Received: 24 December 2019 | Revised: 29 March 2020 | Accepted: 15 June 2020

DOI: 10.1002/mana.201900559

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Riesz potential and its commutators on generalized weighted Orlicz–Morrey spaces

Vagif S. Guliyev[1,2] | **Fatih Deringoz**[3]

[1]Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan

[2]Peoples Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya St, Moscow 117198, Russian Federation

[3]Department of Mathematics, Ahi Evran University, Kirsehir, Turkey

Correspondence
Fatih Deringoz, Department of Mathematics, Ahi Evran University, Kirsehir, Turkey.
Email: deringoz@hotmail.com

Abstract

In the present paper, we shall give a characterization for the Adams-type boundedness of the Riesz potential and its commutators on the generalized weighted Orlicz–Morrey spaces. We also give a characterization for the BMO space via the boundedness of the commutator of the Riesz potential.

KEYWORDS

BMO, commutator, generalized weighted Orlicz–Morrey space, Riesz potential

MSC (2020)

32A37, 42B20, 42B25, 46E30

1 | INTRODUCTION

The classical Morrey spaces were introduced by Morrey [39] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [17, 38, 40] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see also [18, 19, 55]); Komori and Shirai [36] defined weighted Morrey spaces $L^{p,\kappa}(w)$; Guliyev [20] gave a concept of the generalized weighted Morrey spaces $M^{p,\varphi}_w(\mathbb{R}^n)$ which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}(w)$. In [20], the boundedness of the classical operators and their commutators in spaces $M^{p,\varphi}_w$ was also studied. For other boundedness results on these spaces, see [3, 21, 24, 25, 28, 44] for example.

The spaces $M^{p,\varphi}_w(\mathbb{R}^n)$ defined by the norm

$$\|f\|_{M^{p,\varphi}_w} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L^p_w(B(x,r))},$$

where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w is a non-negative measurable function on $\mathbb{R}^n$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

The Orlicz spaces L^Φ were first introduced by Orlicz in [49, 50] as generalizations of Lebesgue spaces L^p. Since then, the theory of Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis.

In [7], the generalized Orlicz–Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz and generalized Morrey spaces. Other definitions of generalized Orlicz–Morrey spaces can be found in [41] (see also [43]) and [57]. In words of [23], our generalized Orlicz–Morrey space is the third kind and the ones in [41] and [57] are the first kind and the second kind, respectively. The first kind and the second kind are different and that the second kind and the third kind are different according to [14]. Notice that the definition of the space of the third kind relies only on the fact that L^Φ is a normed linear space, which is independent of the condition that it is generated by modulars. On the other hand, the space of the first kind is defined via the family of modulars.

Various versions of generalized weighted Orlicz–Morrey spaces were introduced in [26, 31, 37, 53]. The spaces in [31, 37] can be seen as the weighted version of generalized Orlicz–Morrey spaces of the first kind and the spaces in [53] can be seen as the weighted version of generalized Orlicz–Morrey spaces of the second kind. We used the definition of [26] which can be seen as the weighted version of generalized Orlicz–Morrey spaces of the third kind.

There are two remarkable results on the Morrey boundedness of Riesz potential. The first result is due to Spanne [52]. The second milestone result is due to Adams [1]. Since the inclusion relations between Morrey spaces, we can say that Adams improved the result by Spanne. Recently many people are studying these operators from a various points of view [12, 19, 32–34, 42, 46, 48].

In this paper, we shall investigate the Adams-type boundedness of the Riesz potential and its commutators on generalized weighted Orlicz–Morrey spaces. We also give a characterization for the BMO space via the boundedness of the commutator of the Riesz potential. In other words, we obtain weighted versions of previous results appeared in [4, 6, 22]. Unfortunately, we follow the line of [44] and the paper by Komori and Shirai [36] assuming the somewhat strong condition of A_{i_Φ}. However, as is proved in [11, 45–47], one can say more. At least the result of this paper does not recapture these results.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 | DEFINITIONS AND PRELIMINARY RESULTS

Even though the A_p class is well known, for completeness, we offer the definition of A_p weight functions. Let $\mathcal{B} = \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$.

Definition 2.1. For $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p *weight* if

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}} < \infty.$$

A locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_1 *weight* if

$$\frac{1}{|B|}\int_B w(y)\,dy \le Cw(x), \qquad \text{a.e. } x \in B,$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p\ge1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)\,dx$.

We recall the definition of Young functions.

