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A new result on weighted arithmetic mean summability factors of infinite series involving almost increasing sequences

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Abstract. In this paper, a known theorem dealing with weighted mean summability methods of non-decreasing sequences has been generalized for $|A, p_n; \delta|_k$ summability factors of almost increasing sequences. Also, some new results have been obtained concerning $\left|\bar{N}, p_n\right|_k$, $\left|\bar{N}, p_n; \delta\right|_k$ and $|C, 1; \delta|_k$ summability factors.

Keywords: Summability,infinite series, almost increasing sequences
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INTRODUCTION

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). We denote u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [8]),

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{2}$$

A series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [10]),

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \tag{3}$$

If we take $\delta = 0$, then we have $|C, \alpha|_k$ summability (see [9]).
Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

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defines the sequence (w_n) of the weighted arithmetic mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [11]).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [4]),

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid \Delta w_{n-1} \mid^k < \infty. \tag{6}$$

where

$$\Delta w_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{7}$$

In the special case if we take $\delta = 0$, we have $|\bar{N}, p_n|_k$ summability (see [2]). When $p_n = 1$ for all values of n, $|\bar{N}, p_n; \delta|_k$ summability is the same as $|C, 1; \delta|_k$ summability. Also if we take $\delta = 0$ and $k = 1$, then we have $|\bar{N}, p_n|$ summability. Let $A = (a_{nv})$ be a normal matrix. i.e., a lower triangular matrix of nonzero diagonal entries. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{8}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{9}$$

Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{10}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$\begin{aligned} A_n(s) &= \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i = \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv} \\ &= \sum_{i=0}^{n} a_i \bar{a}_{ni} = \sum_{v=0}^{n} \bar{a}_{nv} a_v. \end{aligned} \tag{11}$$

Since $\bar{a}_{n-1,n} = \sum_{i=n}^{n-1} a_{n-1,i} = 0$,

$$\begin{aligned} \bar{\Delta} A_n(s) &= A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \bar{a}_{nv} a_v - \sum_{v=0}^{n-1} \bar{a}_{n-1,v} a_v \\ &= \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) a_v + \bar{a}_{n-1,n} a_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \end{aligned} \tag{12}$$

The series $\sum a_n$ is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [14])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty \tag{13}$$

where

$$\Delta A_n(s) = A_n(s) - A_{n+1}(s), \quad and \quad \bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

By a weighted mean matrix we state

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq \text{n} \\ 0 & v > n, \end{cases}$$

where (p_n) is a sequence of positive numbers with $P_n = p_0 + p_1 + p_2 + ... + p_n \to \infty$ as $n \to \infty$.
If we take $\delta = 0$, then $|A, p_n; \delta|_k$ summability is the same as $|A, p_n|_k$ summability (see [15]) and if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n|_k$ summability. Also, if we take $\delta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then $|A, p_n; \delta|_k$ summability is the same as $|C, 1|_k$ summability.

The Known Results

Quite recently, Bor has proved the following theorems concerning on weighted arithmetic mean summability factors of infinite series.

Theorem 2.1 [3] Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{14}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{15}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{16}$$

$$|\lambda_n|X_n = O(1). \tag{17}$$

If

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{18}$$

and (p_n) is a sequence that

$$P_n = O(np_n), \tag{19}$$

$$P_n\Delta p_n = O(p_n p_{n+1}), \tag{20}$$

then the series $\sum a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Theorem 2.2 [5] Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n), (β_n), (λ_n), (p_n) satisfy the conditions (14)-(17), (19)-(20) of Theorem 2.1, and

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|s_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{21}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right) \quad as \quad m \to \infty, \tag{22}$$

then the series $\sum a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Theorem 2.3 [6] Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (14)-(17), (19)-(20) of Theorem 2.1, and

$$\sum_{n=1}^{m} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{23}$$

then the series $\sum a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Theorem 2.4 [7] Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n) , (β_n), (λ_n), and (p_n) satisfy the conditions (14)-(17), (19)-(20) of Theorem 2.1, condition (22) of Theorem 2.2, and

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{24}$$

then the series $\sum a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$, $0 \leq \delta < 1/k$.

The Main Results

In this paper we generalize Theorem 2.4 to $|A, p_n; \delta|_k$ summability method using almost incerasing sequences and normal matrix instead of non-decreasing sequences and weighted mean matrix, respectively. The following our main

theorem is generalized the above results concerning $|\bar{N}, p_n|_k$ and $|\bar{N}, p_n; \delta|_k$ summability methods.
Theorem 3.1 Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{n0} = 1, \ n = 0, 1, ..., \tag{25}$$
$$a_{n-1,v} \geq a_{nv}, \text{ for } n \geq v+1, \tag{26}$$
$$a_{nn} = O(\frac{p_n}{P_n}), \tag{27}$$
$$\sum_{v=1}^{n-1} a_{vv}\hat{a}_{n,v+1} = O(a_{nn}), \tag{28}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\} \quad as \quad m \to \infty, \tag{29}$$
$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \quad as \quad m \to \infty. \tag{30}$$

Let (X_n) be an almost increasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy all the conditions of Theorem 2.4, then the series $\sum a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|A, p_n; \delta|_k$, $k \geq 1$, $0 \leq \delta < 1/k$. If we take $\delta = 0$ in Theorem 3.1, then Theorem 3.1 reduces to $|A, p_n|_k$ summability theorem (see [17]).

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