}	5.43×10^{-13}	7.71×10^{-14}
0.6	2.29×10^{-5}	8.40×10^{-7}	8.40×10^{-13}	1.12×10^{-13}
0.8	6.79×10^{-5}	5.67×10^{-6}	1.16×10^{-12}	1.44×10^{-13}
1	1.89×10^{-3}	2.56×10^{-4}	1.19×10^{-10}	1.74×10^{-11}

Examples 5.3 and 5.4, which are obtained by taking $\Gamma_1 = 0$, are integral equations. The numerical results of these examples are reported in Table 3. Also, Figs. 3 and 4 show the exact and approximate solution of Examples 5.3 and 5.4 for $M = 2$ and $N = 3$, respectively.

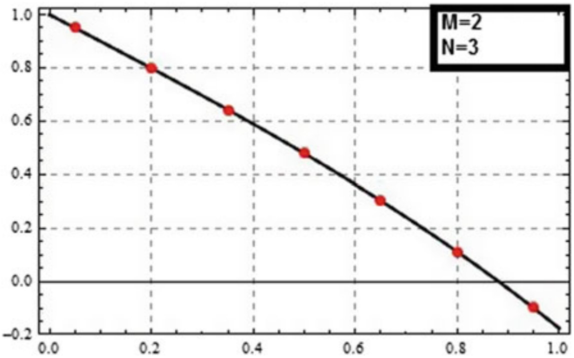

Fig. 3 Exact and approximate solution of Example 5.3 for M = 2 and N = 3 using presented method

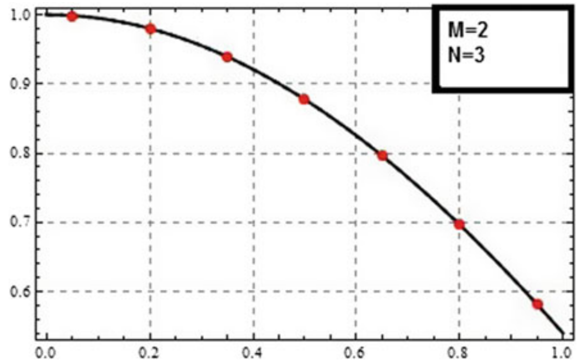

Fig. 4 Exact and approximate solution of Example 5.4 for M = 2 and N = 3 using presented method

Table 4 Numerical results of Example 5.4 using Shannon approximation

x	Absolute errors	
	N = 4	N = 5
0	2.64×10^{-6}	2.61×10^{-9}
0.2	1.19×10^{-8}	1.16×10^{-10}
0.4	8.70×10^{-9}	2.92×10^{-10}
0.6	7.13×10^{-9}	2.33×10^{-10}
0.8	6.00×10^{-9}	3.07×10^{-10}
1	1.30×10^{-16}	1.68×10^{-15}

Example 5.4 Consider the following equation

$$\begin{cases} u(x) = f(x) + \int_0^x k(x,t)u(t)dt, \\ f(x) = 1, \\ k(x,t) = -x + t, \end{cases} \tag{23}$$

with the exact solution $u(x) = \cos(x)$.

Example 5.5 Consider the following equation

$$\begin{cases} u(x) = f(x) + \int_0^1 k(x,t)(u(t))^{-1}dt, \\ f(x) = \dfrac{21 - 11e^{10}}{100} e^{(-10(1+x))} + \dfrac{1}{1+x}, \\ k(x,t) = e^{-10(x+t)}, \end{cases} \tag{24}$$

with the exact solution $u(x) = \dfrac{1}{1+x}$. This problem is a nonlinear Hammerstein integral equation which arising from chemical phenomenon. By choosing Shannon scaling functions, Example 5.5 has been solved. The reported results in Table 4 show that the Shannon approximation has produced highly numerical results. Good numerical results can be achieved by additional numerical experiments (e.g. with $N \geq 2$). This problem has been solved by $u(x) \simeq \sum_{k=1}^{2^N} \alpha_k \varphi_{N,k}(x)$.

6 Conclusions

In this present work, we applied an accurate and efficient method for solving a class of IDEs. We consider a special class of IE, which is a quantum chemistry, by the Shannon scaling functions. Our obtained results are in a good agreement with the exact solutions and are given to demonstrate the applicability of our proposed method.

References

1. R.P. Agarwal, D.O. Regan, *Ordinary and Partial Differential Equations* (Springer, Berlin, 2009)
2. A.D. Boozer, Advanced action in classical electrodynamics. J. Phys. A: Math. Theor. **41** (2008)
3. C. Cattani, Connection coefficients of Shannon wavelets. Math. Model. Anal. **11**(2), 117–132 (2006)
4. C. Cattani, Shannon wavelets theory. Math. Probl. Eng. Article ID 164808, 1–24 (2008)
5. M. Dehghan, F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic fields using He's Homptopy perturbation method. Prog. Electromagn. Res. **78**, 361–376 (2008)
6. G.N. Elnagar, M. Kazemi, Chebyshev spectral solution of nonlinear Volterra-Hammerestein integral equations. J. Comput. Appl. Math. **76**, 147–158 (1996)
7. M.A. Fariborzi Araghi, Gh. Kazemi Gelian, Numerical solution of integro-differential equations based on double exponential transformation in the Sinc-collocation method. App. Math. Comp. Intel. **1**, 48–55 (2012)
8. J.M. Machado, M. Tsuchida, Solution for a class of integro-differential equations with time periodic coefficients. Appl. Math. E-Notes **2**, 66–71 (2002)
9. K. Maleknejad, M. Attary, An efficient numerical approximation for the linear class of Fredholm integro-differential equations based on Cattani's method. Commun. Nonlinear. Sci. Numer. Simulat. **16**, 2672–2679 (2011)