Definition 2.2. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a *Young function*, if Φ is convex, left-continuous, $\lim_{r\to0^+}\Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty}\Phi(r) = \infty$.

The convexity and the condition $\Phi(0) = 0$ force any Young function to be increasing. In particular, if there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then it follows that $\Phi(r) = \infty$ for $r \ge s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty.$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) \equiv \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r), \qquad r > 0,$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition and it fails the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but it fails the Δ_2-condition.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) \equiv \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & \text{if } r \in [0,\infty), \\ \infty, & \text{if } r = \infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and it satisfies $\widetilde{\widetilde{\Phi}} = \Phi$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$.

It is also known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r, \qquad r \ge 0. \tag{2.1}$$

We recall an important pair of indices used for Young functions. For any Young function Φ, write

$$h_\Phi(t) = \sup_{s>0} \frac{\Phi(st)}{\Phi(s)}, \qquad t > 0.$$

The lower and upper dilation indices of Φ are defined by

$$i_\Phi = \lim_{t\to 0^+} \frac{\log h_\Phi(t)}{\log t} \quad \text{and} \quad I_\Phi = \lim_{t\to\infty} \frac{\log h_\Phi(t)}{\log t},$$

respectively.

A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $t \in [1,\infty)$ (resp. $t \in [0,1]$) and $s \in [0,\infty)$,

$$\Phi(st) \le Ct^p\Phi(s). \tag{2.2}$$

Remark 2.3. It is well known that if Φ is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\widetilde{\Phi}$ is of lower type p_1' and upper type p_0' and Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $\Phi \in \Delta_2 \cap \nabla_2$.

Remark 2.4. It is easy to see that Φ is of lower type $i_\Phi - \varepsilon$, and of upper type $I_\Phi + \varepsilon$ for every $\varepsilon > 0$, where the constant appearing in (2.2) may depend on ε. We also mention that i_Φ and I_Φ may be viewed as the supremum of the lower types of Φ and the infimum of upper types, respectively.

Definition 2.5. For a Young function Φ and $w \in A_\infty$, the set

$$L_w^\Phi(\mathbb{R}^n) \equiv \left\{ f\text{-measurable} : \int_{\mathbb{R}^n} \Phi(k|f(x)|)w(x)\,dx < \infty \text{ for some } k > 0 \right\}$$

is called the *weighted Orlicz space*. The local weighted Orlicz space $L^{\Phi}_{w,\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{_B} \in L^{\Phi}_{w}(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Note that $L^{\Phi}_{w}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}_{w}(\mathbb{R}^n)} \equiv \|f\|_{L^{\Phi}_{w}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)\,dx \le 1\right\}$$

and

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L^{\Phi}_{w}}}\right) w(x)\,dx \le 1. \tag{2.3}$$

The following analogue of the Hölder inequality is known.

$$\left|\int_{\mathbb{R}^n} f(x)g(x)w(x)\,dx\right| \le 2\|f\|_{L^{\Phi}_{w}} \|g\|_{L^{\widetilde{\Phi}}_{w}}. \tag{2.4}$$

We refer, for instance, to [54] for details on Orlicz space.

For a weight w, a measurable function f and $t > 0$, let

$$m(w,\ f,\ t) = w(\{x \in \mathbb{R}^n \ :\ |f(x)| > t\}).$$

Definition 2.6. The weak weighted Orlicz space

$$WL^{\Phi}_{w}(\mathbb{R}^n) = \{f\text{-measurable} \ :\ \|f\|_{WL^{\Phi}_{w}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}_{w}(\mathbb{R}^n)} \equiv \|f\|_{WL^{\Phi}_{w}} = \inf\left\{\lambda > 0 \ :\ \sup_{t>0} \Phi(t) m\Big(w,\ \frac{f}{\lambda},\ t\Big) \le 1\right\}.$$

We can prove the following by a direct calculation:

$$\|\chi_{_B}\|_{L^{\Phi}_{w}} = \|\chi_{_B}\|_{WL^{\Phi}_{w}} = \frac{1}{\Phi^{-1}(w(B)^{-1})}, \quad B \in \mathcal{B}, \tag{2.5}$$

where $\chi_{_B}$ denotes the characteristic function of the B.

The Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy, \qquad x \in \mathbb{R}^n$$

for a locally integrable function f on $\mathbb{R}^n$.

