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ORIGINAL ARTICLE

The development and application of an interview structure on determining preservice mathematics teachers' competence in proportional reasoning

Muhammet Arican[1]

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Abstract

Determining preservice teachers' competence in proportional reasoning is a very complex task, and existing research methods do not provide effective tools in doing that. This study describes an interview structure and application of it in determining preservice teachers' competence in proportional reasoning. The interview structure consists of *in-depth questioning* and *cognitive conflicts* and is situated in the *knowledge-in-pieces* epistemological perspective. The preservice teachers' competence in proportional reasoning was determined by examining the *knowledge resources* that they drew upon when reasoning about proportional and nonproportional situations. Semi-structured interviews were conducted with six preservice teachers to demonstrate the effectiveness of the interview structure. The case analysis indicated the preservice teachers' attention to the various fine-grained knowledge resources. However, they mostly drew upon the *qualitative relationships*, *cross-multiplication*, and *across-multiplication*. The preservice teachers' over-attention to these three knowledge resources hindered their ability to distinguish proportional relationships from nonproportional relationships. Furthermore, the productivity of knowledge resources was easily influenced by the relationships presented in problems and participants' past learning experiences. Implications of the application of the interview structure in teacher education programs and further research suggestions are discussed.

Keywords Competence in proportional reasoning · Knowledge-in-pieces perspective · Knowledge resources · Preservice teacher education · Proportional relationships

✉ Muhammet Arican
muhammetarican@gmail.com

[1] Department of Mathematics and Science Education, Kirsehir Ahi Evran University, Kirsehir, Turkey

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Introduction

Ratio, proportion, and proportional reasoning constitute a fundamental area of school mathematics that are essential for students to learn but difficult for teachers to teach (Lamon, 2007; Lobato & Ellis, 2010). In the literature, *ratio* is defined as a multiplicative comparison of two quantities with the same or different units (Lobato & Ellis, 2010). For instance, mixing cups of sugar and cups of flour, which have the same units, in a ratio of 2:3 tells us that the amount of sugar is two thirds of the amount of flour. On the other hand, paying 2 dollars per five apples is a multiplicative comparison of two quantities with different units. Moreover, *proportion* is a mathematical statement showing the equivalence of two ratios (i.e., a/b = c/d; Fisher, 1988). Finally, *proportional reasoning* "consists of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities" (Lamon, 2007, pp. 637–638). Therefore, proportional reasoning has been regarded as a specific form of multiplicative reasoning (Lesh et al., 1988).

There are two types of proportional relationships between quantities: directly proportional and inversely proportional. A directly proportional relationship exists between two quantities if "the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor" (Lobato & Ellis, 2010, p. 11). The directly proportional relationship is modeled by the equation $y = k*x$, in which y and x are quantities compared and k refers to the constant of proportionality. Hence, the directly proportional relationship is a special form of linear (affine) relationships, which are represented by the equation $y = k*x + b$. On the other hand, if the ratio formed by within values of a quantity is equal to the inverse of the ratio formed by within values of the second quantity (i.e., a/b = 1/[c/d]) then an inversely proportional relationship exists between these two quantities. An inversely proportional relationship is modeled by the equation $y*x = k$ that represents constancy of products.

Proportional reasoning plays a significant role in the development of students' mathematics, and it has been regarded as an important concept in their elementary mathematics and in advanced mathematics and as a benchmark for students' mathematical competence (Kilpatrick et al., 2001). Hence, the necessity of the development of students' proportional reasoning has been emphasized in national teaching and learning standards of many countries (e.g., Australian Curriculum, Assessment & Reporting Authority [ACARA], 2017; National Council of Teachers of Mathematics [NCTM], 2000).

Although proportional reasoning is an essential concept that students need throughout their education, many studies (e.g., Arican, 2019; Fisher, 1988; Izsák & Jacobson, 2017; Johnson, 2017; Lim, 2009; Modestou & Gagatsis, 2007; Singh, 2000) reported students' difficulties and poor performances on these concepts. Some researchers (e.g., Ben-Chaim et al., 2007; Hull, 2000; Simon & Blume, 1994) also noted that preservice (PSTs) and in-service teachers' difficulties with these concepts are often similar to students' difficulties. Among these difficulties, studies (e.g., Atabas & Oner, 2017; Arican, 2019; Hilton et al., 2016;

Izsák & Jacobson, 2017; Johnson, 2017; Lim, 2009; Modestou & Gagatsis, 2007; Van Dooren et al., 2003, 2007) mostly stated students' and teachers' difficulties in distinguishing directly and inversely proportional relationships from each other and from nonproportional relationships.

Existing studies on proportional reasoning mostly concentrate on the development of students' proportional reasoning and difficulties that they have with this complex concept, and relatively few studies (e.g., Johnson, 2017; Lim, 2009; Livy & Herbert, 2013; Riley, 2010) have been conducted with PSTs and in-service teachers. Moreover, research on teachers' knowledge in this domain assess and make sense of teachers' understanding of proportional reasoning by means of bootstrapping models of students' proportional reasoning (Lobato et al., 2011). Hence, there is a clear need for investigating teachers' proportional reasoning because determining knowledge needed for teaching mathematics (e.g., Ball et al., 2008) goes well beyond understandings required of students (Weiland et al., 2020). On the other hand, "research on teachers' knowledge of proportions has often focused on measuring particular kinds of knowledge that teachers exhibit, and the findings have been presented in a way that implies deficits in teachers' understandings" (Weiland et al., 2020, p. 2). Furthermore, there are also studies relying on the psychometric models, especially on the Item Response Theory, to assess teachers' knowledge of proportional reasoning. Weiland et al. (2020) reported the Rational Numbers Project (e.g., Cramer & Lesh, 1988; Post et al., 1991) and the Learning Mathematics for Teaching assessments (e.g., Hill, 2007; Hill et al., 2008) as the best examples for these two types of research, respectively.

Although the two types of research above assess teachers' difficulties with proportional reasoning and knowledge needed for teaching it, they lack information on how teachers develop their understanding of proportional reasoning into "richer and more interconnected networks of understanding" (Weiland et al., 2020, p. 2). Hence, it is important to apply more effective assessment tools to understand teachers' proportional reasoning. Thus, this study contributes to the literature by proposing an interview structure that uses *in-depth questioning* and *cognitive conflicts* to determine PSTs' competence in proportional reasoning. The interview structure is situated in the knowledge-in-pieces (KiP) epistemological perspective (diSessa, 1988, 2006) that assumes knowledge is developed as fine-grained resources (Weiland et al., 2020). The interview structure aims to assist researchers in determining PSTs' competence in proportional reasoning by means of examining the knowledge resources that they draw upon when reasoning about proportional and nonproportional situations.

Theoretical background

In this section, the theoretical foundations of the interview structure are described with respect to the existing literature on proportional reasoning. After describing these theoretical foundations, competence in proportional reasoning is defined in terms of the KiP perspective, and the interview structure and features of it are explained in detail.

Knowledge-in-pieces perspective

The interview structure follows the KiP epistemological perspective (diSessa, 1988, 2006). Developed originally in Newtonian mechanics (e.g., diSessa, 1988, 1993), KiP has its roots in science education research on conceptual change and has also been applied in several areas of mathematics, including whole-number multiplication, fractions, functions, and probability (Izsák, 2005). The KiP perspective acknowledges that elements of knowledge are "more diverse and smaller in grain size than those presented in textbooks" (Izsák, 2005, pp. 361–362). These grain-sized elements of the knowledge are referred to as *knowledge resources* that a person draw upon when explaining a situation or an event. Knowledge resources are the components that create complex knowledge systems (Brown et al., 2019). Knowledge resources can consist of well-established understandings and intuitions of individuals. Individuals can solve a variety of problems by connecting several knowledge resources, and the connections among these knowledge resources can be formed by perturbations and other experiences (Brown et al., 2019). Furthermore, as individuals "develop more meaningful understandings of a concept, more knowledge resources are developed and/or existing knowledge resources are refined" (Brown et al., 2019, p. 4). Therefore, the KiP perspective is effective in the sense of analyzing PSTs' "contextually sensitive" (Wagner, 2006, p. 7) intuitive knowledge and characterizing the evolution of this knowledge from novice to expert.