10. K. Maleknejad, M. Hadizadeh, M. Attary, On the approximate solution of Integro-differential equations arising in oscillating magnetic fields. Appl. Math. **5**, 595–607 (2013)
11. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999)
12. M. Razzaghi, Y. Ordokhani, Solution of nonlinear Volterra-Hammerstein integral equations via rationalized Haar functions. Math. Probl. Eng. **7**, 205–219 (2001)
13. A. Saadatmandi, M. Dehghan, M.R. Azizi, The Sinc-Legendre collocation method for a class of fractional convection-diffusion equations with variable coefficients. Commun. Nonlinear. Sci. **17**, 4125–4136 (2012)
14. Sh. Wang, Dislocation equation from the lattice dynamics. J. Phys. A: Math. Theor. **41** (2008)

Inverse Source Problem for Multi-term Fractional Mixed Type Equation

E.T. Karimov, S. Kerbal and N. Al-Salti

Abstract In this work, we investigate an inverse source problem for multi-term fractional mixed type equation in a rectangular domain. We seek solutions in a form of series expansions using orthogonal basis obtained by using the method of a separation of variables. The obtained solutions involve multi-variable Mittag-Leffler functions, and hence, certain properties of the multi-variable Mittag-Leffler function needed for our calculations were established and proved. Imposing certain conditions to the given data, the convergence of the infinite series solutions was proved as well.

Keywords Caputo operator · Mixed type equation · Mittag-Leffler function

2010 AMS Math. Subject Classification Primary 35M10 · Secondary 35R30

1 Introduction and Preliminaries

Fractional differential equations (FDEs) become one of the interesting targets in mathematics due to their essential role in modelling of many problems of physics, chemistry, mechanics, geology, medicine, and other applied fields. Most of the fractional models of these problems derived from the classical equations by replacing the integer order time derivative with non-integer order derivatives. For instance, in [1, 2], fractional models in Maxwell fluid and generalized Oldroyd B-fluid have been investigated, respectively. Many other interesting applications of FDEs can be found in [3].

E.T. Karimov (✉)
Institute of Mathematics, National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: erkinjon@gmail.com

S. Kerbal · N. Al-Salti
Sultan Qaboos University, Muscat, Oman
e-mail: skerbal@hotmail.com

N. Al-Salti
e-mail: nalsalti@gmail.com

M. Ruzhansky et al. (eds.), *Advances in Real and Complex Analysis with Applications*, Trends in Mathematics, DOI 10.1007/978-981-10-4337-6_13

Skipping huge amount of works, devoted to studying both theoretical and practical aspects of various FDEs, we would like to mention some work related to multi-term time-fractional differential equations. Luchko and Gorenflo [4] studied multi-term time-fractional differential equation by operational method explicitly representing its solution in terms of multi-variate Mittag-Leffler function. Later on, maximum principles for such equations involving Riemann–Liouville and Caputo fractional derivatives have been studied by many authors. For instance, maximum principle for multi-term time-fractional diffusion equation with Riemann–Liouville derivative was investigated by Al-Refai and Luchko [5]. Recently, strong maximum principle for multi-term time-fractional diffusion equations and its application to an inverse problem studied by Liu [6].

Many other boundary-value problems for various multi-term FDEs were investigated by Daftardar-Gejji and Bhalekar [7], using the method of separation of variables, and Ming et al. [8], generalizing the existing results for classical Navier–Stokes, Oltroyd-B, Maxwell, and second-grade fluids.

Inverse source problems for multi-term time-fractional partial differential equations with singularity studied in [9, 10], where authors expanded solutions of the investigated problems in a form of Fourier–Bessel series and represented them in an explicit form in terms of multi-variate Mittag-Leffler function.

We also note the work by Karimov and Feng [11], where inverse source problem for mixed type equation with Caputo fractional derivative was studied for weak solvability. Recently, in [12], a non-local inverse source problem was investigated for time-fractional mixed type equation in rectangular domain. Solutions were represented in a series form using bi-orthogonal basis and involve Mittag-Leffler-type functions of two variables.

In the present work, we aim to investigate inverse source problem for multi-term time-fractional mixed type equation in a rectangular domain. Using the method of separation of variables, we represent solutions of the problem in a form of infinite series, involving particular case of multi-variate Mittag-Leffler function. We proved new estimation for this function, which allows us to impose less conditions in order to provide uniform convergence of certain infinite series.

The rest of the paper is organized as follows. Further, in this section, we give definition of the Caputo fractional derivative and represent several properties of aforementioned Mittag-Leffler function of two variable. In the next section, we formulate the problem and give formal representation of solutions by expanding them into sine-Fourier series. Next, we provide detailed proofs for uniform convergence of the obtained series solutions. In the last section, we present main results and conclusion.

1.1 Caputo Fractional Derivatives

If $\alpha \notin \mathbb{N} \cup \{0\}$, the Caputo fractional derivatives ${}_C D^{\alpha}_{ax}\, y$ and ${}_C D^{\alpha}_{xb}\, y$ of order α are defined by [13, p. 92]

$$\left({}_C D_{ax}^{\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)\,dt}{(x-t)^{\alpha-n+1}} \qquad (n = [\mathrm{Re}(\alpha)]+1,\ x > a),$$

$$\left({}_C D_{xb}^{\alpha} y\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)\,dt}{(t-x)^{\alpha-n+1}} \qquad (n = [\mathrm{Re}(\alpha)]+1,\ x < b), \tag{1}$$

respectively, while for $\alpha = n \in \mathbb{N} \cup \{0\}$, we have

$$\left({}_C D_{ax}^{0} y\right)(x) = y(x),\ \left({}_C D_{xb}^{0} y\right)(x) = y(x),$$
$$\left({}_C D_{ax}^{n} y\right)(x) = y^{(n)}(x),\ \left({}_C D_{xb}^{n} y\right)(x) = (-1)^n y^{(n)}(x).$$