Let $0 < \alpha < n$. The Riesz potential operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Theorem 2.7 [16, Proposition 2.4]. *Let Φ be a Young function. Assume in addition $w \in A_{i_\Phi}$. Then, there is a constant $C > 1$ such that*

$$\Phi(t) m\Big(w,\ Mf,\ t\Big) \le C \int_{\mathbb{R}^n} \Phi(C|f(x)|) w(x)\,dx$$

for every locally integrable f and every $t > 0$.

Remark 2.8. For a sublinear operator S, weak modular inequality

$$\Phi(t)m\Big(w,\, Sf,\, t\Big) \le C\int_{\mathbb{R}^n}\Phi(C|f(x)|)w(x)\,dx \tag{2.6}$$

implies the corresponding norm inequality. Indeed, let (2.6) holds. Then, we have

$$\Phi(t)w\left(\{x\in\mathbb{R}^n : \frac{|Sf(x)|}{C^2\|f\|_{L^\Phi_w}} > t\}\right) = \Phi(t)w\left(\{x\in\mathbb{R}^n : \left|S\left(\frac{f}{C^2\|f\|_{L^\Phi_w}}\right)(x)\right| > t\}\right)$$
$$\le C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{C\|f\|_{L^\Phi_w}}\right)w(x)\,dx \le 1,$$

which implies $\|Sf\|_{WL^\Phi_w} \lesssim \|f\|_{L^\Phi_w}$.

Lemma 2.9. *Let Φ be a Young function and $f\in L^\Phi_{w,\mathrm{loc}}(\mathbb{R}^n)$. Assume in addition $w\in A_{i_\Phi}$. For a ball B, the following inequality is valid:*

$$\|f\|_{L^1(B)} \lesssim |B|\Phi^{-1}\big(w(B)^{-1}\big)\|f\|_{L^\Phi_w(B)},$$

where $\|f\|_{L^\Phi_w(B)} \equiv \|f\chi_B\|_{L^\Phi_w}$.

Proof. Let

$$\mathfrak{M}f(x) = \sup_{B\in\mathcal{B}}\frac{\chi_B(x)}{|B|}\int_B|f(y)|\,dy, \quad x\in\mathbb{R}^n,$$

and let $\tilde f$ denote the extension of f from B to $\mathbb{R}^n$ by zero. It is well known that $\mathfrak{M}f(x)\le 2^nMf(x)$ for all $x\in\mathbb{R}^n$. Then taking into account Remark 2.8 and using Theorem 2.7, we have

$$\frac{\|f\|_{L^1(B)}}{|B|}\|\chi_B\|_{WL^\Phi_w(B)} = \frac{\|\tilde f\|_{L^1(B)}}{|B|}\|\chi_B\|_{WL^\Phi_w(B)} \lesssim \|\mathfrak{M}\tilde f\|_{WL^\Phi_w(B)}$$
$$\lesssim \|M\tilde f\|_{WL^\Phi_w(B)} \le \|M\tilde f\|_{WL^\Phi_w(\mathbb{R}^n)} \lesssim \|\tilde f\|_{L^\Phi_w(\mathbb{R}^n)} = \|f\|_{L^\Phi_w(B)}.$$

So, Lemma 2.9 is proved. □

Lemma 2.10 [13]. *If $B_0 := B(x_0, r_0)$, then $r_0^\alpha \le CI_\alpha\chi_{B_0}(x)$ for every $x\in B_0$.*

3 | GENERALIZED WEIGHTED ORLICZ–MORREY SPACES

In this section, we give the definition of the generalized weighted Orlicz–Morrey spaces $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ and investigate the fundamental structure of $M^{\Phi,\varphi}_w(\mathbb{R}^n)$. In the sequel we use the notation $\varphi(B)\equiv\varphi(x,r)$ and $cB\equiv B(x,cr)$ for $B = B(x,r)\in\mathcal{B}$ and $c > 0$.