Hammer (2000) reported that knowledge resources can be productive and counterproductive in different situations. Productive knowledge resources facilitate learners in understanding any given situation or solving it. On the other hand, counterproductive knowledge resources do not facilitate learners in understanding the situation. For example, although some knowledge resources such as attention to the qualitative relationships (i.e., simultaneous increases and/or decreases), linearity (i.e., the graph being linear), constancy of the rate of change, and rote computations (i.e., cross-multiplication and across-multiplication) can be effective in determining proportional relationships, they may not be sufficient to distinguish proportional relationships from nonproportional ones (Arican, 2019; Izsák & Jacobson, 2017). Hence, these knowledge resources can be regarded as counterproductive in nonproportional situations.

Studies (e.g., Brown et al., 2019; Burke et al., 2017; Glassmeyer et al., 2021; Izsák & Jacobson, 2017; Orrill et al., 2017; Weiland et al., 2016, 2020) reported a variety of knowledge resources that teachers draw upon when determining proportional and nonproportional situations. These knowledge resources may include knowledge of an algorithm (e.g., cross-multiplication and across-multiplication) and knowledge of a particular understanding (e.g., multiplicative relationship and within and between measure space reasoning) (Weiland et al., 2020). Moreover, depending on the problem situation, teachers may draw upon a knowledge resource productively or counterproductively. Hence, in the current study, both productive and counterproductive knowledge resources are taken in consideration when determining PSTs' competence in proportional reasoning. In addition, in the interview structure, knowledge resources have been also used when creating cognitive conflicts.

Defining competence in proportional reasoning

Competence in proportional reasoning requires identifying multiplicative relationships presented in proportion problems and representing these relationships using a variety of mathematical representations (i.e., graphs, formulas, tables, figures). Moreover, the ability to differentiate proportional and nonproportional relationships from each other has been considered as a sign of individuals' competency in proportional reasoning (Lim, 2009; Orrill et al., 2017). Hence, competence in proportional reasoning entails developing a robust understanding of this complex concept. Therefore, this study uses the robust understanding of proportional reasoning for teaching framework provided by Weiland et al. (2020) to determine PSTs' competence in proportional reasoning. The framework was developed by means of examining the relevant literature and considering their past experiences with teachers. As presented in Table 1, the framework consisted of 10 categories.

Based on their definition of a robust understanding of proportional reasoning, Weiland et al. (2020) stated that they identified an initial list of knowledge resources by examining literature on proportional reasoning and applying grounded theory techniques (e.g., Charmaz, 2014). Next, by conducting semi-structured interviews with 32 in-service teachers, they refined their initial list and presented 19 operationalized productive knowledge resources that they expected to be related to a robust understanding of teachers. Weiland et al. (2020) stated that this list of knowledge resources could be used as an analytic tool when determining teachers' robust understanding of proportional reasoning. These 19 knowledge resources are reported as follows: (1) comparison of quantities, (2) batches, (3) multiplicative comparison, (4) covariance, (5) ratio as measure, (6) ratios as part:part and part:whole, (7) ratios ≠ fractions, (8) unit rate, (9) equivalence, (10) constant ratio, (11) scaling up/down, (12) partitioning and tiling, (13) horizon knowledge, (14) relative thinking, (15) proportional situation, (16) variable parts, (17) distortion, (18) fluidity with symbolic representations, and (19) rules.

Most recently, Glassmeyer et al. (2021) used robust understanding framework to examine 51 mathematics teachers' solutions to a comparison problem and found that 50 out of 51 teachers drew upon four knowledge resources: proportional situation, ratios as part: part or part: whole, unit rates, and ratio as measure. However, Weiland et al. (2020) and Glassmeyer et al. (2021) did not examine knowledge resources that teachers draw upon when solving inverse proportion problems. Furthermore, although the two studies reported knowledge resources needed for robust understanding of proportional reasoning, they did not discuss in detail if exhibiting all these 19 resources or only some of them would be necessary in determining robust understanding of teachers. Moreover, while Weiland et al. (2020) only focused on the productive knowledge resources, Glassmeyer et al. (2021) presented both productive and counterproductive resources. Therefore, building on Weiland et al. (2020) and Glassmeyer et al. (2021), the current study uses direct, inverse, and additive problems and considers both productive and counterproductive knowledge resources of PSTs when determining their competence. In addition, this study diagnoses PSTs' competence in proportional reasoning by providing insights into how they refine existing knowledge resources and develop new resources. As stated

Table 1 The robust understanding of proportional reasoning for teaching framework (adapted from Weiland et al., 2020)

Category	Description
Appropriateness	*Appropriateness* category highlights an understanding of not all situations are proportional. Hence, competence in proportional reasoning necessitates distinguishing the directly and inversely proportional relationships from each other and as well as from nonproportional relationships.
Reasoning	*Reasoning* category emphasizes that proportional relationships can be reasoned about. Following Lamon's (2007) definition of proportional reasoning, a robust understanding of proportional reasoning requires "supplying reasons in support of claims made about the structural relationships among four quantities" (p. 637).
Structure	*Structure* category points out the importance of understanding mathematical structures in proportional situations.
Comparison of quantities	*Comparison of quantities* category highlights the possession of the knowledge; a ratio is a comparison of two quantities.
Abstractable quantity	*Abstractable quantity* category emphasizes an understanding that there is a constant relationship between quantities forming a ratio.
Multiplicative	*Multiplicative* category points out that a robust understanding of proportional reasoning requires detecting multiplicative relationships (i.e., constant ratio and constant product) presented in proportional. relationships
Variable parts	*Variable parts* category highlights that a competent teacher should understand that proportional situations can be reasoned from a variable parts perspective (e.g., Beckmann & Izsák, 2015), which emphasizes fixed number of parts with varying sizes.
Fraction/ratio relationship	*Fraction/ratio relationship* category implies understanding similarities and differences between fractions and ratios and converting one to another when appropriate.
Multiple representation	*Multiple representation* category highlights using multiple representations (e.g., graphs, formulas, tables, figures) when representing proportional relationships.
Connections	*Connections* category emphasizes understanding connections between proportional reasoning and other mathematics topics (e.g., similarity, scale factor, probability).

by Scheiner (2020), individuals develop new understandings of concepts from their productive knowledge elements by "continually refining and extending their knowledge system" (p. 135).

