A note on the structure constants of Leibniz algebras

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Nil Mansuroğlu and Mücahit Özkaya

A Note On The Structure Constants of Leibniz Algebras

Nil Mansuroğlu[a)] and Mücahit Özkaya[b)]

Department of Mathematics, Kırşehir Ahi Evran University, Kırşehir, Turkey

[a)]Corresponding author: nil.mansuroglu@ahievran.edu.tr
[b)]muco.ozk@icloud.com

Abstract. Leibniz algebras are certain generalization of Lie algebras. In literature, there are many articles on low dimensional Leibniz algebras. In this note our main goal is to investigate low dimensional Leibniz algebras and we give some examples for three dimensional non-Lie Leibniz algebras. Moreover, we focus on the structure constants of Leibniz algebras and to give some properties of the structure constants of Leibniz algebras. In particular, some conditions are investigated for non-Lie Leibniz algebras.

Keywords: Lie algebra, Leibniz algebra, structure constants.
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INTRODUCTION

Leibniz algebras are certain non-(anti)commutatible analogs of Lie algebras were discovered by J.L. Loday [1]. Finite dimensional Leibniz algebras is an interesting area, which was studied in varies papers (see [2, 3]). Our main starting point is given by the paper [4] İ. Demir, K.C. Misra and E. Stitzinger in 2014 which studied on some results on Leibniz algebras analogous to results on Lie algebras. Moreover, they gave some non-Lie and nilpotent Leibniz algebras which are isomorphic. The classification of Leibniz algebras is still an open problem. The classification of Leibniz algebras with dimension less than three or equal to three is known (see [1, 5]). In this study we try to give some results on the classification of three dimensional non-Lie Leibniz algebras.

PRELIMINARIES

In this section we give some necessary definitions and notation. Recall that a Lie algebra L over a field F is a non-associative algebra with a bilinear map, the Lie bracket $[,] : L \times L \to L$ defined by $(x, y) \mapsto [x, y]$, satisfying the following properties:

$$(L1)\ [x, x] = 0, \text{ for all } x \in L \text{ (anti-commutative)}$$
$$(L2)\ [[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \text{for all } x, y, z \in L \text{ (Jacobi identity)}$$

(see [6]). A Leibniz algebra L over a field F is a non-associative algebra with multiplication which is called bracket $[,] : L \times L \to L$ satisfying the Leibniz identity

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

for all $x, y, z \in L$.

Let L be a Lie algebra over a field F. Then, L satisfies the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

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It follows that $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$ which, shows that every Lie algebra is a Leibniz algebra. Suppose that L is a Leibniz algebra with $[x, x] = 0$ for all $x \in L$. Then we have

$$0 = [x + y, x + y] = [x, y] + [y, x].$$

It follows that $[x, y] = -[y, x]$. Then we obtain

$$\begin{aligned} 0 &= [[x, y], z] - [x, [y, z]] + [y, [x, z]] \\ &= [[x, y], z] + [[y, z], x] + [[z, x], y] \end{aligned}$$

for all $x, y, z \in L$. Hence, a Leibniz algebra L is a Lie algebra if and only if $[x, x] = 0$ for every element x in L.

A Leibniz algebra L is said to be abelian if $[x, y] = 0$ for all $x, y \in L$. A subspace A is called a Leibniz subalgebra of L, if $[x, y] \in A$ for all $x, y \in A$. A subspace A is called a left (respectively right) ideal, if $[y, x] \in A$ (respectively $[x, y] \in A$) for all $x \in A$ and $y \in L$. If a subspace A is both a left and a right ideal of L, then it is called an ideal, that is, $[x, y], [y, x] \in A$ for all $x \in A$ and $y \in L$. By $Leib(L)$, we donote the subspace generated by the elements $[x, x]$, for some $x \in L$. This subspace is an ideal of L and it is said to be the Leibniz kernel of L. Since for $[x, x] \in Leib(L)$ and $y \in L$,

$$[[x, x], y] = [x, [x, y]] - [x, [x, y]] = 0,$$

the Leibniz kernel of L is an abelian Leibniz algebra.

If a Leibniz algebra L has an ideal A, then the factor algebra L/A is a Leibniz algebra. Say $K = Leib(L)$. If the Leibniz algebra L/K is abelian, then

$$[x + K, x + K] = [x, x] + K = K$$

for all $x \in L$. This means that L/K is a Lie algebra.

Let L_1 and L_2 be two Leibniz algebras. A map $\varphi : L_1 \to L_2$ is called a homomorphism if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L_1$ and φ is a linear map. If φ is bijective, we say that φ is an isomorphism.

Let L be a Leibniz algebra. The subspace $[L, L]$ which is called derived subalgebra is generated by the elements $[x, y]$ for all $x, y \in L$. Define the composition chain of ideals $L^1 = L, L^2 = [L, L], \ldots, L^{k+1} = [L^k, L]$ for $k \geq 1$. Then the Leibniz algebra L is called a nilpotent Leibniz algebra if there exists a positive integer $k \geq 1$ such that $L^k = 0$. Now, we define the composition chain of ideals $L^{(1)} = L, L^{(2)} = [L^{(1)}, L^{(1)}], \ldots, L^{(n+1)} = [L^{(n)}, L^{(n)}]$ for $n \geq 1$. If for some positive integer $n \geq 1$, we have $L^{(n)} = 0$, the Leibniz algebra L is said to be a solvable Leibniz algebra.

SOME RESULTS ON LOW DIMENSIONAL NON-LIE LEIBNIZ ALGEBRAS

Suppose that L is a non-Lie Leibniz algebra over a field F. Then $Leib(L) \neq \emptyset$ and $Leib(L) \neq L$. In the light of this, it is obvious that there does not exist any one dimensional non-Lie Leibniz algebra. Hence, L should be an algebra of dimension greater than two or equal to two.

Theorem 0.1 ([4], Theorem 6.2) *Let L be a non-Lie Leibniz algebra and* $dimL \leq 4$. *Then L is solvable.*

Proposition 0.2 ([4], Proposition 6.3) *If the Leibniz algebra L is nilpotent of dimension n and* $dim[L, L] = n - 1$, *then L is a cyclic Leibniz algebra generated by a single element.*

The proof of following theorem is similar to the proof of Theorem 6.4 in [4].

Theorem 0.3 *Let L be a non-Lie nilpotent Leibniz algebra with* $dimL = 3$. *Then L is isomorphic to a Leibniz algebra spanned by* $\{x, y, z\}$ *with the non-zero products given by one of the following:*

(1) $[x, x] = z, [y, z] = x.$
(2) $[x, x] = z, [x, z] = y.$
(3) $[x, x] = z, [x, y] = z.$
(4) $[y, y] = z, [y, z] = x.$

Theorem 0.4 *Let L be a non-Lie non-nilpotent Leibniz algebra with* $dimL = 3$. *Then L is isomorphic to a Leibniz algebra spanned by* $\{x, y, z\}$ *with the non-zero products given by one of the following:*

(1) $[y, y] = x, [z, y] = y, [x, y] = y.$
(2) $[z, x] = x, [z, y] = y, [y, z] = y.$
(3) $[x, x] = y, [y, x] = x.$
(4) $[x, x] = y, [x, y] = z, [y, x] = x.$
(5) $[y, y] = z, [y, z] = y.$

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