1.2 Two-Variable Mittag-Leffler Function

A particular case of multi-variate Mittag-Leffler function (see [4], formula (39)) in two variables can be presented as

$$E_{(\alpha-\beta,\alpha),\rho}(x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \frac{x^i y^{n-i}}{\Gamma(\rho+\alpha n-\beta i)}. \tag{2}$$

Lemma 1.1 *For $\alpha > \beta > 0$, the following properties are true:*

(I) $\frac{d}{dt}\left[t^{\alpha} E_{(\alpha-\beta,\alpha),\alpha+1}\left(m_1 t^{\alpha-\beta}, m_2 t^{\alpha}\right)\right] = t^{\alpha-1} E_{(\alpha-\beta,\alpha),\alpha}\left(m_1 t^{\alpha-\beta}, m_2 t^{\alpha}\right);$

(II) $\int_0^t z^{\alpha-1} E_{(\alpha-\beta,\alpha),\alpha}\left(m_1 z^{\alpha-\beta}, m_2 z^{\alpha}\right) = t^{\alpha} E_{(\alpha-\beta,\alpha),\alpha+1}\left(m_1 t^{\alpha-\beta}, m_2 t^{\alpha}\right);$

(III) $m_1 t^{\alpha-\beta} E_{(\alpha-\beta,\alpha),\alpha-\beta+\rho}\left(m_1 t^{\alpha-\beta}, m_2 t^{\alpha}\right) + m_2 t^{\alpha} E_{(\alpha-\beta,\alpha),\alpha+\rho}\left(m_1 t^{\alpha-\beta}, m_2 t^{\alpha}\right) =$

$E_{(\alpha-\beta,\alpha),\rho}\left(m_1 t^{\alpha-\beta}, m_2 t^{\alpha}\right) - \frac{1}{\Gamma(\rho)}.$

Here, $m_1,\ m_2$ are nonzero constants.

We have also established new properties of the above given Mittag-Leffler function in two variables, which will be used later in our calculations. These properties are formulated in the following two lemmas along with their proofs.

Lemma 1.2 *For $1 < \beta < \alpha < 2$ and $0 < \rho \le 1$ the following statements are true:*

(I) ${}_C D_{t0}^{\rho}\left(t E_{(\alpha-\beta,\alpha),2}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right)\right) =$
$(-t)^{1-\rho} E_{(\alpha-\beta,\alpha),2-\rho}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right),$

(II) ${}_C D_{t0}^{\rho}\left(E_{(\alpha-\beta,\alpha),1}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right)\right) =$
$(-t)^{-\rho}\left[E_{(\alpha-\beta,\alpha),1-\rho}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right) - \frac{1}{\Gamma(1-\rho)}\right],$

(III) ${}_C D_{t0}^{\rho}\left((-t)^{\alpha} E_{(\alpha-\beta,\alpha),\alpha+1}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right)\right) =$
$(-t)^{\alpha-\rho} E_{(\alpha-\beta,\alpha),\alpha+1-\rho}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right).$

Proof Here, we prove the third property, and the proof of the first two can be done similarly. Using the definition (1) and representation (2), one can get

$$
{}_C D_{t0}^{\rho}\left((-t)^{\alpha} E_{(\alpha-\beta,\alpha),\alpha+1}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right)\right) =
$$

$$
\frac{1}{\Gamma(1-\rho)}\int_t^0 (z-t)^{-\rho}\frac{d}{dz}\left[(-z)^{\alpha}\left(\frac{1}{\Gamma(1+\alpha)}+\frac{m_2(-z)^{\alpha})}{\Gamma(2\alpha+1)}+\frac{m_1(-z)^{\alpha-\beta}}{\Gamma(2\alpha+1-\beta)}+\right.\right.
$$

$$
\left.\left.\frac{(m_2(-z)^{\alpha})^2}{\Gamma(3\alpha+1)}+\frac{2m_1m_2(-z)^2\alpha-\beta}{\Gamma(3\alpha+1-\beta)}+\frac{(m_1(-z)^{\alpha-\beta})^2}{\Gamma(3\alpha+1-2\beta)}+...\right)\right]dz=
$$

$$
\frac{1}{\Gamma(\alpha)\Gamma(1-\rho)}\int_0^t (-z)^{\alpha-1}(z-t)^{-\rho}dz+\frac{m_2}{\Gamma(2\alpha)\Gamma(1-\rho)}\int_0^t (-z)^{2\alpha-1}(z-t)^{-\rho}dz+
$$

$$
\frac{m_1}{\Gamma(2\alpha-\beta)\Gamma(1-\rho)}\int_0^t (-z)^{2\alpha-\beta-1}(z-t)^{-\rho}dz+...=(-t)^{\alpha-\rho}\left[\frac{1}{\Gamma(\alpha+1-\rho)}+\right.
$$

$$
\left.\frac{m_2(-t)^{\alpha}}{\Gamma(\alpha+1-\rho+\alpha)}+\frac{m_1(-t)^{\alpha-\beta}}{\Gamma(\alpha+1-\rho+\alpha-\beta)}+...\right]=
$$

$$
(-t)^{\alpha-\rho}E_{(\alpha-\beta,\alpha),\alpha+1-\rho}\left(m_1(-t)^{\alpha-\beta}, m_2(-t)^{\alpha}\right).
$$

□

Note that here we have used the well-known Beta-function

$$
\int_0^1 \xi^{a-1}(1-\xi)^{b-1}d\xi=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
$$

Lemma 1.3 *If* $\rho>0,\ 0<\alpha-\beta<2,\ \lambda\le|arg(x+y)|\le\pi$ *such that* $\pi(\alpha-\beta)/2<\lambda<min(\pi,\pi(\alpha-\beta))$, *and*

$$
\Gamma(\rho+n(\alpha-\beta)+k\beta)>\Gamma(\rho+n(\alpha-\beta)),\ n,k\in\mathbb{N},\ n\ge k, \tag{3}
$$

then

$$
\left|E_{(\alpha-\beta,\alpha),\rho}(x,y)\right|\le\frac{c}{1+|x+y|}. \tag{4}
$$

Here, c is any constant.