The interview structure for determining competence in proportional reasoning

The interview structure (Fig. 1) was developed for facilitating researchers' determination of PSTs' competence in proportional reasoning. It was developed by examining the relevant literature on proportional reasoning and considering experiences with PSTs. In the studies (e.g., Arican, 2018, 2019) that I conducted with PSTs, I realized the complexity of determining their competence in proportional reasoning. During these studies, I observed the PSTs' tendency to use rote computations to solve problems and over-reliance on certain knowledge resources when determining

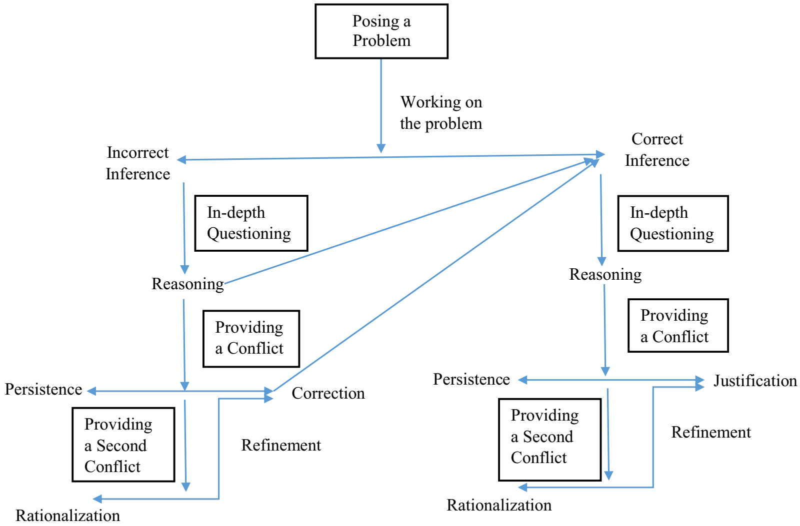


Fig. 1 The interview structure for determining competence in proportional reasoning

(directly and inversely) proportional relationships. These studies also indicated that when PSTs were provided with some cognitive conflicts, which contradicted with their existing understanding of a relationship, they either refined the knowledge resources that they relied on or persisted on using these knowledge resources by rationalizing their incorrect or partially correct understanding. Furthermore, in some PSTs, providing cognitive conflicts resulted in their over-generalization of certain understandings to inappropriate situations. Hence, considering my previous experiences with PSTs, I decided to apply a more effective interview structure that could combine in-depth questioning and cognitive conflicts to determine PSTs' competence in proportional reasoning. The interview structure is effective in terms of providing researchers with a systematic approach in determining PSTs' competence.

The interview structure consisted of three researcher activities (problem posing, in-depth questioning, and providing cognitive conflicts), which are presented inside the rectangles, and PSTs' reactions (working on the problem, inference, reasoning, persistence, justification, refinement, and rationalization) to these activities. The features of the interview structure are presented in the following pages.

Problem posing. The first step of the interview structure includes posing a problem. This researcher-driven activity consists of the development of well-designed problems and interview protocols. Developing well-designed problems is a very critical step in determining PSTs' competence in proportional reasoning because problems with rich contexts that have links to the real-life can evoke their use of a variety of knowledge resources. Hence, using problems with rich real-life contexts can facilitate researchers' determination of PSTs' competence in proportional reasoning. Furthermore, it is important develop separate problems that include proportional and nonproportional relationships. The inclusion of these two types of problems helps researchers to understand PSTs' ability to distinguish proportional relationships from nonproportional relationships. In addition, missing-value problems (i.e., three of the four values are presented, and PSTs are asked to determine the missing value) and comparison problems (i.e., two ratios are compared to determine whether they are equal, or if one is larger or smaller) can precipitate different levels of proportional reasoning and evoke different knowledge resources. Therefore, both types of problems should be included in interview settings.

Besides well-designed problems, it is also important to have well-developed interview protocols to examine PSTs' proportional reasoning in detail. When developing these interview protocols, researchers should anticipate possible solutions and knowledge resources of PSTs, so that they can include potential conflicts in their protocols for in-depth examination. To anticipate possible solutions and knowledge resources, researchers should refer to the relevant literature on proportional reasoning.

After provided with a problem, PSTs start working on the problem and present a solution or solutions. Next, researchers should ask PSTs to explain their solutions. By doing that researchers can understand how PSTs solved the given problem and detect possible knowledge resources used in solving the problem. Based on their solution strategies and knowledge resources, PSTs are expected to provide an initial inference, which can be either correct or wrong, about the relationship presented in the problem. If PSTs explain their solutions without inferring a relationship, researchers should ask questions such as "Do you think there is a relationship between the two quantities in this problem?" to reveal their initial inferences.

In-depth questioning. After determining PSTs' initial inferences, researchers can start in-depth questioning to understand how they decided those relationships. In this step, researchers can ask questions such as "Can you tell me how you decided on this relationship?" or they may ask questions about specific relationships such as "How do you know that there is a directly proportional relationship between the two quantities?" Besides understanding PSTs' reasoning for their initial inferences, in-depth questioning also assists researchers in detecting knowledge resources that PSTs draw upon when determining those relationships. In their responses, PSTs may rely on a variety of productive or counterproductive knowledge resources including but not limited to the attention to qualitative relationships, multiplicative relationships, constancy of the rate of change, linearity, or attend to the other type of knowledge resources (e.g., textual features and graphical features). Examining relevant literature on proportional reasoning, researchers may anticipate PSTs' knowledge resources beforehand, so that they can easily develop appropriate cognitive conflicts that correspond to these knowledge resources.

Providing cognitive conflicts. A cognitive conflict, which is also referred to as conceptual conflict, is a "psychological state involving a discrepancy between cognitive structures and experience, or between various cognitive structures (i.e., mental representations that organize knowledge, beliefs, values, motives, and needs)" (Waxer & Morton, 2012, p. 585). Cognitive conflicts have been used by researchers (e.g., De Bock et al., 2002; Forman & Cazden, 1998; Limón, 2001) to create a cognitive disequilibrium (Piaget, 1985) in learners with a purpose of leading learners to either discover or develop new ideas. In this study, cognitive conflicts used during semi-structured interviews aimed at confronting learners with an information contradicting with their existing understanding of proportional relationships. Cognitive conflicts may create cognitive disequilibrium in certain PSTs, so that they can refine their initial understanding of a relationship. However, some PSTs may not be affected by those conflicts and persist on using their incorrect or partially correct understanding of the relationship. Therefore, researchers should consider individual differences among PSTs and their previous experiences with proportional relationships when providing cognitive conflicts.

Based on the knowledge resource or resources that PSTs use when determining relationships in proportion tasks, researchers can create cognitive conflicts that contradict with PSTs' existing knowledge resources. Hence, these conflicts can assist researchers to understand if PSTs have a robust understanding of the presented relationships. Cognitive conflicts can be presented in the form of a new information or features of the existing problems can be used as conflicts. Cognitive conflicts can be presented in many forms such as in tables, figures, graphs, equations, or diagrams.

Based on the correctness of PSTs' initial inferences of a relationship, various cognitive conditions may arise. For instance, if PSTs determine the correct relationship by attending to a counterproductive knowledge resource or resources, after providing a cognitive conflict, they may either *persist* on using these knowledge resources or *refine* their understanding of the relationship and *justify* it by providing correct evidence. Hence, a second conflict should be provided to the PSTs who persist on using counterproductive knowledge resources. After this second conflict, the PSTs may either *rationalize* their incorrect or partially correct understanding of the relationship or refine it by providing a justification. If PSTs determine the correct relationship by attending to a productive knowledge resource or resources, researchers expect these PSTs to recognize conflicting information and to justify their understanding of the relationship by providing correct evidence. If PSTs' initial inferences are not correct, they may *correct* their incorrect inferences during in-depth questioning or when faced with cognitive conflicts.

Figure 1 presents the two phases of providing cognitive conflicts. Depending on the responses of PSTs, one conflict can be sufficient, or more than two conflicts can be required to determine their competence in proportional reasoning. It is up to researchers to decide the number of conflicts. If PSTs state that they have no idea about how to solve a problem or no idea about the relationship presented, researchers should understand these statements as signals for changing the subject of the conversation.

