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Bernoulli F-polynomials and Fibo–Bernoulli matrices

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Abstract

In this article, we define the Euler–Fibonacci numbers, polynomials and their exponential generating function. Several relations are established involving the Bernoulli F-polynomials, the Euler–Fibonacci numbers and the Euler–Fibonacci polynomials. A new exponential generating function is obtained for the Bernoulli F-polynomials. Also, we describe the Fibo–Bernoulli matrix, the Fibo–Euler matrix and the Fibo–Euler polynomial matrix by using the Bernoulli F-polynomials, the Euler–Fibonacci numbers and the Euler–Fibonacci polynomials, respectively. Factorization of the Fibo–Bernoulli matrix is obtained by using the generalized Fibo–Pascal matrix and a special matrix whose entries are the Bernoulli–Fibonacci numbers. The inverse of the Fibo–Bernoulli matrix is also found.

MSC: Primary 11B68; 11B39; secondary 15A60

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1 Introduction

Many mathematicians have recently studied various matrices and analogs of these matrices. Especially, these matrices are the Bernoulli, Pascal and Euler matrices [1–11]. These matrices and their analogs are obtained using numbers and polynomials such as the Bernoulli, Euler, q-Bernoulli, and q-Euler expressions [5, 12–18].

In this study we are interested in some matrices whose entries are the Bernoulli F-polynomials, Bernoulli–Fibonacci numbers, Euler–Fibonacci numbers and Euler–Fibonacci polynomials.

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = \begin{cases} F_{n+2} = F_{n+1} + F_n, \\ F_0 = 0, \qquad F_1 = 1. \end{cases}$$

For convenience of the reader, we provide a summary of the mathematical notations and some basic definitions of the Fibonomial coefficient.

The F-factorial is defined as follows:

$$F_n! = F_n F_{n-1} F_{n-2} \cdots F_1, \qquad F_0! = 1.$$

The Fibonomial coefficients are defined $n \geq k \geq 1$ as

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}!F_k!},$$

with $\binom{n}{0}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$. Fibonomial coefficients have the following properties:

$$\binom{n}{k}_F = \binom{n}{n-k}_F$$

and

$$\binom{n}{k}_F \binom{k}{j}_F = \binom{n}{j}_F \binom{n-j}{k-j}_F.$$

The binomial theorem for the F-analog is given by

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}. \tag{1}$$

The F-exponential function e_F^t is defined by

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!} \tag{2}$$

in [19, 20].

2 The Bernoulli F-polynomials and some of its properties

Firstly, we mention the Bernoulli F-polynomials. Krot [19] defined the Bernoulli F-polynomials. In this section, we obtain an exponential generating function of the Bernoulli F-polynomials. Then we give some properties of the Bernoulli F-polynomials.

Definition 1 ([19]) Let $\binom{n}{k}_F$ be Fibonomial coefficients and F_n be the nth Fibonacci numbers, and we use Bernoulli's F-polynomials of order 1; we define

$$B_{n,F}(x) = \sum_{k\geq 0} \frac{1}{F_{k+1}} \binom{n}{k}_F x^{n-k}. \tag{3}$$

The first few Bernoulli's F-polynomials are as follows:

$$B_{0,F}(x) = 1,$$

$$B_{1,F}(x) = x + 1,$$

$$B_{2,F}(x) = x^2 + x + \frac{1}{2},$$

$$B_{3,F}(x) = x^3 + 2x^2 + x + \frac{1}{3},$$

$$B_{4,F}(x) = x^4 + 3x^3 + 3x^2 + x + \frac{1}{5},$$

$$B_{5,F}(x) = x^5 + 5x^4 + \frac{15}{2}x^3 + 5x^2 + x + \frac{1}{8}.$$

Theorem 1 *The exponential generating function of the Bernoulli F-polynomial $B_{n,F}(x)$ is*

$$g(x) = \frac{e_F^{xt}(e_F^t - 1)}{t}. \tag{4}$$

Proof For the proof, we use the F-exponential function e_F^t.

$$\begin{aligned}
\frac{e_F^{xt}(e_F^t - 1)}{t} &= \frac{1}{t}\left(\sum_{n=0}^{\infty} x^n \frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} \frac{t^n}{F_n!} - 1\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} x^n \frac{t^n}{F_n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{F_{k+1}!} \frac{x^{n-k}}{F_{n-k}!}\right) t^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{F_{k+1}} \binom{n}{k}_F x^{n-k}\right) \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} B_{n,F}(x) \frac{t^n}{F_n!}.
\end{aligned}$$

□

Theorem 2 *Let $B_{n,F}(x + y)$ be the Bernoulli F-polynomials, we have*

$$B_{n,F}(x + y) = \sum_{k=0}^{n} \binom{n}{k}_F B_{k,F}(x) y^{n-k}, \tag{5}$$

where $B_{n,F}(x + y) = \sum_{k\geq 0} \frac{1}{F_{k+1}} \binom{n}{k}_F (x +_F y)^{n-k}$ for all nonnegative integers n.