Proof According to (2), we have

$$\left|E_{(\alpha-\beta,\alpha),\rho}(x,y)\right| = \left|\frac{1}{\Gamma(\rho)} + \left(\frac{y}{\Gamma(\rho+\alpha)} + \frac{x}{\Gamma(\rho+\alpha-\beta)}\right) + \left(\frac{y^2}{\Gamma(\rho+2\alpha)} + \frac{2xy}{\Gamma(\rho+2\alpha-\beta)} + \frac{x^2}{\Gamma(\rho+2\alpha-2\beta)}\right) + \left(\frac{y^3}{\Gamma(\rho+3\alpha)} + \frac{3xy^2}{\Gamma(\rho+3\alpha-\beta)} + \frac{3x^2y}{\Gamma(\rho+3\alpha-2\beta)} + \frac{x^3}{\Gamma(\rho+3\alpha-3\beta)}\right) + ...\right|$$

Here, we will replace $\Gamma(\rho+\alpha)$ with $\Gamma(\rho+\alpha-\beta)$ and $\Gamma(\rho+2\alpha)$ with $\Gamma(\rho+2(\alpha-\beta))$. Generally, we replace $\Gamma(\rho+n(\alpha-\beta)+k\beta)$ with $\Gamma(\rho+n(\alpha-\beta))$, where $n, k = 1, 2, ...,$ such that $n \geq k$. Imposing the condition (3), we obtain

$$\left|E_{(\alpha-\beta,\alpha),\rho}(x,y)\right| \leq \left|\frac{1}{\Gamma(\rho)} + \frac{x+y}{\Gamma(\rho+\alpha-\beta)} + \frac{(x+y)^2}{\Gamma(\rho+2(\alpha-\beta))} + ... + \frac{(x+y)^n}{\Gamma(\rho+n(\alpha-\beta))} + ...\right| \leq \left|\sum_{n=0}^{\infty} \frac{(x+y^n)}{\Gamma(\rho+n(\alpha-\beta))}\right| = \left|E_{\alpha-\beta,\rho}(x+y)\right| \leq \frac{C}{1+|x+y|}.$$

In the last step, we have used an estimation of the two parametric Mittag-Leffler function $E_{m,n}(z) = \sum\limits_{i=0}^{\infty} \frac{z^i}{\Gamma(n+mi)}$, given in [14]. □

2 Formulation of Problem and Formal Solution

2.1 Formulation of Problem

Consider fractional order mixed type equation

$$\frac{1+sgn(t)}{2}\left({}_CD_{0t}^{\alpha_1}u + \mu_1 {}_CD_{0t}^{\beta_1}u\right) + \frac{1-sgn(t)}{2}\left({}_CD_{t0}^{\alpha_2}u + \mu_2 {}_CD_{t0}^{\beta_2}u\right) - u_{xx} = f(x) \tag{5}$$

in a rectangular domain $\Omega = \{(x,t) : 0 < x < 1, -p < t < q\}$. Here, $0 < \beta_1 < \alpha_1 < 1$, $1 < \beta_2 < \alpha_2 < 2$, $\mu_1, \mu_2 \in \mathbb{R}$.

Problem. Find a pair of functions $\{u(x,t), f(x)\}$, satisfying

- $u(x,t) \in C(\overline{\Omega})$, $u_{xx} \in C(\Omega^+ \cup \Omega^-)$, ${}_CD_{0t}^{\alpha_1}u \in C(\Omega^+)$, ${}_CD_{t0}^{\alpha_2}u \in C(\Omega^-)$, $f(x) \in C(0,1)$;
- Equation (5) in Ω^+ and Ω^-;
- the boundary conditions

$$u(0,t) = u(1,t) = 0, \ -p \leq t \leq q, \tag{6}$$

$$u(x,-p) = \psi(x), \ 0 \leq x \leq 1; \quad {}_CD_{t0}^{\gamma}u(x,-p) = \phi(x), \ 0 < x < 1, \tag{7}$$

$$u(x,q) = \varphi(x), \ 0 \leq x \leq 1. \tag{8}$$

Here, $\Omega^+ = \Omega \cap \{t > 0\}, \Omega^- = \Omega \cap \{t < 0\}, \psi(x), \phi(x)$ and $\varphi(x)$ are given functions such that $\psi(0) = \psi(1) = \varphi(0) = \varphi(1) = 0, 0 < \gamma \leq 1$.

2.2 Formal Solution

Using the method of separation of variables leads to the spectral problem in space-variable x:

$$X''(x) - \lambda X(x) = 0,\ \ X(0) = X(1) = 0. \tag{9}$$

It is well known that the problem (9) is self-adjoint and its solutions, $X_n(x) = \sin n\pi x,\ n \in \mathbb{N}$, form a complete orthogonal basis in $L_2(0, 1)$. Based on this, we look for a solution of problem (5)–(8) as follows:

$$u(x,t) = \sum_{k=1}^{\infty} T_k^+(t) \sin k\pi x,\ \ t \geq 0, \tag{10}$$

$$u(x,t) = \sum_{k=1}^{\infty} T_k^-(t) \sin k\pi x,\ \ t \leq 0, \tag{11}$$

$$f(x) = \sum_{k=1}^{\infty} f_k \sin k\pi x, \tag{12}$$

where

$$\begin{aligned} T_k^+(t) &= \int_0^1 u(x,t) \sin k\pi x dx,\ \ t \geq 0, \\ T_k^-(t) &= \int_0^1 u(x,t) \sin k\pi x dx,\ \ t \leq 0, \\ f_k &= \int_0^1 f(x) \sin k\pi x dx \end{aligned} \tag{13}$$

are the Fourier coefficients of series (10)–(12), respectively.