Definition 3.1. Let φ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$, let w be a non-negative measurable function on $\mathbb{R}^n$ and Φ any Young function. Denote by $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ the generalized weighted Orlicz–Morrey space, the space of all functions $f \in L^{\Phi}_{w,\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M^{\Phi,\varphi}_w(\mathbb{R}^n)} \equiv \|f\|_{M^{\Phi,\varphi}_w} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\big(w(B(x,r))^{-1}\big)\, \|f\|_{L^{\Phi}_w(B(x,r))}$$

$$\equiv \sup_{B\in\mathcal{B}} \varphi(B)^{-1}\, \Phi^{-1}\big(w(B)^{-1}\big)\, \|f\|_{L^{\Phi}_w(B)} < \infty.$$

We denote by $WM^{\Phi,\varphi}_w(\mathbb{R}^n)$ the weak generalized weighted Orlicz–Morrey space, the space of all functions $f \in WL^{\Phi}_{w,\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{WM^{\Phi,\varphi}_w} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\big(w(B(x,r))^{-1}\big)\, \|f\|_{WL^{\Phi}_w(B(x,r))} < \infty.$$

Example 3.2. Let $1 \le p < \infty$ and $0 < \varkappa < 1$.

- If $\Phi(r) = r^p$ and $\varphi(x,r) = w(B(x,r))^{-1/p}$, then $M^{\Phi,\varphi}_w(\mathbb{R}^n) = L^p_w(\mathbb{R}^n)$ and $WM^{\Phi,\varphi}_w(\mathbb{R}^n) = WL^p_w(\mathbb{R}^n)$.
- If $\Phi(r) = r^p$ and $\varphi(x,r) = w(B(x,r))^{\frac{\varkappa-1}{p}}$, then $M^{\Phi,\varphi}_w(\mathbb{R}^n) = L^{p,\varkappa}(w)$ and $WM^{\Phi,\varphi}_w(\mathbb{R}^n) = WL^{p,\varkappa}(w)$.
- If $\Phi(r) = r^p$, then $M^{\Phi,\varphi}_w(\mathbb{R}^n) = M^{p,\varphi}_w(\mathbb{R}^n)$ and $WM^{\Phi,\varphi}_w(\mathbb{R}^n) = WM^{p,\varphi}_w(\mathbb{R}^n)$.
- If $\varphi(x,r) = \Phi^{-1}\big(w(B(x,r))^{-1}\big)$, then $M^{\Phi,\varphi}_w(\mathbb{R}^n) = L^{\Phi}_w(\mathbb{R}^n)$ and $WM^{\Phi,\varphi}_w(\mathbb{R}^n) = WL^{\Phi}_w(\mathbb{R}^n)$.

For a Young function Φ and a non-negative measurable function w, we denote by $\mathcal{G}^w_\Phi$ the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that

$$\inf_{B\in\mathcal{B};\, r_B \le r_{B_0}} \varphi(B) \gtrsim \varphi(B_0) \quad \text{for all } B_0 \in \mathcal{B}$$

and

$$\inf_{B\in\mathcal{B};\, r_B \ge r_{B_0}} \frac{\varphi(B)}{\Phi^{-1}\big(w(B)^{-1}\big)} \gtrsim \frac{\varphi(B_0)}{\Phi^{-1}\big(w(B_0)^{-1}\big)} \quad \text{for all } B_0 \in \mathcal{B},$$

where r_B and r_{B_0} denote the radius of the balls B and B_0, respectively.

The following lemma was proved in [5].

Lemma 3.3. *Let $B_0 := B(x_0, r_0)$. If $\varphi \in \mathcal{G}^w_\Phi$, then there exists $C > 0$ such that*

$$\frac{1}{\varphi(B_0)} \le \|\chi_{B_0}\|_{WM^{\Phi,\varphi}_w} \le \|\chi_{B_0}\|_{M^{\Phi,\varphi}_w} \le \frac{C}{\varphi(B_0)}.$$

The following boundedness result for the Hardy–Littlewood maximal operator on generalized weighted Orlicz–Morrey spaces is valid.

Theorem 3.4 [5]. *Let Φ be a Young function, $w \in A_{i_\Phi}$ then the operator M is bounded from $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ to $WM^{\Phi,\varphi}_w(\mathbb{R}^n)$ under the condition*

$$\sup_{r<t<\infty} \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(w(B(x,s))^{-1}\big)} \right) \Phi^{-1}\big(w(B(x,t))^{-1}\big) \le C\, \varphi(x,r), \tag{3.1}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where C does not depend on x and r. Moreover, if $\Phi \in \nabla_2$, M is bounded on $M^{\Phi,\varphi}_w(\mathbb{R}^n)$.

If we assume $\varphi \in \mathcal{G}^w_\Phi$ in Theorem 3.4, we get the following result.