Proof By virtue of the definition of the Bernoulli F-polynomials we get

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} B_{n,F}(x) \frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} y^n \frac{t^n}{F_n!}\right) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{B_{k,F}(x)}{F_k!} \frac{y^{n-k}}{F_{n-k}!}\right) t^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k}_F B_{k,F}(x) y^{n-k}\right) \frac{t^n}{F_n!}.
\end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} B_{n,F}(x) \frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} y^n \frac{t^n}{F_n!}\right) &= \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{F_{k+1}} \binom{n}{k}_F x^{n-k} \frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} y^n \frac{t^n}{F_n!}\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} x^n \frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} y^n \frac{t^n}{F_n!}\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{t^n}{F_{n+1}!}\right)\left(\sum_{n=0}^{\infty} (x +_F y)^n \frac{t^n}{F_n!}\right)
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F (x +_F y)^{n-k}\right)\frac{t^n}{F_n!}$$

$$= \sum_{n=0}^{\infty} B_{n,F}(x+y)\frac{t^n}{F_n!}. \tag{7}$$

Comparing the coefficients of $\frac{t^n}{F_n!}$ on both sides of Eqs. (6) and (7), we arrive at the desired result. □

3 The Euler–Fibonacci polynomials and their relation with Bernoulli F-polynomials

In this section, we define the Euler–Fibonacci numbers and the Euler–Fibonacci polynomials. Then we obtain their exponential functions and the relationship between the Bernoulli F-polynomials and these polynomials.

Definition 2 For all nonnegative integer n, the Euler–Fibonacci numbers $E_{n,F}$ are defined by

$$E_{n,F} = -\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}, \tag{8}$$

where $E_{0,F} = 1$.

The first few Euler–Fibonacci numbers are as follows:

$E_{0,F}$	$E_{1,F}$	$E_{2,F}$	$E_{3,F}$	$E_{4,F}$	$E_{5,F}$
1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{11}{8}$	$\frac{17}{16}$

Theorem 3 *The exponential generating function of Euler–Fibonacci numbers $E_{n,F}$ is defined by*

$$\sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!} = \frac{2}{e_F^t + 1}. \tag{9}$$

Proof For the proof, we show that

$$\left(\sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!}\right)\left(e_F^t + 1\right) = 2.$$

From (2), we have

$$\left(\sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{F_n!} + 1\right) = \left(\sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!}\right)\left(2 + \sum_{n=1}^{\infty}\frac{t^n}{F_n!}\right)$$

$$= 2\sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!} + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\frac{E_{k,F}}{F_k!}\frac{1}{F_{n-k}!}\right)t^n$$

$$= 2\sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!} + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F} - E_{n,F}\right)\frac{t^n}{F_n!}$$

$$= 2\sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!} + \sum_{n=1}^{\infty}(-2E_{n,F})\frac{t^n}{F_n!}$$
$$= 2,$$

which is the desired result. □

Definition 3 The Euler–Fibonacci polynomials $E_{n,F}(x)$ are defined by

$$E_{n,F}(x) = \sum_{k=0}^{n} \binom{n}{k}_F E_{k,F}x^{n-k},$$

where $E_{0,F}(x) = 1$ and $E_{n,F}$ are the nth Euler–Fibonacci numbers.

The first few Euler–Fibonacci polynomials are as follows:

$$E_{0,F}(x) = 1,$$
$$E_{1,F}(x) = x - \frac{1}{2},$$
$$E_{2,F}(x) = x^2 - \frac{x}{2} - \frac{1}{4},$$
$$E_{3,F}(x) = x^3 - x^2 - \frac{x}{2} - \frac{1}{4},$$
$$E_{4,F}(x) = x^4 - \frac{3}{2}x^3 - \frac{3}{2}x^2 - \frac{3}{4}x + \frac{11}{8},$$
$$E_{5,F}(x) = x^5 - \frac{5}{2}x^4 - \frac{15}{4}x^3 - \frac{15}{4}x^2 + \frac{55}{8}x + \frac{17}{16}.$$