Substituting (10)–(12) into Eq. (5), we formally obtain

$${}_C D_{0t}^{\alpha_1} T_k^+(t) + \mu_1 {}_C D_{0t}^{\beta_1} T_k^+(t) + (k\pi)^2\, T_k^+(t) = f_k, \tag{14}$$

$${}_C D_{t0}^{\alpha_2} T_k^-(t) + \mu_2 {}_C D_{t0}^{\beta_2} T_k^-(t) + (k\pi)^2\, T_k^-(t) = f_k. \tag{15}$$

According to [4], solution of (14) satisfying the condition $T_k^+(0) = A_k$ has a form

$$\begin{aligned}&T_k^+(t) = f_k \int\limits_0^t z^{\alpha_1-1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1}\left(-\mu_1 z^{\alpha_1-\beta_1}, -(k\pi)^2 z^{\alpha_1}\right) dz+\\ &A_k\left[1-\mu_1 t^{\alpha_1-\beta_1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1-\beta_1+1}\left(-\mu_1 t^{\alpha_1-\beta_1}, -(k\pi)^2 t^{\alpha_1}\right)-\right.\\ &\left.(k\pi)^2 t_1^{\alpha} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}\left(-\mu_1 t^{\alpha_1-\beta_1}, -(k\pi)^2 t^{\alpha_1}\right)\right].\end{aligned} \tag{16}$$

Solution of (15), which satisfies conditions

$$T_k^-(0) = B_k, \;\; T_k^{-'}(0) = C_k$$

has a form

$$\begin{aligned}&T_k^-(t) = f_k \int\limits_t^0 (-z)^{\alpha_2-1} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2}\left(-\mu_2(-z)^{\alpha_2-\beta_2}, -(k\pi)^2(-z)^{\alpha_2}\right) dz+\\ &B_k\left[1-\mu_2(-t)^{\alpha_2-\beta_2} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2-\beta_2+1}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)-\right.\\ &\left.(k\pi)^2(-t)_2^{\alpha} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)\right]\\ &-C_k t\left[1-\mu_2(-t)^{\alpha_2-\beta_2} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2-\beta_2+2}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)-\right.\\ &\left.(k\pi)^2(-t)_2^{\alpha} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+2}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)\right].\end{aligned} \tag{17}$$

Here, $A_k,\ B_k$, and C_k are unknown constants, which should be determined.

Using the second and third statements in Lemma 1.1, we rewrite (16) and (17) as follows:

$$\begin{aligned}&T_k^+(t) = f_k t^{\alpha_1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}\left(-\mu_1 t^{\alpha_1-\beta_1}, -(k\pi)^2 t^{\alpha_1}\right)+\\ &A_k E_{(\alpha_1-\beta_1,\alpha_1),1}\left(-\mu_1 t^{\alpha_1-\beta_1}, -(k\pi)^2 t^{\alpha_1}\right),\end{aligned} \tag{18}$$

$$\begin{aligned}&T_k^-(t) = f_k(-t)^{\alpha_2} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)+\\ &B_k E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)-\\ &C_k t E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right).\end{aligned} \tag{19}$$

In order to find the unknown constants $A_k,\ B_k$, and C_k, we use the boundary conditions (7)–(8), which can be written in terms of $T_k^{\pm}(t)$ as follows:

$$T_k^-(-p) = \psi_k, \;\; T_k^+(q) = \varphi_k, \tag{20}$$

$${}_C D_{t0}^{\gamma} T_k^-(t)\Big|_{t=-p} = \phi_k, \tag{21}$$

where

$$\varphi_k = \int\limits_0^1 \varphi(x)\sin k\pi x dx, \;\; \psi_k = \int\limits_0^1 \psi(x)\sin k\pi x dx, \;\; \phi_k = \int\limits_0^1 \phi(x)\sin k\pi x dx$$

are Fourier coefficients of the Fourier series of the given functions $\varphi(x)$, $\psi(x)$, and $\phi(x)$, respectively.

From (18)–(21), we obtain

$$\begin{array}{l}\varphi_k = f_k q^{\alpha_1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}\left(-\mu_1 q^{\alpha_1-\beta_1}, -(k\pi)^2 q^{\alpha_1}\right) + \\ A_k E_{(\alpha_1-\beta_1,\alpha_1),1}\left(-\mu_1 q^{\alpha_1-\beta_1}, -(k\pi)^2 q^{\alpha_1}\right),\end{array} \tag{22}$$

$$\begin{array}{l}\psi_k = f_k p^{\alpha_2} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}\left(-\mu_2 p^{\alpha_2-\beta_2}, -(k\pi)^2 p^{\alpha_2}\right) + \\ B_k E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\mu_2 p^{\alpha_2-\beta_2}, -(k\pi)^2 p^{\alpha_2}\right) + \\ C_k p E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\mu_2 p^{\alpha_2-\beta_2}, -(k\pi)^2 p^{\alpha_2}\right).\end{array} \tag{23}$$

In order to use condition (21), we will first evaluate the expression ${}_C D_{t0}^{\gamma} T_k^-(t)$ based on (19):

$$\begin{array}{l}{}_C D_{t0}^{\gamma} T_k^-(t) = f_k {}_C D_{t0}^{\gamma}\left((-t)^{\alpha_2} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)\right) + \\ B_k {}_C D_{t0}^{\gamma}\left(E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)\right) - \\ C_k {}_C D_{t0}^{\gamma}\left(t E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right)\right).\end{array} \tag{24}$$