Methods

Research design

The purpose of this qualitative study is to examine PSTs' competence in proportional reasoning by means of conducting in-depth analysis. As stated by Yin (2009), "A case study is an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context" (p. 18). Moreover, the case study methodology allows researchers to explore a real-life phenomenon in depth (Yin, 2009).

Therefore, an exploratory multiple-case study methodology was best suited with the scope of this study in which each PST constituted a case.

Participants and data sources

This study was developed as a part of a larger research project that investigated 48 first-year PSTs' (35 female and 13 male) definitions, formulas, and graphs of the directly and inversely proportional relationships. During spring semester of 2019, the PSTs who were attending to the middle school mathematics program of a university were given a paper-and-pencil test that aimed at collecting their definitions and representations of the directly and inversely proportional relationships. The PSTs did not have any university level instruction on directly and inversely proportional relationships. In Turkey, the instruction on proportional relationships usually focuses on rule memorization and routine computations (i.e., cross-multiplication and across-multiplication). Hence, the PSTs defined and represented proportional relationships using their previous knowledge on these relationships that they learned in middle and high school.

After collecting the PSTs' responses to the paper-and-pencil test, I conducted a content analysis (e.g., Hsieh & Shannon, 2005) in which I coded the PSTs' responses for their correctness and knowledge resources. Based on the analysis, I conducted semi-structured interviews with six selected PSTs (3 females and 3 males) to understand their proportional reasoning in detail. These six PSTs were selected because their responses to the paper-and-pencil test expressed varying levels of proportional reasoning and suggested some but not complete understanding of proportional relationships. Therefore, a purposive sampling technique was followed when selecting these six PSTs.

The PSTs' definitions and representations of proportional relationships and responses to the two proportion tasks and two linear graphs constituted the data sources of this study. When conducting the interviews, I followed the interview structure described in Fig. 1. I conducted all the interviews by myself, and a colleague helped me operating the video camera. Each interview took between 30 to 70 minutes. After collecting video recordings, I transcribed these videos verbatim. Next, the PSTs' responses were coded according to knowledge resources in which I referred to the list of 19 resources provided by Weiland et al. (2020) and considered newly emerging codes that are not covered in that list.

Problems

During the interviews, the PSTs were given a bicycle problem with two items in which they included a directly proportional relationship and an inversely proportional relationship, respectively (Table 2). Moreover, a candle problem was used to examine the PSTs' understanding of nonproportional relationships. Finally, the PSTs were provided with two linear (affine) graphs with a constant difference and constant sum relationship, respectively (Fig. 2), and were asked to determine the relationships presented in these two graphs. I designed the bicycle problem and adapted the

Table 2 Problem descriptions

Problem	Description
Bicycle	A bicycle has a pedal gear with 6-cm radius and a rear gear with 2-cm radius. If there are 30 notches around the pedal gear, how many notches are there around the rear gear?
	To complete a certain distance with this bicycle, the rear gear rotates 120 times. Please calculate the number of rotations made by the pedal gear.
Candle	Two different but identical candles, A and B, are burning at the same constant rate but they are lit at different times. We know that when candle B has burned 16 mm, candle A has burned 10 mm. When 24 mm of candle B has burned, how many millimeters will candle A have burned?

candle problem from Lim (2009). During the interviews, I paid enough attention to the PSTs' definitions and representations of directly and inversely proportional relationships to create cognitive conflicts. Similarly, the candle problem and two additive graphs were used as conflicting information to examine their proportional reasoning.

In the bicycle problem, there was a directly proportional relationship between the sizes of two gears and number of notches around them. Hence, the PSTs were expected to recognize that when the radius of a gear increased, more notches could be placed around it. For this specific problem, the ratios formed by the radius and number of notches were equal, $\frac{6\text{ cm}}{30\text{ notches}} = \frac{2\text{ cm}}{10\text{ notches}}$, and yielding a constant ratio,

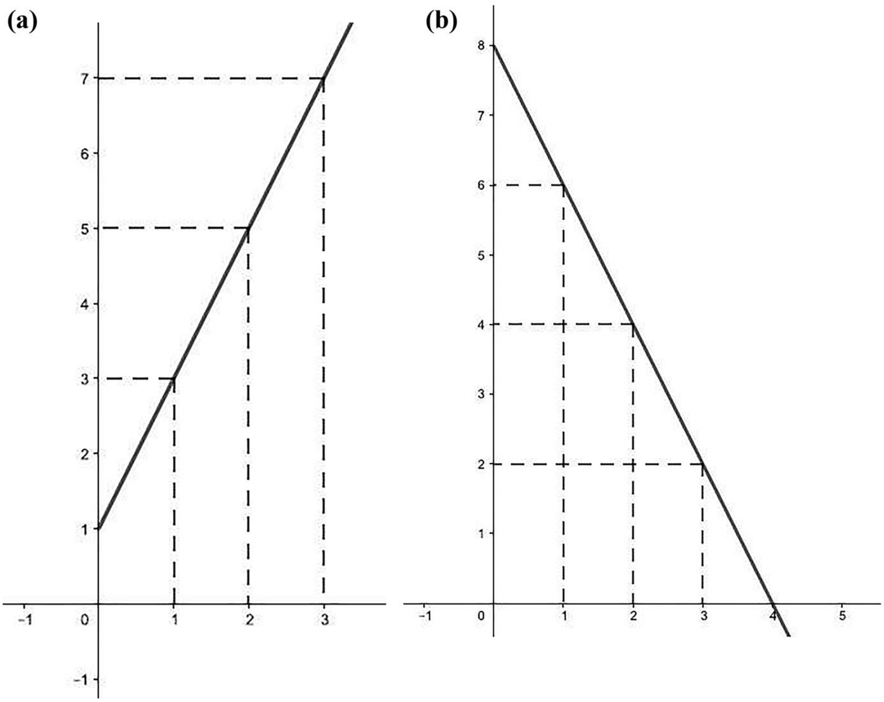


Fig. 2 (**a**) A linear increasing graph. (**b**) A linear decreasing graph

$\frac{1\text{ cm}}{5\text{ notches}}$. This constant ratio implied that there were 5 notches per 1-cm radius. On the other hand, there was an inversely proportional relationship between the radius and number of rotations. Therefore, I expected the PSTs to recognize that when the radius of a gear increased, it should have made less rotations. Similarly, the PSTs should recognize that since the distance was taken as constant, the product of radius and number of notches was equal, $2\text{ cm} \times 120\ \textit{rotations} = 5\text{ cm} \times 48\ \textit{rotations}$.

In the candle problem, since candles A and B were identical and had the same burning rates, the difference between the heights of their burning parts were equal to a constant number, 6 mm. Hence, the PSTs were expected to recognize this constant difference relationship and to calculate that candle A had burned 18 mm. On the other hand, the additive relationships in Fig. 2a, b could be represented by the equations $y = 2x + 1$ and $y = -2x + 8$. The PSTs were expected to understand that these two graphs were representing additive relationships and to distinguish these additive relationships from proportional relationships.

In the following section, an empirical evidence on the effectiveness of the interview structure in determining PSTs' competence in proportional reasoning is presented by providing a summary of the six PSTs' knowledge resources that they used when defining and representing proportional and nonproportional relationships and solving interview tasks. To better understand the functionality of the interview structure, two PSTs' responses are presented in-detail. When reporting findings, the PSTs' knowledge resources are presented by referring to the 19 knowledge resources provided by Weiland et al. (2020). Moreover, the current study also reported knowledge resources that are not covered in Weiland et al. (2020). In the transcripts, pauses were shown with ellipses and actions were described between square brackets. Pseudonyms are used when reporting the names of PSTs.