Theorem 4 *The exponential generating function of Euler–Fibonacci polynomials $E_{n,F}(x)$ is defined by*

$$\sum_{n=0}^{\infty} E_{n,F}(x)\frac{t^n}{F_n!} = \frac{2e_F^{xt}}{(e_F^t + 1)}. \tag{10}$$

Proof By virtue of the definition of the Euler–Fibonacci polynomials, we get

$$\frac{2e_F^{xt}}{(e_F^t + 1)} = \sum_{n=0}^{\infty} E_{n,F}\frac{t^n}{F_n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{F_n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{E_{k,F}}{F_k!}\frac{x^{n-k}}{F_{n-k}!}\right)t^n$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}x^{n-k}\right)\frac{t^n}{F_n!}$$
$$= \sum_{n=0}^{\infty} E_{n,F}(x)\frac{t^n}{F_n!}.$$
□

In the following proposition, we will give a relationship between the Bernoulli F-polynomials $B_{n,F}(x)$ and the Euler–Fibonacci polynomials $E_{n,F}(x)$.

Proposition 1 *Let n be a nonnegative integer,*

$$B_{n,F}(x) = \frac{x^{n+1} - E_{n+1,F}(x)}{F_{n+1}} + \sum_{k=0}^{n} \frac{1}{F_{k+1}} \binom{n}{k}_F \left(x^{k+1} - E_{k+1,F}(x)\right). \tag{11}$$

Proof For the proof, we use the exponential generating functions for the Bernoulli F-polynomial and the Euler–Fibonacci polynomials. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_{n,F}(x) \frac{t^n}{F_n!} \\
&= \frac{e_F^{xt}(e_F^t - 1)}{t} \\
&= \frac{(e_F^t + 1)}{t} \left(e_F^{xt} - \frac{2e_F^{xt}}{e_F^t + 1} \right) \\
&= \left(\sum_{n=0}^{\infty} \frac{t^n}{F_n!} + 1 \right) \left(\sum_{n=0}^{\infty} \frac{x^n t^{n-1}}{F_n!} - \sum_{n=0}^{\infty} E_{n,F}(x) \frac{t^{n-1}}{F_n!} \right) \\
&= \left(\sum_{n=0}^{\infty} \frac{t^n}{F_n!} + 1 \right) \left(\sum_{n=0}^{\infty} \left(x^{n+1} - E_{n+1,F}(x)\right) \frac{t^n}{F_{n+1}!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{x^{k+1} - E_{k+1,F}(x)}{F_{k+1}!} \frac{1}{F_{n-k}!} \right) t^n + \sum_{n=0}^{\infty} \left(\frac{x^{n+1} - E_{n+1,F}(x)}{F_{n+1}} \right) \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{F_{k+1}} \binom{n}{k}_F \left(x^{k+1} - E_{k+1,F}(x)\right) \right) t^n + \sum_{n=0}^{\infty} \left(\frac{x^{n+1} - E_{n+1,F}(x)}{F_{n+1}} \right) \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{x^{n+1} - E_{n+1,F}(x)}{F_{n+1}} + \sum_{k=0}^{n} \frac{1}{F_{k+1}} \binom{n}{k}_F \left(x^{k+1} - E_{k+1,F}(x)\right) \right) \frac{t^n}{F_n!}.
\end{aligned}$$

Comparing the coefficients of $t^n/F_n!$ on both sides of the above equations we arrive at the desired result. □

Also,

$$B_{n,F}(x) = 2\left(\frac{x^{n+1} - E_{n+1,F}(x)}{F_{n+1}} \right) + \sum_{k=0}^{n-1} \frac{1}{F_{k+1}} \binom{n}{k}_F \left(x^{k+1} - E_{k+1,F}(x)\right). \tag{12}$$

For example, if we take $n = 2$ in Proposition 1, we have

$$\begin{aligned}
B_{2,F}(x) &= \frac{x^3 - E_{3,F}(x)}{F_3} + \sum_{k=0}^{2} \frac{1}{F_{k+1}} \binom{2}{k}_F \left(x^{k+1} - E_{k+1,F}(x)\right) \\
&= \frac{1}{2}\left(x^2 + \frac{x}{2} - \frac{1}{4} \right) + x - \left(x - \frac{1}{2} \right) + x^2 - \left(x^2 - \frac{x}{2} - \frac{1}{4} \right) \\
&\quad + \frac{1}{2}\left(x^3 - \left(x^3 - x^2 + \frac{x}{2} + \frac{1}{4} \right) \right) \\
&= x^2 + x + \frac{1}{2}.
\end{aligned}$$