According to Lemma 1.2, we get

$$\begin{array}{l}{}_C D_{t0}^{\gamma} T_k^-(t) = f_k (-t)^{\alpha_2-\gamma} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1-\gamma}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right) + \\ B_k(-t)^{-\gamma}\left[E_{(\alpha_2-\beta_2,\alpha_2),1-\gamma}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right) - \frac{1}{\Gamma(1-\gamma)}\right] - \\ C_k(-t)^{1-\gamma} E_{(\alpha_2-\beta_2,\alpha_2),2-\gamma}\left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right).\end{array}$$

Now, substituting this into (21), we get

$$\begin{array}{l}\phi_k = f_k p^{\alpha_2-\gamma} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1-\gamma}\left(-\mu_2 p^{\alpha_2-\beta_2}, -(k\pi)^2 p^{\alpha_2}\right) + \\ B_k p^{-\gamma}\left[E_{(\alpha_2-\beta_2,\alpha_2),1-\gamma}\left(-\mu_2 p^{\alpha_2-\beta_2}, -(k\pi)^2 p^{\alpha_2}\right) - \frac{1}{\Gamma(1-\gamma)}\right] - \\ C_k p^{1-\gamma} E_{(\alpha_2-\beta_2,\alpha_2),2-\gamma}\left(-\mu_2 p^{\alpha_2-\beta_2}, -(k\pi)^2 p^{\alpha_2}\right).\end{array} \tag{25}$$

To find the unknown constant f_k, we need another condition. For this aim, we rewrite the transmitting condition $u(x,+0) = u(x,-0)$, which follows from $u(x,t) \in C(\overline{\Omega})$, as $T_k^+(0) = T_k^-(0)$. This leads to $A_k = B_k$. One can easily check this fact by evaluating $\lim\limits_{t\to+0} T_k^+(t)$ and $\lim\limits_{t\to-0} T_k^-(t)$ using (18) and (19).

First, from (22) we find f_k as

$$f_k = \frac{\varphi_k - A_k E_{(\alpha_1-\beta_1,\alpha_1),1}\left(-\mu_1 q^{\alpha_1-\beta_1}, -(k\pi)^2 q^{\alpha_1}\right)}{q^{\alpha_1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}\left(-\mu_1 q^{\alpha_1-\beta_1}, -(k\pi)^2 q^{\alpha_1}\right)}. \tag{26}$$

Further, considering that $A_k = B_k$ from (23) and (25) we obtain system of algebraic equations with respect to unknown constants B_k and C_k:

$$\begin{cases} \phi_k = \frac{p^{\alpha_2-\gamma}E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1-\gamma}(..p..)}{q^{\alpha_1}E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}(..q..)}\left[\varphi_k - B_k E_{(\alpha_1-\beta_1,\alpha_1),1}(..q..)\right]+ \\ B_k p^{-\gamma}\left[E_{(\alpha_2-\beta_2,\alpha_2),1-\gamma}(..p..) - \frac{1}{\Gamma(1-\gamma)}\right] - C_k p^{1-\gamma}E_{(\alpha_2-\beta_2,\alpha_2),2-\gamma}(..p..), \\ \\ \psi_k = \frac{p^{\alpha_2}E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}(..p..)}{q^{\alpha_1}E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}(..q..)}\left[\varphi_k - B_k E_{(\alpha_1-\beta_1,\alpha_1),1}(..q..)\right]+ \\ B_k E_{(\alpha_2-\beta_2,\alpha_2),1}(..p..) + C_k p E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..). \end{cases} \tag{27}$$

If

$$\begin{aligned} &\Delta_k = p^{1-\gamma}\Big\{E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..)\left[E_{(\alpha_2-\beta_2,\alpha_2),1-\gamma}(..p..) - \tfrac{1}{\Gamma(1-\gamma)}\right]- \\ &E_{(\alpha_2-\beta_2,\alpha_2),2-\gamma}(..p..)E_{(\alpha_2-\beta_2,\alpha_2),1}(..p..) - \tfrac{p^{\alpha_2}E_{(\alpha_1-\beta_1,\alpha_1),1}(..q..)}{q^{\alpha_1}E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}(..q..)}\times \\ &\big[E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..)E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1-\gamma}(..p..)+ \\ &E_{(\alpha_2-\beta_2,\alpha_2),2-\gamma}(..p..)E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}(..p..)\big]\Big\} \neq 0, \end{aligned} \tag{28}$$

then we can find B_k, C_k as follows:

$$\begin{aligned} &B_k = A_k = \tfrac{1}{\Delta_k}\Big\{p\phi_k E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..) + p^{1-\gamma}\psi_k E_{(\alpha_2-\beta_2,\alpha_2),2-\gamma}(..p..)- \\ &\tfrac{\varphi_k p^{\alpha_2+1-\gamma}}{q^{\alpha_1}E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}(..q..)}\Big[E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..)E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1-\gamma}(..p..)+ \\ &E_{(\alpha_2-\beta_2,\alpha_2),1}(..p..)\Big(E_{(\alpha_2-\beta_2,\alpha_2),1-\gamma}(..p..) - \tfrac{1}{\Gamma(1-\gamma)}\Big)\Big]\Big\} \end{aligned} \tag{29}$$

$$\begin{aligned} &C_k = \tfrac{1}{pE_{(\alpha_2-\beta_2,\alpha_2),2}(..p..)}\Big\{\psi_k - \tfrac{p^{\alpha_2}E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}(..p..)}{q^{\alpha_1}E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}(..q..)}\varphi_k + \\ &\tfrac{1}{\Delta_k}\big[p^{\alpha_2}E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1}(..p..)E_{(\alpha_1-\beta_1,\alpha_1),1}(..q..) - q^{\alpha_1}E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}(..q..)\times \\ &E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..)\big]\Big\{p\phi_k E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..) + p^{1-\gamma}\psi_k E_{(\alpha_2-\beta_2,\alpha_2),2-\gamma}(..p..)- \\ &\tfrac{p^{\alpha_2+1-\gamma}\varphi_k}{q^{\alpha_1}E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1+1}(..q..)}\Big[E_{(\alpha_2-\beta_2,\alpha_2),2}(..p..)E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2+1-\gamma}(..p..)+ \\ &E_{(\alpha_2-\beta_2,\alpha_2),1}(..p..)\Big(E_{(\alpha_2-\beta_2,\alpha_2),1-\gamma}(..p..) - \tfrac{1}{\Gamma(1-\gamma)}\Big)\Big]\Big\}\Big\}. \end{aligned} \tag{30}$$