Corollary 3.5. *If Φ be a Young function, $w \in A_{i_\Phi}$ and $\varphi \in \mathcal{G}^w_\Phi$, then the operator M is bounded from $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ to $WM^{\Phi,\varphi}_w(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then the operator M is bounded on $M^{\Phi,\varphi}_w(\mathbb{R}^n)$.*

4 | THE OPERATOR I_α IN THE SPACES $M^{\Phi,\varphi}_w$

The following pointwise estimate plays a key role where we prove our main results.

Lemma 4.1. *Let $0 < \alpha < n$, Φ be a Young function, $w \in A_{i_\Phi}$ and $\varphi(x,t)$ satisfies the condition*

$$t^\alpha \varphi(x,t) + \int_t^\infty r^\alpha \varphi(x,r)\,\frac{dr}{r} \le C\varphi(x,t)^\beta \tag{4.1}$$

for some $\beta \in (0,1)$ and for every $x \in \mathbb{R}^n$ and $t > 0$. Then for the operator I_α we have the following pointwise estimate

$$|I_\alpha f(x)| \lesssim (Mf(x))^\beta \|f\|^{1-\beta}_{M^{\Phi,\varphi}_w}. \tag{4.2}$$

Proof. For arbitrary ball $B = B(x,t)$ we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_B(y), \quad f_2(y) = f(y)\chi_{\complement_B}(y),$$

and have

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x).$$

For $I_\alpha f_1(x)$, following Hedberg's trick, see [29], we obtain $|I_\alpha f_1(x)| \lesssim t^\alpha Mf(x)$. For $I_\alpha f_2(x)$ by Lemma 2.9 we have

$$\begin{aligned}
\int_{\complement(B(x,t))} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy &\approx \int_{\complement(B(x,t))} |f(y)| \int_{|x-y|}^\infty \frac{dr}{r^{n+1-\alpha}}\,dy \\
&\approx \int_t^\infty \int_{t\le|x-y|<r} |f(y)|\,dy\,\frac{dr}{r^{n+1-\alpha}} \\
&\lesssim \int_t^\infty \Phi^{-1}\big(w(B(x,r))^{-1}\big) r^{\alpha-1} \|f\|_{L^\Phi_w(B(x,r))}\,dr.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
|I_\alpha f(x)| &\lesssim t^\alpha Mf(x) + \int_t^\infty \Phi^{-1}\big(w(B(x,r))^{-1}\big) r^{\alpha-1} \|f\|_{L^\Phi_w(B(x,r))}\,dr \\
&\lesssim t^\alpha Mf(x) + \|f\|_{M^{\Phi,\varphi}_w} \int_t^\infty r^\alpha \varphi(x,r)\,\frac{dr}{r}.
\end{aligned}$$

Thus, the technique in [56, p. 6492] by (4.1) we obtain

$$\begin{aligned}
|I_\alpha f(x)| &\lesssim \min\left\{\varphi(x,t)^{\beta-1} Mf(x), \varphi(x,t)^\beta \|f\|_{M^{\Phi,\varphi}_w}\right\} \\
&\lesssim \sup_{s>0} \min\left\{s^{\beta-1} Mf(x), s^\beta \|f\|_{M^{\Phi,\varphi}_w}\right\} \\
&= (Mf(x))^\beta\, \|f\|^{1-\beta}_{M^{\Phi,\varphi}_w},
\end{aligned}$$

where we have used that the supremum is achieved when the minimum parts are balanced. □

Remark 4.2. Conditions of type (4.1) go back to the work of Gunawan [27]. Eridani et al. [12] (see also [19]) expanded this technique.

The following theorem is one of our main results.

Theorem 4.3. *Let $0 < \alpha < n$, $w \in A_\infty$, Φ be a Young function, $\beta \in (0,1)$ and $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.*

1. *If $\Phi \in \nabla_2$, $w \in A_{i_\Phi}$ and $\varphi(x,t)$ satisfies (3.1), then the condition (4.1) is sufficient for the boundedness of the operator I_α from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.*
2. *If $\varphi \in \mathcal{G}_\Phi^w$, then the condition*

$$t^\alpha \varphi(x,t) \le C\varphi(x,t)^\beta \tag{4.3}$$

for all $x \in \mathbb{R}^n$ and $t > 0$, where $C > 0$ does not depend on x and t, is necessary for the boundedness of the operator I_α from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.