Proposition 2 *Let $E_{n,F}$ be the nth Euler–Fibonacci number. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}_F B_{k,F}(x)E_{n-k,F}=\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F E_{n-k,F}(x). \tag{13}$$

Proof We have

$$\begin{aligned}\left(\sum_{n=0}^{\infty}E_{n,F}(x)\frac{t^n}{F_n!}\right)\left(\frac{e_F^t-1}{t}\right)&=\left(\sum_{n=0}^{\infty}E_{n,F}(x)\frac{t^n}{F_n!}\right)\left(\sum_{n=1}^{\infty}\frac{t^{n-1}}{F_n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{F_{k+1}!}\frac{E_{n-k,F}(x)}{F_{n-k}!}\right)t^n\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F E_{n-k,F}(x)\right)\frac{t^n}{F_n!},\end{aligned} \tag{14}$$

$$\begin{aligned}\left(\sum_{n=0}^{\infty}E_{n,F}(x)\frac{t^n}{F_n!}\right)\left(\frac{e_F^t-1}{t}\right)&=\left(\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{F_n!}\right)\left(\sum_{n=1}^{\infty}\frac{t^{n-1}}{F_n!}\right)\\&=\left(\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{F_{k+1}!}\frac{x^{n-k}}{F_{n-k}!}\right)t^n\right)\\&=\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\sum_{n=0}^{\infty}B_{n,F}(x)\frac{t^n}{F_n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F B_{k,F}(x)E_{n-k,F}\right)\frac{t^n}{F_n!}.\end{aligned} \tag{15}$$

From (14) and (15), we get

$$\sum_{k=0}^{n}\binom{n}{k}_F B_{k,F}(x)E_{n-k,F}=\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F E_{n-k,F}(x).$$ □

For example

$$\begin{aligned}\sum_{k=0}^{2}\binom{2}{k}_F B_{k,F}(x)E_{2-k,F}&=-\frac{1}{4}+(x+1)\left(-\frac{1}{2}\right)+\left(x^2+x+\frac{1}{2}\right)1\\&=x^2+\frac{1}{2}x-\frac{1}{4}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{2}\frac{1}{F_{k+1}}\binom{2}{k}_F E_{2-k,F}(x)&=x^2-\frac{x}{2}-\frac{1}{4}+x-\frac{1}{2}+\frac{1}{2}\\&=x^2+\frac{1}{2}x-\frac{1}{4}.\end{aligned}$$

4 The Bernoulli–Fibonacci numbers and the Bernoulli–Fibonacci polynomials

In [20], the author defined the nth Bernoulli–Fibonacci numbers and the Bernoulli–Fibonacci polynomials. For all nonnegative integers n, the nth Bernoulli–Fibonacci polynomials $B_n^F(x)$ are given with the exponential generating function as follows:

$$\sum_{n=0}^{\infty} B_n^F(x) \frac{t^n}{F_n!} = \frac{t e_F^{tx}}{e_F^t + 1}, \tag{16}$$

where $B_n^F(0) = B_n^F$.

Let the nth Bernoulli–Fibonacci number be $B_n^F(0) = B_n^F$, its exponential generating function is

$$\sum_{n=0}^{\infty} B_n^F \frac{t^n}{F_n!} = \frac{t}{e_F^t + 1}. \tag{17}$$

Proposition 3 ([20]) *Let the nth Bernoulli–Fibonacci numbers be B_n^F having defined $B_0^F = 1$ and*

$$B_n^F = -\sum_{k=0}^{n} \frac{1}{F_{n-k+1}} \binom{n}{k}_F B_k^F. \tag{18}$$

The first few Bernoulli–Fibonacci numbers are as follows:

B_0^F	B_1^F	B_2^F	B_3^F	B_4^F	B_5^F	B_6^F	B_7^F
1	-1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{3}{10}$	$-\frac{5}{8}$	$\frac{101}{39}$	$-\frac{323}{21}$

Proposition 4 ([20]) *The recurrence formula of the nth Bernoulli–Fibonacci polynomials is*

$$B_n^F(x) = \sum_{k=0}^{n} \binom{n}{k}_F B_k^F x^{n-k}. \tag{19}$$

The first few Bernoulli–Fibonacci polynomials are as follows:

$$\begin{aligned}
B_0^F(x) &= 1, \\
B_1^F(x) &= x + 1, \\
B_2^F(x) &= x^2 - x + \frac{1}{2}, \\
B_3^F(x) &= x^3 - 2x^2 + x - \frac{1}{3}, \\
B_4^F(x) &= x^4 - 3x^3 + 3x^2 - x + \frac{3}{10}, \\
B_5^F(x) &= x^5 - 5x^4 + \frac{15}{2}x^3 - 5x^2 + \frac{3}{2}x - \frac{5}{8}.
\end{aligned}$$

Now, we give the relationship of the first few Bernoulli F-polynomials $B_{n,F}(x)$ and Bernoulli–Fibonacci polynomials $B_n^F(x)$ and the classical Bernoulli polynomials $B_n(x)$ with graphics in Fig. 1.