Here in order to avoid bulky expressions, we have introduced the following short notations:

$$(..q..) = \left(-\mu_1 t^{\alpha_1-\beta_1}, -(k\pi)^2 t^{\alpha_1}\right), \; (..p..) = \left(-\mu_2(-t)^{\alpha_2-\beta_2}, -(k\pi)^2(-t)^{\alpha_2}\right).$$

3 Convergence of Infinite Series

Imposing certain conditions on the given functions, we will prove uniform convergence of the infinite series corresponding to the functions $u(x,t)$, $f(x)$, $u_{xx}(x,t)$, ${}_C D_{0t}^{\alpha_1} u(x,t)$, and ${}_C D_{t0}^{\alpha_2} u(x,t)$.

First, we start with the series corresponding to u_{xx} since it requires stronger conditions due to the appearance of the term $(k\pi)^2$. Precisely, we need to prove the convergence of the series $\sum\limits_{k=1}^{\infty}(k\pi)^2|T_k^+(t)|$ and $\sum\limits_{k=1}^{\infty}(k\pi)^2|T_k^-(t)|$.

Using estimation (4), we get from (18)

$$|T_k^+(t)| \le \frac{c}{(k\pi)^2}\left[|f_k| + |A_k|\right].$$

and from (29) and (26), we deduce

$$|A_k| \le \frac{1}{(k\pi)^2}\frac{1}{|\Delta_k|}\left(c_1|\phi_k| + c_2|\psi_k| + \frac{c_3}{(k\pi)^2}|\varphi_k|\right),$$

$$|f_k| \le c_4|\varphi_k| + \frac{c_5}{(k\pi)^2}|A_k| \le \frac{1}{(k\pi)^2}\left(c_4|{}_2\varphi_k| + c_5|A_k|\right).$$

Here, ${}_2\varphi_k = \int\limits_0^1 \varphi''(x)\sin k\pi x dx.$

Therefore, one can easily state that

$$|T_k^+(t)| \le \frac{1}{(k\pi)^4}\left[c_4|{}_2\varphi_k| + \frac{1}{|\Delta_k|}\left(\frac{c_5}{(k\pi)^2}+1\right)\left(c_1|\phi_k| + c_2|\psi_k| + \frac{c_3}{(k\pi)^2}|\varphi_k|\right)\right]. \tag{31}$$

In order to get this estimation, we impose the following conditions on the given functions:

$$\varphi(x) \in C[0,1]\cap C^1(0,1),\ \varphi(0)=\varphi(1)=0,\ \varphi''(x)\in L(0,1),\ \ \psi(x),\phi(x)\in C[0,1].$$

Now, let us estimate $T_k^-(t)$. Again, using the estimation (4), from (19) we get

$$|T_k^-(t)| \le \frac{1}{(k\pi)^2}\left(c_6|f_k| + c_7|B_k| + c_8|C_k|\right)$$

and from (30), we obtain

$$|C_k| \le c_9|\psi_k| + c_{10}|\varphi_k| + \frac{1}{|\Delta_k|}\left(\frac{c_{11}}{(k\pi)^6}|\phi_k| + \frac{c_{12}}{(k\pi)^6}|\psi_k| + \frac{c_{13}}{(k\pi)^6}|\varphi_k|\right)$$

or

$$|C_k| \le \frac{1}{(k\pi)^2}\left[c_9|{}_2\psi_k| + c_{10}|{}_2\varphi_k| + \frac{1}{|\Delta_k|}\left(\frac{c_{11}}{(k\pi)^4}|\phi_k| + \frac{c_{12}}{(k\pi)^4}|\psi_k| + \frac{c_{13}}{(k\pi)^4}|\varphi_k|\right)\right],$$

where ${}_2\psi_k = \int\limits_0^1 \psi''(x)\sin k\pi x dx$. Based on this estimation, we finally get

$$\begin{aligned}|T_k^-(t)| \le \tfrac{1}{(k\pi)^4}\, [c_4 c_6 |{}_2\varphi_k| + (c_5 c_6 + c_7)\, |A_k| + c_8 c_9 |{}_2\psi_k| + c_8 c_{10} |{}_2\varphi_k| + \\ \tfrac{c_8}{|\Delta_k|}\left(\tfrac{c_{11}}{(k\pi)^4}|\phi_k| + \tfrac{c_{12}}{(k\pi)^4}|\psi_k| + \tfrac{c_{13}}{(k\pi)^4}|\varphi_k|\right)\Big].\end{aligned} \tag{32}$$

Hence, based on (31) and (32) and using the Weierstrass M-test, one can easily prove the uniform convergence of the infinite series corresponding to functions $u(x,t)$, $u_{xx}(x,t)$, and $f(x)$. In order to prove the uniform convergence of infinite series corresponding to ${}_C D_{0t}^{\alpha_1} u(x,t)$ and ${}_C D_{t0}^{\alpha_2} u(x,t)$, we need the following estimates:

$$\left|{}_C D_{0t}^{\alpha_1} T_k^+(t)\right| \le \frac{1}{(k\pi)^2}\left(c_{13}|f_k| + c_{14}|A_k|\right), \tag{33}$$

$$\left|{}_C D_{t0}^{\alpha_2} T_k^-(t)\right| \le \frac{1}{(k\pi)^2}\left(c_{15}|f_k| + c_{16}|B_k| + c_{17}|C_k|\right). \tag{34}$$

Uniqueness of solution to problem (5)–(8) easily follows from the completeness property of the system $\{\sin n\pi x\}_{n\in\mathbb{N}}$ in $L_2(0,1)$.