3. *Let $\Phi \in \nabla_2$ and $w \in A_{i_\Phi}$. If $\varphi \in \mathcal{G}_\Phi^w$ satisfies the condition*

$$\int_t^\infty r^\alpha \varphi(x,r)\frac{dr}{r} \le Ct^\alpha \varphi(x,t) \tag{4.4}$$

for all $x \in \mathbb{R}^n$ and $t > 0$, where $C > 0$ does not depend on x and t, then the condition (4.3) is necessary and sufficient for the boundedness of the operator I_α from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.

Proof. By using the pointwise estimate (4.2) we have have for an arbitrary ball B

$$\|I_\alpha f\|_{L_w^\Psi(B)} \lesssim \left\|(Mf)^\beta\right\|_{L_w^\Psi(B)} \|f\|_{M_w^{\Phi,\varphi}}^{1-\beta}.$$

Note that from (2.3) we get

$$\int_B \Psi\left(\frac{(Mf(x))^\beta}{\|Mf\|_{L_w^\Phi(B)}^\beta}\right) w(x)\,dx = \int_B \Phi\left(\frac{Mf(x)}{\|Mf\|_{L_w^\Phi(B)}}\right) w(x)\,dx \le 1.$$

Thus $\|(Mf)^\beta\|_{L_w^\Psi(B)} \le \|Mf\|_{L_w^\Phi(B)}^\beta$. Consequently by using this inequality we have

$$\|I_\alpha f\|_{L_w^\Psi(B)} \lesssim \|Mf\|_{L_w^\Phi(B)}^\beta \|f\|_{M_w^{\Phi,\varphi}}^{1-\beta}. \tag{4.5}$$

From Theorem 3.4 and (4.5), we get

$$\begin{aligned}
\|I_\alpha f\|_{M_w^{\Psi,\eta}} &= \sup_B \eta(B)^{-1}\Psi^{-1}\big(w(B)^{-1}\big)\|I_\alpha f\|_{L_w^\Psi(B)} \\
&\lesssim \|f\|_{M_w^{\Phi,\varphi}}^{1-\beta} \sup_B \eta(B)^{-1}\Psi^{-1}\big(w(B)^{-1}\big)\|Mf\|_{L_w^\Phi(B)}^\beta \\
&= \|f\|_{M_w^{\Phi,\varphi}}^{1-\beta}\left(\sup_B \varphi(B)^{-1}\Phi^{-1}\big(w(B)^{-1}\big)\|Mf\|_{L_w^\Phi(B)}\right)^\beta \\
&\lesssim \|f\|_{M_w^{\Phi,\varphi}}.
\end{aligned}$$

We shall now prove the second part. Let $B_0 = B(x_0, t_0)$ and $x \in B_0$. By Lemma 2.10 we have $t_0^\alpha \lesssim I_\alpha \chi_{B_0}(x)$. Therefore, by (2.5) and Lemma 3.3 we have

$$t_0^\alpha \lesssim \Psi^{-1}\big(w(B_0)^{-1}\big)\|I_\alpha \chi_{B_0}\|_{L_w^\Psi(B_0)} \le \eta(B_0)\|I_\alpha \chi_{B_0}\|_{M_w^{\Psi,\eta}}$$

$$\lesssim \eta(B_0)\|\chi_{B_0}\|_{M_w^{\Phi,\varphi}} \le \frac{\eta(B_0)}{\varphi(B_0)} = \varphi(B_0)^{\beta-1}.$$

Since this is true for every $B_0 > 0$, we are done.

The third statement of the theorem follows from the first and second parts of the theorem. □

If we take $\Phi(t) = t^p$, $p \in [1, \infty)$ and $\beta = \frac{p}{q}$ with $p < q < \infty$ at Theorem 4.3 we get the following new result for generalized Morrey spaces.

Corollary 4.4. *Let $0 < \alpha < n$, $w \in A_\infty$, $1 \le p < q < \infty$.*

1. *If $1 < p < q < \infty$, $w \in A_p$ and $\varphi(x,t)$ satisfies*

$$\sup_{t<r<\infty} \Big(\operatorname*{ess\,inf}_{r<s<\infty} \varphi(x,s) w(B(x,s))^{1/p} \Big)\, w(B(x,r))^{-1/p} \le C\, \varphi(x,t), \tag{4.6}$$

then the condition

$$t^\alpha \varphi(x,t) + \int_t^\infty r^\alpha\, \varphi(x,r)\, \frac{dr}{r} \le C \varphi(x,t)^{\frac{p}{q}} \tag{4.7}$$

for all $x \in \mathbb{R}^n$ and $t > 0$, where $C > 0$ does not depend on x and t, is sufficient for the boundedness of the operator I_α from $M_w^{p,\varphi}(\mathbb{R}^n)$ to $M_w^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_p^w$, then the condition*

$$t^\alpha \varphi(x,t) \le C \varphi(x,t)^{\frac{p}{q}} \tag{4.8}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of the operator I_α from $M_w^{p,\varphi}(\mathbb{R}^n)$ to $M_w^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