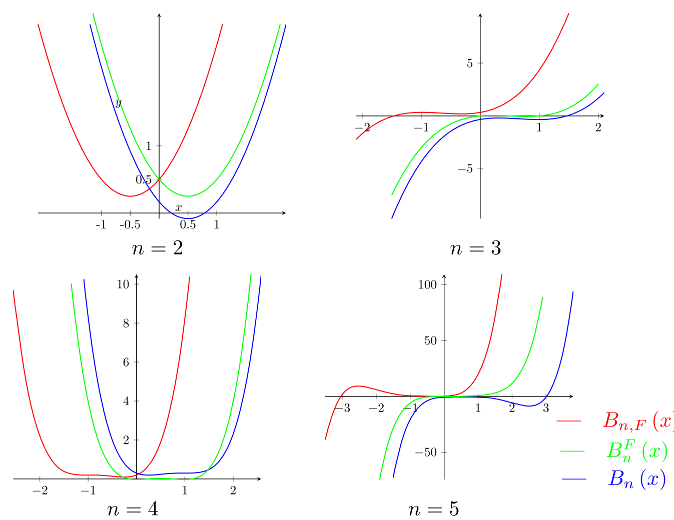

Figure 1 Graphs of $B_{n,F}(x)$, $B_n^F(x)$ and $B_n(x)$ for $n = 2, 3, 4, 5$

5 Fibo–Bernoulli matrices

In this section, we define an interesting Fibo–Bernoulli matrix by using the Bernoulli F-polynomials. Then we obtain a factorization of the Fibo–Bernoulli matrix by using a generalized Fibo–Pascal matrix. Moreover, we obtain the inverse of the Fibo–Bernoulli matrix. We define the Fibo–Euler matrix, the Fibo–Euler polynomial matrix and their inverses. Also, we show a relationship of the Fibo–Bernoulli matrix, Fibo–Euler matrix and Fibo–Euler polynomial matrix.

Definition 4 ([5]) The generalized Fibo–Pascal matrix $\mathcal{U}_{n+1}[x] = (\mathcal{U}_{n+1}(x;i,j))$ is defined by

$$\mathcal{U}_{n+1}(x;i,j) = \begin{cases} \binom{i}{j}_F x^{i-j} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Example 1 We have

$$\mathcal{U}_6[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 \\ x^2 & x & 1 & 0 & 0 & 0 \\ x^3 & 2x^2 & 2x & 1 & 0 & 0 \\ x^4 & 3x^3 & 6x^2 & 3x & 1 & 0 \\ x^5 & 5x^4 & 15x^3 & 15x^2 & 5x & 1 \end{bmatrix}.$$

Definition 5 ([5]) For $n \geq 2$, the inverse of the generalized Fibo–Pascal matrix $V(F) = (v_{ij})$ is defined by

$$v_{ij} = \begin{cases} b_{i-j+1}\binom{i}{j}_F x^{i-j} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

where $b_1 = 1$ and $b_n = -\sum_{k=1}^{n-1} b_k \binom{n}{k}_F$.

Example 2 For $n = 5$, the inverse of the generalized Fibo–Pascal matrix $V(F)$ is as follows:

$$V(F) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ x^3 & 0 & -2x & 1 & 0 & 0 \\ -x^4 & 3x^3 & 0 & -3x & 1 & 0 \\ -6x^5 & -5x^4 & 15x^3 & 0 & -5x & 1 \end{bmatrix}.$$

Definition 6 Let $B_{n,F}(x)$ be the nth Bernoulli's F-polynomial. $(n+1) \times (n+1)$; the Fibo–Bernoulli matrix $\mathcal{B}(x,F) = [b_{ij}(x,F)]$ is defined by

$$b_{ij}(x,F) = \begin{cases} \binom{i}{j}_F B_{i-j,F}(x) & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

where $0 \leq i,j \leq n$.

For $n = 3$, the Fibo–Bernoulli matrix is as follows:

$$\mathcal{B}(x,F) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x+1 & 1 & 0 & 0 \\ x^2 + x + \frac{1}{2} & x+1 & 1 & 0 \\ x^3 + 2x^2 + x + \frac{1}{3} & 2x^2 + 2x + 1 & 2x+2 & 1 \end{bmatrix}.$$

Now, we define a special matrix by using the Fibonomial coefficient. Then we obtain the factorization Fibo–Bernoulli matrix by using the generalized Fibo–Pascal matrix.