Findings obtained from an empirical study

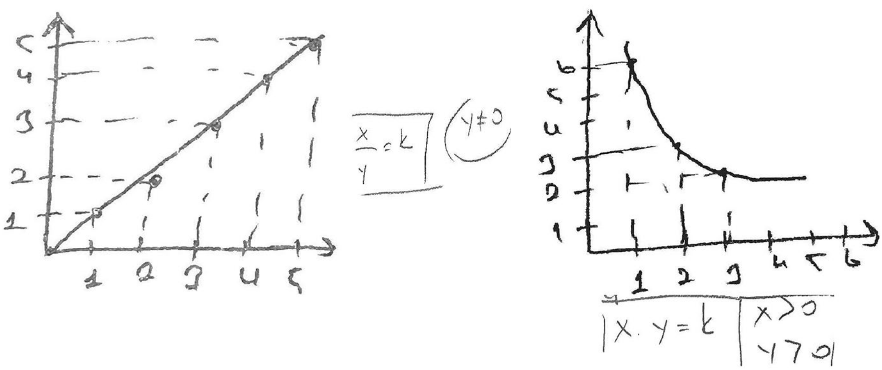
The semi-structured interviews showed that except for Neda the remaining five PSTs correctly solved the first bicycle problem and determined the directly proportional relationship. On the other hand, Neda determined this relationship as inversely proportional. The second bicycle problem was correctly solved by all six PSTs who also determined the inversely proportional relationship. In the Candle problem, again except Neda, who inappropriately expected a directly proportional relationship, the remaining five PSTs solved this problem correctly; however, only two of them recognized that this relationship was not proportional. Although the remaining three PSTs recognized the constant additive difference between the heights of burning parts, they inferred a directly proportional relationship. Similarly, only two PSTs recognized that the relationships presented in Fig. 2a and Fig. 2b were not proportional.

The analysis suggested that although the PSTs used a variety of knowledge resources when defining, representing, and determining proportional and nonproportional relationships, they mostly attended to the *qualitative relationships* (i.e., x and y increases together, or x increases, and y decreases) between quantities. Besides *qualitative relationships*, the PSTs also attended to the *cross-multiplication* and *across-multiplication* algorithms regularly to solve the given problems and to formulate the relationships. On the other hand, *multiplicative comparison*, *proportional situation*, *covariation*, *slope*, *unit rate*, *equivalences*, *constant ratio*, *constant product*, and *horizon knowledge* are also used by the PSTs as knowledge resources when determining proportional and nonproportional situations. However, the PSTs' over-attention to the *qualitative relationships* and *cross-multiplication* and *across-multiplication* algorithms hindered their understanding of the proportional relationships and ability to distinguish proportional relationships from nonproportional relationships.

In the following pages, Neda's and Murat's definitions and representations of proportional relationships and responses to the interview tasks are presented in detail in the light of the interview structure. As stated above, Neda was the only PST who expected incorrect relationships in the first bicycle problem and Candle problem. She also inferred proportional relationships in Fig. 2a and Fig. 2b. On the other hand, Murat developed an interesting reasoning technique to rationalize his inappropriate understanding about proportional relationships.

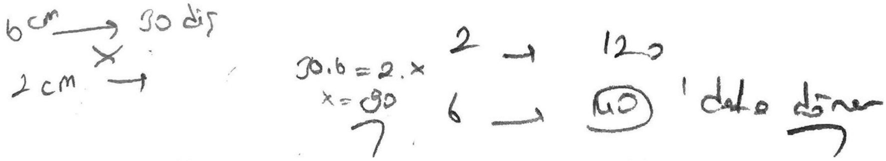
Case 1: Neda

Neda defined the directly and inversely proportional relationships by attending to the *qualitative relationships* and *equivalence* knowledge resources: "Equivalences that increase or decrease regularly are called direct proportion" and "Equivalences that do not increase or decrease regularly are called inverse proportion." Neda provided correct graphs and formulas of these two relationships (Fig. 3). In her graphs and formulas, she attended to the *constant ratio* and *constant product*, which is not reported in Weiland et al. (2020), knowledge resources that suggested her understanding of the directly and inversely proportional relationships, respectively.



(a) (b)

Fig. 3 **(a)** Neda's directly proportional graph and formula. **(b)** Neda's inversely proportional graph and formula



(a) **(b)**

Fig. 4 **(a)** Neda's incorrect response for the number of notches. **(b)** Neda's correct response for the number of rotations

In the first Bicycle problem, assuming an inversely proportional relationship, Neda applied an across-multiplication strategy (e.g., Arican, 2018, 2019) and obtained an incorrect answer, 90 notches (Fig. 4a). Neda explained her answer as follows: "If a radius with 6 cm radius has 30 notches, I thought that the gear with 2 cm radius should have more notches. Hence, I used not the direct proportion method but the inverse proportion method to obtain the number of notches." When I asked Neda, how she decided that there were more notches on the gear with 2-cm radius, she responded as follows: "I thought like that if 6 cm takes 2 steps, then the other should have taken more steps so that the rate of movement could be higher. So, I decided that it has more notches." Next, she incorrectly inferred an inversely proportional relationship between the radius and number of notches. Neda's responses above suggested that she might be thinking about notches completing some constant distance. To complete a certain distance, the small gear should rotate more than the big gear. Hence, she appeared to confuse the inversely proportional relationship between the sizes of gears and number of rotations with the directly proportional relationship between the sizes and number of notches. Therefore, Neda's attention to the *qualitative relationships* and *horizon knowledge* on the movements of gears acted as counterproductively in this situation.

As a cognitive conflict to Neda's incorrect inference of the relationship, I paid her attention to the sample bicycle picture provided in the interview form. I asked Neda that if it was visually okay that the small gear to have more notches than the larger gear. She responded as follows: "I obviously thought that the small gear had more notches. When I looked at the picture, I may calculate it incorrectly. This idea does not make sense right now." Hence, I asked Neda why her initial idea was not making sense, she responded as follows: "It does not make sense because of the picture of gears. In the picture, there are less notches on the small gear. So, I do not think it makes sense." Although the bicycle figure created a cognitive conflict, she could not correct her wrong inference about the inversely proportional relationship and could not calculate the correct answer. Therefore, we moved on to the next problem.

Neda correctly solved the second bicycle problem using the across-multiplication strategy (Fig. 4b). When I asked her if there was a relationship between the radius and number of rotations, she inferred an inversely proportional relationship and explained it as follows:

N: Because I thought that the small one moves more in terms of work. The large one moves less because I thought this as taking steps. It is like, the large wheel of a tractor takes less steps or moves less than the small wheel. This situation uses the same reasoning.

Neda used her previous knowledge of the movements of tractor wheels (*horizon knowledge*) to explain her understanding of the inverse relationship. Although this horizon knowledge facilitated Neda's understanding of the inversely proportional relationship in the current problem, it hindered her understanding of the directly proportional relationship in the first problem. Therefore, this example demonstrated that depending on the problem situation, the same knowledge resource can be productive or counterproductive. Furthermore, Neda explained that the product of 120 notches by 2 cm was equal to the product of 40 notches by 6 cm. When I asked her what the equality of these products implies, she explained it as follows: "…can it be because the distance they take are the same? I do not know." Although Neda was not sure about her answer, she related the equality of products to the concept of constant distance. Hence, the knowledge resource, *constant product*, appeared to facilitate her understanding of the inversely proportional relationship in this situation.