3. *If $1 < p < q < \infty$, $w \in A_p$ and $\varphi \in \mathcal{G}_p^w$ satisfies the condition (4.4), then the condition (4.8) is necessary and sufficient for the boundedness of the operator I_α from $M_w^{p,\varphi}(\mathbb{R}^n)$ to $M_w^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.*

Remark 4.5. If we take $\varphi(x,t) = t^{\frac{\lambda-n}{p}}$ at Corollary 4.4, then condition (4.4) is equivalent to $0 < \lambda < n - \alpha p$ and condition (4.8) is equivalent to $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Therefore, we get the following result for weighted Morrey spaces.

Corollary 4.6. *Let $0 < \alpha < n$, $1 < p < q < \infty$, $w \in A_p$ and $0 < \lambda < n - \alpha p$. Then I_α is bounded from $M_w^{p,\lambda}(\mathbb{R}^n)$ to $M_w^{q,\lambda}(\mathbb{R}^n)$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.*

In the case $w \equiv 1$ we have the following classical result of Adams.

Corollary 4.7 [1]. *Let $0 < \alpha < n$, $1 < p < q < \infty$ and $0 < \lambda < n - \alpha p$. Then I_α is bounded from $M^{p,\lambda}(\mathbb{R}^n)$ to $M^{q,\lambda}(\mathbb{R}^n)$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.*

We also have the following weak type result:

Theorem 4.8. *Let $0 < \alpha < n$, $w \in A_\infty$, Φ be a Young function, $\beta \in (0,1)$ and $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi\big(t^{1/\beta}\big)$.*

1. *If $w \in A_{i_\Phi}$ and $\varphi(x,t)$ satisfies (3.1), then the condition (4.1) is sufficient for the boundedness of the operator I_α from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $WM_w^{\Psi,\eta}(\mathbb{R}^n)$.*
2. *If $\varphi \in \mathcal{G}_\Phi^w$, then the condition (4.3) is necessary for the boundedness of the operator I_α from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $WM_w^{\Psi,\eta}(\mathbb{R}^n)$.*
3. *Let $w \in A_{i_\Phi}$. If $\varphi \in \mathcal{G}_\Phi^w$ satisfies the condition (4.4) then the condition (4.3) is necessary and sufficient for the boundedness of the operator I_α from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $WM_w^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. The proof is similar to the proof of Theorem 4.3. We omit the details. □

5 | THE COMMUTATOR $[b, I_\alpha]$ IN $M_w^{\Phi,\varphi}$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 5.1. Suppose that $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|\, dy,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)\, dy.$$

Define

$$BMO(\mathbb{R}^n) = \big\{ b \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \ :\ \|b\|_* < \infty \big\}.$$

To prove main results of this section, we need the following lemmas.

Lemma 5.2 [35]. *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$\big|b_{B(x,r)} - b_{B(x,t)}\big| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{5.1}$$

where C is independent of b, x, r and t.

Lemma 5.3 [30]. *If $w \in A_\infty$, $b \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$, then*

$$\sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(w(B(x,r))^{-1}\big) \big\|b - b_{B(x,r)}\big\|_{L_w^\Phi(B(x,r))} \lesssim \|b\|_*. \tag{5.2}$$

Lemma 5.4 [30]. *Let $0 < p < \infty$, $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Then for any ball B, we have*

$$\left(\frac{1}{w(B)} \int_B |b(y) - b_B|^p w(y)\, dy \right)^{\frac{1}{p}} \lesssim \|b\|_*.$$

The commutators generated by a suitable function b and the operator I_α is formally defined by

$$[b, I_\alpha]f = I_\alpha(bf) - bI_\alpha(f).$$

Given a measurable function b the operators $|b, I_\alpha|$ and M_b are defined by

$$|b, I_\alpha| f(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y)\, dy,$$

and

$$M_b(f)(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x) - b(y)||f(y)|\, dy$$

respectively. We refer to [2, 15] for details on these operators.