Definition 7 Let the nth Fibonacci numbers be F_n. For $1 \leq i,j \leq n+1$, the $W(F) = [w_{ij}]$ matrix is defined as follows:

$$w_{ij} = \begin{cases} \frac{1}{F_{i-j+1}} \binom{i}{j}_F & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

For $n = 5$, the $W(F)$ matrix is

$$W(F) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 2 & 1 & 0 & 0 \\ \frac{1}{5} & 1 & 3 & 3 & 1 & 0 \\ \frac{1}{8} & 1 & 5 & \frac{15}{2} & 5 & 1 \end{bmatrix}.$$

Proposition 5 ([4]) *We have*

$$\sum_{k=0}^{n} \binom{n}{k}_F B_{n-k}^F \frac{1}{F_{k+1}} = F_n! \delta_{n,0}. \tag{24}$$

Theorem 5 *Let B_n^F be the nth Bernoulli–Fibonacci numbers. $T(F) = [t_{ij}]_{(n+1)\times(n+1)}$, the inverse of the $W(F)$ matrix, is*

$$t_{ij} = \begin{cases} \binom{i}{j}_F B_{i-j}^F & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Proof We have

$$\begin{aligned}
\bigl(T(F)W(F)\bigr)_{ij} &= \sum_{k=j}^{i} t_{ik} w_{kj} \\
&= \sum_{k=j}^{i} \binom{i}{k}_F B_{i-k}^F \frac{1}{F_{k-j+1}} \binom{k}{j}_F \\
&= \sum_{k=j}^{i} \binom{i}{j}_F \binom{i-j}{k-j}_F B_{i-k}^F \frac{1}{F_{k-j+1}} \\
&= \binom{i}{j}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F B_{i-j-k}^F \frac{1}{F_{k+1}} \\
&= \binom{i}{j}_F F_{i-j}! \delta_{i-j,0}.
\end{aligned}$$

Hence, $(T(F)W(F))_{ij} = 1$ for $i = j$ and $(T(F)W(F))_{ij} = 0$ for $i \neq j$. □

For $n = 5$, $T(F)$ is as follows:

$$T(F) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & -2 & 1 & 0 & 0 \\ \frac{3}{10} & -1 & 3 & -3 & 1 & 0 \\ -\frac{5}{8} & \frac{3}{2} & -5 & \frac{15}{2} & -5 & 1 \end{bmatrix}.$$

Theorem 6 *Let $\mathcal{B}(x,F)$ be the Fibo–Bernoulli matrix and $U_{n+1}[x]$ be a generalized Fibo–Pascal matrix, then*

$$\mathcal{B}(x,F) = U_{n+1}[x]W(F).$$

Proof We have

$$\begin{aligned}
\bigl(U[x]\cdot W(F)\bigr)_{ij} &= \sum_{k=j}^{i} u_{ik} w_{kj} \\
&= \sum_{k=j}^{i} \binom{i}{k}_F x^{i-k} \frac{1}{F_{k-j+1}} \binom{k}{j}_F \\
&= \binom{i}{j}_F \sum_{k=j}^{i} \frac{1}{F_{k-j+1}} \binom{i-j}{k-j}_F x^{i-k}
\end{aligned}$$

$$\begin{aligned}
&= \binom{i}{j}_F \sum_{k=0}^{i-j} \frac{1}{F_{k+1}} \binom{i-j}{k}_F x^{i-j-k} \\
&= \binom{i}{j}_F B_{i-j,F}(x) \\
&= \bigl[\mathcal{B}(x,F)\bigr]_{ij}.
\end{aligned}$$

□

Example 3 For $n = 3$, we have

$$\begin{aligned}
U_{n+1}[x]W(F) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & x & 1 & 0 \\ x^3 & 2x^2 & 2x & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{3} & 1 & 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x+1 & 1 & 0 & 0 \\ x^2+x+\frac{1}{2} & x+1 & 1 & 0 \\ x^3+2x^2+x+\frac{1}{3} & 2x^2+2x+1 & 2x+2 & 1 \end{bmatrix} \\
&= \mathcal{B}(x,F).
\end{aligned}$$

Theorem 7 *Let $\mathcal{D}(x,F) = [d_{ij}]$ be the $(n+1) \times (n+1)$ matrix defined by*

$$d_{ij} = \begin{cases} \binom{i}{j}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F B^F_{i-j-k} b_{k+1} x^k & \textit{if } i \geq j, \\ 0 & \textit{otherwise}. \end{cases} \tag{26}$$