In the Candle problem, Neda calculated an incorrect answer, 15 mm, by erroneously applying the cross-multiplication strategy. Hence, she inferred a directly proportional relationship between the heights of burning parts in two candles. Although I provided the information that "two candles have the same burning rate" and "Candle B was lit before Candle A" as cognitive conflicts, Neda could not recognize the additive relationship between heights of these two candles. Therefore, Neda's over-attention to *the cross-multiplication strategy*, which she used as a knowledge resource, hindered her understanding of this additive relationship. Neda rationalized her incorrect inference by explaining the fact that Candle B must have burned a piece longer than Candle A burned (*qualitative relationships*).

In her response to the relationship between x and y values in Fig. 2a, Neda wrote that "To me, there is a relationship if we take the starting point as 1. The difference of 3 and 1 is 2 and from here, we have $\frac{2}{1}$. If we look at other sections of the graph, we have $\frac{7-1}{3-0} = \frac{6}{3} = 2$. It increases as directly proportional." Neda's writing suggested that she attended to the *slope* when determining this directly proportional relationship. To create a cognitive conflict, I reminded that her original direct proportion graph passed from the origin and highlighted that the graph in Fig. 2a does not pass from the origin. She responded as follows: "We would not need to pass the line from the origin…For instance, rather than passing this [pointed at her original directly proportional graph in Fig. 3a] from the origin, I could have started it from 1 [pointed at (0, 1)]. Again, it would be increasing in terms of equivalent ratios. In my view, the starting point is not important." Neda's responses suggested that she attended to the *slope*, which she indicated by mentioning the term *equivalent ratios* (i.e., $\frac{3-1}{1-0} = \frac{7-1}{3-0}$), when determining the directly proportional relationship. Therefore, Neda persisted on using *slope* and could not recognize the additive relationship between x and y. In Fig. 2a, there was a directly proportional relationship between $y-1$ and x (i.e., $y-1=2x$), taking start point as (0, 1). Thus, Neda erroneously relied on this fact to *rationalize* her incorrect inference of the directly proportional relationship.

In Fig. 2b, Neda incorrectly inferred an inversely proportional relationship between x and y. She wrote that "There is a relationship. It is inversely proportional relationship. We can prove this relationship as follows, $\frac{8-6}{1} = 2$, and if we consider triangles $\frac{6-4}{1} = 2$." Consistent with her determination of the directly proportional relationship in Fig. 2a, Neda attended to the equality of *slopes* in congruent triangles that she drew under the inverse proportion line. However, Neda recognized that the idea of slope was not compatible with the *constant product* idea that she explained earlier for second bicycle problem. She explained this incompatibility as follows: "I found congruent triangles but…if we say 2 to 3 then it is 6, but when we take 4 and 2, it is 8. So, these are not equal." Hence, Neda's previous knowledge resource *constant product* functioned as a cognitive conflict. Next, Neda stated that she was not sure if the relationship was inversely proportional or not. Thus, although *constant product* idea created a disequilibrium in Neda's understanding of the inversely proportional relationship, she could not recognize that the relationship was additive. Neda's difficulties with identifying additive relationships and confusion of the directly proportional relationship in the first bicycle problem suggested constraints in her understanding of the directly and inversely proportional relationships and distinguishing these two relationships from each other and as well as from additive relationships.

Case 2: Murat

In the paper-and-pencil test, Murat described the directly and inversely proportional relationships attending to the *qualitative relationships*: "Depending on quantity x increases or decreases, quantity y also increases or decreases" and "It is the increases and decreases being in a negative direction. So, when one quantity increases, the second quantity decreases." Although Murat provided a correct direct proportion graph, his inverse proportion graph was representing an additive decreasing relationship (Fig. 5). Moreover, using the *cross-multiplication* and *across-multiplication* algorithms as his knowledge resources, Murat provided these two algorithms as his direct and inverse proportion formulas, respectively. Murat's initial definitions and representations suggested some but limited understanding about proportional relationships.

In the first Bicycle problem, Murat calculated the correct answer (i.e., 10 notches) by representing the relationship between radius and number of notches with the $\frac{cm}{notches} = k$ formula and showing that $\frac{6[cm]}{30[notches]}$ was equal to $\frac{2[cm]}{y[notches]}$ (Fig. 6a). Therefore, he used *equivalence* knowledge resource in calculating the answer. Murat explained his solution as follows:

> Murat [M]: I thought these two [pointed at radius and number of notches] are directly proportional.
> Int: Why do you think they are directly proportional?
> M: Because…if the radius decreases, then its size will decrease too. The notches around the gears have the same size in both gears. It is already mentioned in the problem. So, if radius increases, then its circumference

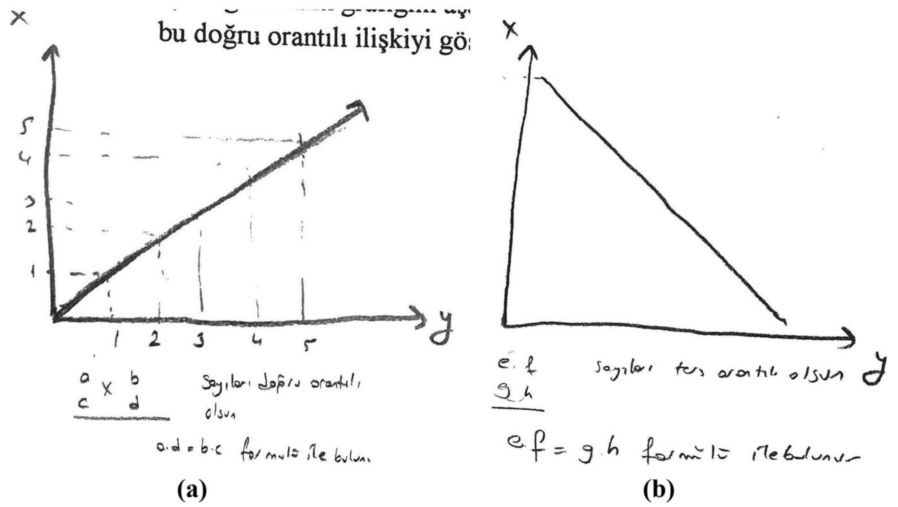


Fig. 5 (**a**) Murat's direct proportion graph and formula. (**b**) Murat's inverse proportion graph and formula

increases. If radius decreases, then circumference decreases, and number of notches decreases. Hence, I thought they are directly proportional.

The exchanges above show Murat's initial inference of the directly proportional relationship. When determining this relationship, he used the knowledge of "notches being at the same size," which can be accepted as a *horizon knowledge*, and attended to the *qualitative relationships*.

I continued in-depth questioning to investigate Murat's understanding of the directly proportional relationship. Hence, I asked him what he implied by writing $\frac{6}{30} = \frac{1}{5}$, and Murat explained that "There are five notches for every 1 cm radius." Next, Murat stated that $\frac{1}{5}$ was representing the constant of proportionality. Therefore, Murat attended to *constant ratio*, *unit rate*, and *qualitative relationships* when explaining the directly proportional relationship. Thus, Murat's explanations and solution in Fig. 6a suggested his understanding of the directly proportional relationship between the radius and number of notches.