4 Main Result and Conclusion

We formulate our main result in the following theorem:

Theorem 4.1 *If all fractional orders of (1) satisfy the conditions of Lemma 1.3 and the condition (28) along with the following conditions:*

$$\varphi(x), \psi(x) \in C[0,1] \cap C^1(0,1),\ \varphi(0) = \varphi(1) = 0,\ \psi(0) = \psi(1) = 0,$$

$$\varphi''(x), \psi''(x) \in L(0,1),\ \phi(x) \in C[0,1]$$

is satisfied, then there exists a unique solution of the problem (5)–(8) represented by (10)–(12), where the coefficients $T_k^+(t)$, $T_k^-(t)$, and f_k are given by (18), (19), and (26), respectively.

Remark 4.1 It is also possible to avoid restrictions to the fractional orders of (1) given in the conditions of Lemma 1.3. In this case, instead of estimation (4), we will use another estimation, given in [15] (see Lemma 3.2), precisely,

$$E_{(\alpha-\beta,\alpha),\rho}(x,y) \le \frac{c}{1+|x|}.$$

However, in this case, we have to impose more conditions on the given functions.

The result for the later case can be formulated as follows:

Theorem 4.2 *If condition (28) and the following conditions*

$$\phi(x) \in C[0,1],\ \varphi(x), \psi(x) \in C^2[0,1] \cap C^3(0,1),\ \varphi(0) = \varphi(1) = 0,\ \psi(0) = \psi(1) = 0,$$

$$\varphi''(0) = \varphi''(1) = 0,\ \psi''(0) = \psi''(1) = 0, \varphi^{iv}(x), \psi^{iv}(x) \in L(0,1)$$

hold, then there exists a unique solution of the problem (5)–(8) represented by (10)–(12), where the coefficients $T_k^+(t)$, $T_k^-(t)$, and f_k are given by (18), (19), and (26), respectively.

Conclusion. In this work, we have considered an inverse source problem for mixed type equation involving two different orders of Caputo fractional derivatives in a rectangular domain. In order to reduce conditions on the given functions, we proved a new estimation for a particular case of the multi-variate Mittag-Leffler function. We also proved some other properties of that Mittag-Leffler function, which is given in Lemma 1.2.

In order to illustrate how important is the new estimation (4), we have presented another result, which was obtained using another estimation. We also have to note that new estimation requires additional restrictions to the fractional order of the equation (5) as given in Lemma 1.3.

We also note that in both cases, condition to the geometry of the considered domain in a form of (28) is essential.

Acknowledgements Authors acknowledge financial support from The Research Council (TRC), Oman. This work is funded by TRC under the research agreement no. ORG/SQU/CBS/13/030.

References

1. M. Jamil, A. Rauf, A.A. Zafar, N.A. Khan, New exact analytical solutions for Stokes' first problem in Maxwell fluid with fractional derivative approach. Comput. Math. Appl. **62**, 1013–1023 (2011)
2. D. Tong, X. Zhang, X. Zhang, Unsteady helical flows of generalized Oldroyd-B fluid. J. Non-Newton. Fluid. Mech. **156**, 75–83 (2009)
3. R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific Press, Singapore, 2000)
4. Y. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives. Acta Mathematica Vietnamica **24**, 207–233 (1999)
5. M. Al-Refai, Y. Luchko, Maximumprinciple for the multi-term time-fractional diffusion equations with the Riemann-Liouville fractional derivatives. Appl. Math. Comput. **257**, 40–51 (2015)
6. Y. Liu, Strong maximum principle for multi-term time-fractional diffusion equations and its application to an inverse source problem. Comput. Math. Appl. **73**, 96–108 (2017)
7. V. Daftardar-Gejji, S. Bhalekar, Boundary value problems for multi-term fractional differential equations. J. Math. Anal. Appl. **345**, 754–765 (2008)

8. Ch. Ming, F. Liu, L. Zheng, I. Turner, V. Anh, Analytical solutions of multi-term time fractional differential equations and application to unsteady flows of generalized viscoelastic fluid. Comput. Math. Appl. **72**, 2084–2097 (2016)
9. P. Agarwal, E. Karimov, M. Mamchuev, M. Ruzhansky, On boundary-value problems for a partial differential equation with Caputo and Bessel operators. in *Novel Methods in Harmonic Analysis*, eds. by I. Pesenson, Le Gia, et al., vol. 2 (Applied and Numerical Harmonic Analysis). (Birkhauser, Basel, 2017), pp. 707–719. arXiv:math/01624
10. E. Karimov, M. Mamchuev, M. Ruzhansky, Non-local initial problem for second order time-fractional and space-singular equation, accepted to Communications in Pure and Applied Analysis (2017). arXiv:1701.01904
11. E.T. Karimov, P. Feng, Inverse source problems for time-fractional mixed parabolic-hyperbolic-type equations. J. Inverse Ill-Posed Probl. **23**, 339–353 (2015)
12. N. Al-Salti, E.T. Karimov, S. Kerbal, An inverse source non-local problem for a mixed type equation with a Caputo fractional differential operator. East Asian J. Appl. Math. **7**(2), 417–438 (2017)
13. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006)
14. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999)
15. Zh Li, Y. Liu, M. Yamamoto, Initial-boundary problems for multi-term time-fractional diffusion equations with positive constant coefficients. Appl. Math. Comput. **257**, 381–397 (2015)