Then $\mathcal{D}(x,F)$ is the inverse of the Fibo–Bernoulli matrix. Thus,

$$\mathcal{B}^{-1}(x,F) = \mathcal{D}(x,F).$$

Proof Let $U_{n+1}[x]$ be a generalized Fibo–Pascal matrix. Using the factorization of $\mathcal{B}(x,F)$ in Theorem 6

$$\mathcal{B}^{-1}(x,F) = W^{-1}(F)U_{n+1}^{-1}[x] = T(F)V(F)$$

and the inverse of the generalized Fibo–Pascal matrix in (21), we obtain

$$\begin{aligned}
\bigl[T(F)V(F)\bigr]_{ij} &= \sum_{k=j}^{i} \binom{i}{k}_F B^F_{i-k} \binom{k}{j}_F b_{k-j+1} x^{k-j} \\
&= \binom{i}{j}_F \sum_{k=j}^{i} \binom{i-j}{k-j}_F B^F_{i-k} b_{k-j+1} x^{k-j} \\
&= \binom{i}{j}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F B^F_{i-j-k} b_{k+1} x^{k-j} \\
&= \bigl[\mathcal{D}(x,F)\bigr]_{ij}.
\end{aligned}$$

□

Example 4 For $n = 4$, $\mathcal{D}(x, F)$ is as follows:

$$\mathcal{D}(x,F) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 & 0 \\ -\frac{1}{3} & 1 & -2 & 1 & 0 \\ \frac{3}{10} & -1 & 3 & -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ x^3 & 0 & -2x & 1 & 0 \\ -x^4 & 3x^3 & 0 & -3x & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -x-1 & 1 & 0 & 0 & 0 \\ x+\frac{1}{2} & -x-1 & 1 & 0 & 0 \\ x^3-x-\frac{1}{3} & 2x+1 & -2x-2 & 1 & 0 \\ -x^4-3x^3+x+\frac{3}{10} & 3x^3-3x-1 & 6x+3 & -3x-3 & 1 \end{bmatrix}.$$

Definition 8 Let $E_{n,F}$ be the Euler–Fibonacci number. For $1 \leq i,j \leq n+1$, then the Fibo–Euler matrix $E_F = (e_F)_{ij}$ is defined as follows:

$$(e_F)_{ij} = \begin{cases} \binom{i}{j}_F E_{i-j,F} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Example 5 For $n = 3$, the Fibo–Euler matrix is

$$E_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{2} & -1 & 1 \end{bmatrix}.$$

Definition 9 ([5]) The Fibo–Pascal matrix $U_{n+1,F} = [u_{i,j}]_{(n+1)\times(n+1)}$ is defined by

$$u_{i,j} = \begin{cases} \binom{i}{j}_F & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6 ([16]) *Let $E_{n,F}$ be the Euler–Fibonacci number*

$$\sum_{k=0}^{n} \binom{n}{k}_F E_{n-k,F} + E_{n,F} = 2\delta_{0,n}. \tag{28}$$

Theorem 8 *Let $U_{n+1,F} = [u_{i,j}]$ be the $(n+1)\times(n+1)$ the Fibo–Pascal matrix, I_{n+1} be the identity matrix, and E_F be the Fibo–Euler matrix, then we get*

$$\frac{1}{2}(U_{n+1,F} + I_{n+1}) = E_F^{-1}.$$

Proof We have

$$\begin{aligned}\left(E_F\frac{1}{2}(U_{n+1,F}+I_{n+1})\right)_{ij} &= \frac{1}{2}(E_F U_{n+1,F}+E_F)_{ij}\\ &= \sum_{k=j}^{i}\binom{i}{k}_F E_{i-k,F}\frac{1}{2}\binom{k}{j}_F+\binom{i}{j}_F E_{i-j,F}\\ &= \frac{1}{2}\binom{i}{j}_F\sum_{k=j}^{i}\binom{i-j}{k-j}_F E_{i-k,F}+\binom{i}{j}_F E_{i-j,F}\\ &= \frac{1}{2}\binom{i}{j}_F\left[\sum_{k=0}^{i-j}\binom{i-j}{k}_F E_{i-j-k,F}+E_{i-j,F}\right]\\ &= \frac{1}{2}\binom{i}{j}_F 2\delta_{0,i-j}\\ &= \binom{i}{j}_F\delta_{0,i-j}.\end{aligned}$$