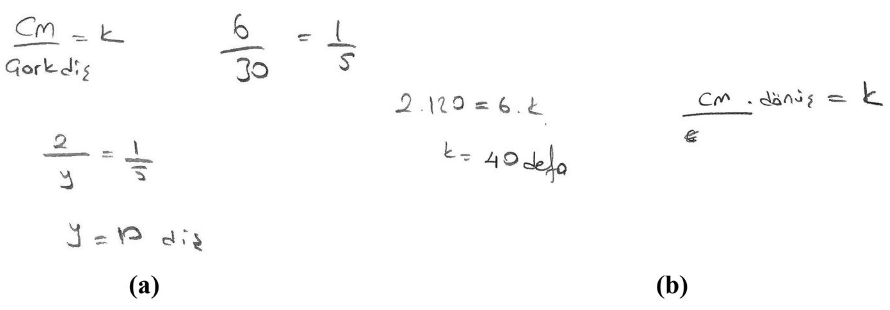


Fig. 6 (**a**) Murat's solution to the number of notches. (**b**) Murat's solution to the number of rotations

To understand how robust Murat's understanding of the directly proportional relationship was, I provided him with Fig. 2a, as a cognitive conflict, and asked him to determine the relationship presented in this graph. Although Murat wrote $y = 2x + 1$ formula besides Fig. 2a, he incorrectly inferred a directly proportional relationship. When I asked Murat how he knew that the relationship was directly proportional, he responded as follows:

> M: According to this graph [pointed at Fig. 2a], when x increases, y increases. So, when x increases by 1, y increases by 3 [he made a mistake], so y increases by $2x + 1$.
> Int: Are you thinking increases as addition?
> M: Yes. When this [pointed at x] increases by 1, this [pointed at y] increases by 2 [corrected his mistake].

Murat's responses above showed his *persistence* on the *qualitative relationships* that he counterproductively drew upon. In the exchange, Murat also attended to the amounts of change in in y with respect to x that he stated by saying "When this [pointed at x] increases by 1, this [pointed at y] increases by 2" that was described as one of the five mental actions of *covariational* reasoning (Carlson et al., 2002).

As a second conflict, I directed Murat's attention to his direct proportion formula (i.e., $cm/notches = k$; see Fig. 6a). I told him that the ratio formed between radius and number of notches was yielding constant number in his formula. Murat responded as follows:

> M: To determine a directly proportional relationship, the most important thing is the existence of increases and decreases. If two quantities are increasing or decreasing together, then they are directly proportional even if they do not satisfy the formula.

Therefore, the exchange above indicated Murat's persistence on the *qualitative relationships*. Later, he *rationalized* his inappropriate understanding of directly proportional relationships by saying that directly proportional relationships can be represented multiplicatively or additively. Murat provided two examples in Fig. 7 to explain his reasoning.

In his first example, Murat wrote $2 \times 3 = 6$ and named numbers 2, 3, and 6 as x, y, and z. Taking $y = 3$ as constant and writing $1 \times 3 = 3$, he explained that "When x decreased, z decreased." Similarly, taking $x = 2$ as constant and writing $2 \times 1 = 2$, Murat stated that "When y decreased, z decreased." Attending to these *qualitative relationships* and *covariation* between the values, Murat correctly inferred the directly proportional relationships between x and z and between y and z. Finally, taking $z = 6$ as constant and writing $1 \times 6 = 6$, Murat stated that "When x decreased, y increased." He initially inferred this relationship as a directly proportional relationship but changed it to inversely proportional relationship after recognizing the directions of changes in the x and y values. Murat followed the same type of reasoning in his second example, $2 + 3 = 5$, in which he replaced numbers 2, 3, and 5 by x, y, and z, respectively. Attending to the *qualitative relationships* and *covariation*, he inferred a directly proportional relationships between x and z and between y and z.

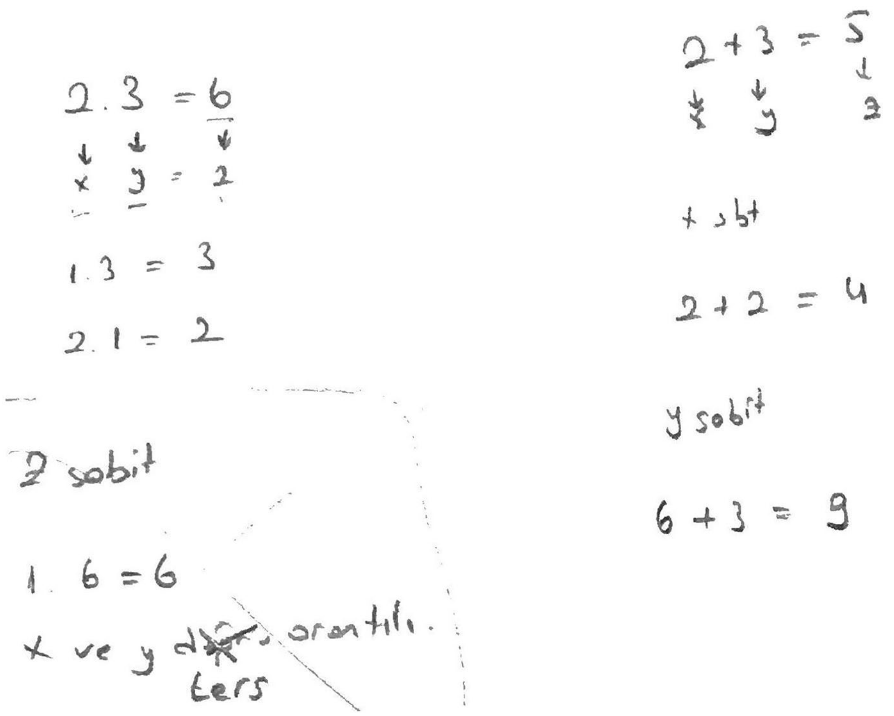


Fig. 7 Murat's explanation of directly proportional relationship from the multiplication and addition perspectives

Besides the direct proportion formula, as a third conflict, I also highlighted that his directly proportional graph (see Fig. 5a) started from the origin but the graph in Fig. 2a started from (0, 1). Murat explained that both graphs were representing directly proportional relationships because in both graphs, "When x increased, y increased" and "x increased by 1, and y increased by 2" (i.e., *covariation*). Moreover, Murat also explained that while the graph in Fig. 5a was satisfying the multiplicative direct proportion formula, the graph in Fig. 2a was satisfying the additive direct proportion formula (see Fig. 5). Therefore, Murat's over-attention to the *qualitative relationships* and *covariation* prevented him from distinguishing a proportional relationship from nonproportional relationship. In Weiland et al. (2020)'s *appropriateness* category, the ability to distinguish proportional from nonproportional relationships is regarded as a sign of competence in proportional reasoning. Thus, the interview structure was effective in terms of determining Murat's incompetence in proportional reasoning.

Murat successfully solved the second bicycle problem and showed that the product of inversely proportional values was equal to a constant (i.e., *constant product*) and represented it with a correct formula (Fig. 6b). He also correctly represented the relationship in Fig. 2b by writing $y = -2x + 8$. Like first bicycle problem and Fig. 2a, Murat inferred inversely proportional relationships by paying attention the *qualitative relationships* in these two situations. When I offered his constant product formula in Fig. 6b as a cognitive conflict, Murat recognized that the products of corresponding x and y values in Fig. 2b were not yielding a constant number. However, Murat persisted on relying on the *qualitative relationships* and rationalized his inappropriate understanding of the inversely proportional relationship by explaining that he was considering simultaneous increases and decreases, as the sole evidence for determining an inversely proportional relationship. Thus, for Murat, even though Fig. 2b did not satisfy the constant product idea, it was representing an inversely proportional relationship between x and y.

To create a second conflict, I directed Murat's attention to his inversely proportional graph (see Fig. 5b) and asked him if he could draw the inversely proportional graph one more time considering the constant product idea. Taking an arbitrary number 12 to represent the constancy of products (i.e., k), Murat drew his new graph as in Fig. 8. Therefore, following my directions and using *constant product* idea, he was able to draw the inversely proportional graph. Therefore, as a second conflict, I asked Murat which one of the graphs, Fig. 2b or Fig. 8, were representing an inversely proportional relationship. Like the directly proportional relationship, Murat stated that both graphs represented inversely proportional relationships. He explained that Fig. 2b represented an inversely proportional relationship that satisfies the addition perspective (i.e., $x+y=z$), and Fig. 6 satisfied the multiplication perspective (i.e., $x \times y=z$).

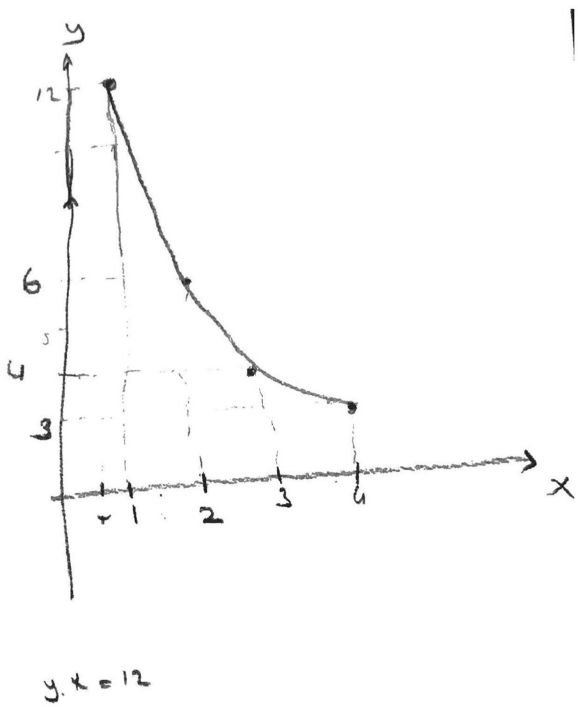


Fig. 8 Murat's second representation of the inversely proportional graph

In the Candle problem, Murat was able to calculate the correct answer 18 mm. In his solution, Murat wrote that "Because two candles were identical, they burn the same amount at the same time. Hence, after Candle B was burned 16 mm, it burned an addition of 8 mm. Similarly, Candle A, which was burned 10 mm, will burn an addition of 8 mm." When I asked Murat if there was a relationship between the heights of burning parts, he responded as follows:

M: Since these are two identical candles, both burn at a certain rate.
Int: What do you imply by the certain rate?
M: If this [pointed at Candle A] burns 1 mm, then this [pointed at Candle B] also burns 1 mm.
Int: Can you represent the relationship between the heights of these burning parts [I pointed at 16 mm, 10 mm, 24 mm, and 18 mm] with a formula?
M: Since Candle B was burned a 6 mm part before, we can write a formula like this [he wrote B = A + 6 mm].
Int: Is there a relationship between A and B in your formula?
M: When A increases, B increases too.
Int: What do these increases tell you?
M: It is directly proportional…
Int: How about if this relationship satisfies the direct proportion formula [I pointed at the *cm/notches* = *k*] that you wrote earlier.
M: ...it does not agree with the formula.
Int: We discussed the formula issue earlier, and you concluded that the formula is not always correct, right?
M: Yes.

Although Murat was able solve the Candle problem and represented the additive relationship, he inferred a directly proportional relationship by paying attention to the *qualitative relationships*. I used his previous direct proportion formula (*cm/notches* = *k*) to create a conflict; however, Murat did not correct his inappropriate understanding of the directly proportional relationship but, as he had done earlier, rationalized it by concluding that "The formula is not always correct."

Discussion and conclusions

Determining PSTs' competence in proportional reasoning is a challenging process since it has been regarded as a complex concept (Lamon, 2007). Hence, a careful examination is needed in determining PSTs' competence in proportional reasoning. Therefore, the purpose of this study was to present an interview structure that provides effective tools for conducting a comprehensive examination for investigating PSTs' competence. To conduct such an examination, the interview structure uses *in-depth questioning* and *cognitive conflicts* to determine PSTs' competence. Situated in the KiP perspective and building on the Weiland et al. (2020) study, this current study aims to assist university educators in diagnosing the competence of PSTs by offering a systematic and holistic approach.

In the empirical example, the analysis indicated the PSTs' attention to the various fragmented knowledge resources. However, they mostly drew upon the three knowledge resources: *qualitative relationships*, *cross-multiplication*, and *across-multiplication*. These three resources are not included Weiland et al. (2020)'s list of 19 resources because drawing upon them is not always productive especially in nonproportional situations. On the other hand, the PSTs also drew upon the following knowledge resources: *multiplicative comparison*, *proportional situation*, *covariation*, *slope*, *unit rate*, *equivalences*, *constant ratio*, *constant product*, and *horizon knowledge*. The knowledge resources that the PSTs drew upon indicated quite differences with Glassmeyer et al. (2021) who reported proportional situation, ratios as part: part or part: whole, unit rates, and ratio as measure as the most attended knowledge resources by the teachers. This difference might be the result of using different types of problems (missing value vs comparison) with differing contexts (gears vs mixture). In addition, the participants' background on proportions might have affected their preferences. Consequently, the PSTs' over-attention to the *qualitative relationships*, *cross-multiplication*, and *across-multiplication* appears to be a result of the instruction on proportions that regularly emphasizes rote computations and rule memorization (Arican, 2018, 2019).

One of the most important findings of this study is the necessity of examining PSTs' competence in proportional reasoning by employing a holistic approach. Hence, rather than focusing on the outcomes of single instances, it is necessary to consider the big picture in investigating proportional reasoning competence. For instance, without careful analysis, one might have concluded that Neda and Murat were competent in proportional reasoning. However, the semi-structured interviews yielded their difficulties with proportional and nonproportional relationships. Therefore, the interview structure was effective in terms of assisting the researcher in determining the two PSTs' difficulties with proportional reasoning.

In addition to the above results, the findings showed that the productivity of a knowledge resource was easily influenced by the relationships presented in problems (i.e., direct, inverse, and additive) and PSTs' past experiences with proportions. In the empirical example, Neda and Murat determined the directly and inversely proportional relationships by productively drawing upon the *qualitative relationships*. However, they erroneously inferred the additive relationships presented in the Candle problem and in Fig. 2a and Fig. 2b as proportional relationships by counterproductively drawing upon this knowledge resource. Similarly, attention to the *slope* can be helpful in determining the directly proportional relationship, as in Neda's case, it may hinder the PSTs' ability to distinguish proportional relationships from nonproportional relationships. In addition, while attention to the *cross-multiplication* and *across-multiplication* algorithms facilitated the PSTs in solving the direct and inverse proportion problems, respectively, they hindered their ability to solve nonproportional problems. Thus, these results also suggest the necessity of a systematic and holistic approach in determining PSTs' competence.

Implications for teacher education programs and further research suggestions

While the main purpose of the interview structure was not to create conceptual changes in the PSTs' proportional reasoning, it appeared to be effective in terms of developing their knowledge of proportional relationships. The interview structure contributed to the PSTs' development of content knowledge through refinement of existing knowledge resources and development of new knowledge resources. Furthermore, the interview structure can also benefit the teacher education programs by allowing researchers to detect PSTs' competence in proportional reasoning and difficulties with this complex concept.

A limitation of the interview structure is that it has been developed by means of conducting studies with PSTs. Hence, I recommend conducting further studies to investigate the effectiveness of the interview structure in determining students' and in-service teachers' competence in proportional reasoning. Another limitation is that knowledge resources are affected by the previous instruction on proportions and problem contexts. Therefore, involving participants with varying learning experiences and using problems with rich contexts can benefit researchers in detecting a wide range of productive and counterproductive knowledge resources.

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Code availability This study did not involve software applications.

Declarations

Conflict of interest The author declares no competing interests.

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