Thus, for $i=j$, $\binom{i}{j}_F\delta_{0,i-j}=1$ and for $i\neq j$ $\binom{i}{j}_F\delta_{0,i-j}=0$. Hence,

$$\frac{1}{2}(U_{n+1,F}+I_{n+1})=E_F^{-1}.$$ □

Definition 10 Let $E_{n,F}$ be the Euler–Fibonacci number. For $1\leq i,j\leq n+1$, then the Fibo–Euler polynomial matrix $E_F(x)=[(\varepsilon_F)_{ij}]$ is defined as follows:

$$(\varepsilon_F)_{ij}=\begin{cases}\binom{i}{j}_F E_{i-j,F}x^{i-j} & \text{if } i\geq j,\\ 0 & \text{otherwise.}\end{cases} \tag{29}$$

Example 6 5 × 5 For $n=4$, the Fibo–Euler polynomial matrix is as follows:

$$E_F(x)=\begin{bmatrix}1&0&0&0&0\\ -\frac{x}{2}&1&0&0&0\\ -\frac{x^2}{4}&-\frac{x}{2}&1&0&0\\ \frac{x^3}{4}&-\frac{x^2}{2}&-x&1&0\\ \frac{5x^4}{8}&\frac{3x^3}{4}&-\frac{3x^2}{2}&-\frac{3x}{2}&1\end{bmatrix}.$$

Theorem 9 *Let $H_F(x)=[(h_F)_{ij}]$ be the inverse of the Fibo–Euler polynomial matrix, then we have*

$$H_F(x)=\frac{1}{2}\big(U_{n+1}[x]+I_{n+1}\big), \tag{30}$$

where $U_{n+1,F}$ is $(n+1)\times(n+1)$ Fibo–Pascal matrix and I_{n+1} is the identity matrix.

Proof

$$\begin{aligned}\big(E_F(x)\big(U_{n+1}[x]+I_{n+1}\big)\big)_{ij} &= \sum_{k=j}^{i}\binom{i}{k}_F E_{i-k,F}x^{i-k}\binom{k}{j}_F x^{k-j}+\binom{i}{j}_F E_{i-j,F}x^{i-j}\\ &= \binom{i}{j}_F\sum_{k=j}^{i}\binom{i-j}{k-j}_F E_{i-k,F}x^{i-j}+\binom{i}{j}_F E_{i-j,F}x^{i-j}\\ &= \binom{i}{j}_F x^{i-j}\Bigg[\sum_{k=0}^{i-j}\binom{i-j}{k}_F E_{i-j-k,F}+E_{i-j,F}\Bigg]\\ &= 2\binom{i}{j}_F x^{i-j}\delta_{0,i-j}\end{aligned}$$

for $i=j$ $\binom{i}{j}_F x^{i-j}\delta_{0,i-j}=1$ and for $i\neq j$ $\binom{i}{j}_F x^{i-j}\delta_{0,i-j}=0$. Thus the proof is completed. □

Now, we obtain the Fibo–Bernoulli matrix factorization by using the inverse of the Fibo–Euler polynomial matrix.

Theorem 10 *Let $\mathcal{B}(x,F)$ be $(n+1)\times(n+1)$ the Fibo–Bernoulli matrix, then we have*

$$\mathcal{B}(x,F)=\big[2H_F(x)-I_{n+1}\big]W(F). \tag{31}$$

Proof We have

$$\big(\big[2H_F(x)-I_{n+1}\big]W(F)\big)_{ij}=\sum_{k=j}^{i}\left(2\frac{1}{2}\binom{i}{k}_F x^{i-k}-\delta_{ik}\right)\binom{k}{j}_F\frac{1}{F_{k-j+1}}$$

for $j<k<i$ $\delta_{ik}=0$, then we get

$$\begin{aligned}\big(\big[2H_F(x)-I_{n+1}\big]W(F)\big)_{ij} &= \sum_{k=j}^{i}\binom{i}{k}_F x^{i-k}\binom{k}{j}_F\frac{1}{F_{k-j+1}}\\ &= \binom{i}{j}\sum_{k=j}^{i}\binom{i-j}{k-j}_F\frac{1}{F_{k-j+1}}x^{i-k}\\ &= \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}_F\frac{1}{F_{k+1}}x^{i-j-k}\\ &= \binom{i}{j}_F B_{i-j,F}(x)\\ &= \big[\mathcal{B}(x,F)\big]_{ij}\end{aligned}$$

and

$$\big(\big[2H_F(x)-\delta\big]W(F)\big)_{ij}=0$$

for $i=j=k$ and $i<k<j$. Thus the proof is completed. □

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