Theorem 5.5 [5]. *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$, φ and Φ satisfy the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big)\Bigg(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(w(B(x,s))^{-1}\big)}\Bigg)\Phi^{-1}\big(w(B(x,t))^{-1}\big) \le C\,\varphi(x,r) \tag{5.3}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where C does not depend on x and r. Then the operator M_b is bounded from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi}(\mathbb{R}^n)$.

Remark 5.6. Theorem 5.5 was considered under the condition $w \in A_1$ in [5]. One can easily extend this result to the case $w \in A_{p_0}$ by using the technique given in Theorem 5.9.

The following lemma is the analogue of the Hedberg's trick for $[b, I_\alpha]$.

Lemma 5.7. *If $0 < \alpha < n$ and $f, b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ and $r > 0$ we get*

$$\int_{B(x,r)} \frac{|f(y)|}{|x - y|^{n-\alpha}} |b(x) - b(y)|\, dy \lesssim r^\alpha M_b f(x). \tag{5.4}$$

Proof.

$$\begin{aligned}\int_{B(x,r)} \frac{|f(y)|}{|x - y|^{n-\alpha}} |b(x) - b(y)|\, dy &= \sum_{j=0}^\infty \int_{2^{-j-1}r \le |x-y| < 2^{-j}r} \frac{|f(y)|}{|x - y|^{n-\alpha}} |b(x) - b(y)|\, dy \\ &\lesssim \sum_{j=0}^\infty (2^{-j}r)^\alpha (2^{-j}r)^{-n} \int_{|x-y|<2^{-j}r} |f(y)||b(x) - b(y)|\, dy \\ &\lesssim r^\alpha M_b f(x).\end{aligned}$$

□

Lemma 5.8. *If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $B_0 := B(x_0, r_0)$, then*

$$r_0^\alpha\, |b(x) - b_{B_0}| \lesssim |b, I_\alpha| \chi_{B_0}(x)$$

for every $x \in B_0$.

Proof. If $x, y \in B_0$, then $|x - y| \le |x - x_0| + |y - x_0| < 2r_0$. Since $0 < \alpha < n$, we get $(2r_0)^{\alpha-n} \le |x - y|^{\alpha-n}$. Therefore

$$\begin{aligned}|b, I_\alpha| \chi_{B_0}(x) = \int_{B_0} |b(x) - b(y)||x - y|^{\alpha-n}\, dy &\ge (2r_0)^{\alpha-n} \int_{B_0} |b(x) - b(y)|\, dy \\ &\ge (2r_0)^{\alpha-n} \left| \int_{B_0} (b(x) - b(y))\, dy \right| \approx r_0^\alpha\, |b(x) - b_{B_0}|.\end{aligned}$$

□

Theorem 5.9. *Let $b \in BMO(\mathbb{R}^n)$, $\beta \in (0,1)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$. Let $\varphi(x,r)$ satisfy the conditions (5.3) and*

$$r^\alpha \varphi(x,r) + \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi(x,t)t^\alpha \frac{dt}{t} \le C\varphi(x,r)^\beta \tag{5.5}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where C does not depend on x and r. Define $\eta(x,r) \equiv \varphi(x,r)^\beta$ and $\Psi(r) \equiv \Phi\big(r^{1/\beta}\big)$. Then, the operator $[b, I_\alpha]$ is bounded from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

For $x \in B$ we have

$$\begin{aligned}
\big|[b, I_\alpha]f_2(x)\big| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x-y|^{n-\alpha}}|f_2(y)|\,dy \approx \int_{\complement_{(2B)}} \frac{|b(y) - b(x)|}{|x_0-y|^{n-\alpha}}|f(y)|\,dy \\
&\lesssim \int_{\complement_{(2B)}} \frac{|b(y) - b_B|}{|x_0-y|^{n-\alpha}}|f(y)|\,dy + \int_{\complement_{(2B)}} \frac{|b(x) - b_B|}{|x_0-y|^{n-\alpha}}|f(y)|\,dy \\
&= J_1 + J_2(x),
\end{aligned}$$

since $x \in B$ and $y \in {}^{\complement}(2B)$ implies $|x-y| \approx |x_0 - y|$.

By an argument similar to that used in the estimate (2.25) in [37], we have

$$\big\||b(\cdot) - b_B|w^{-1}(\cdot)\big\|_{L_w^{\widetilde{\Phi}}(B)} \lesssim \Phi^{-1}\big(w(B)^{-1}\big)|B|. \tag{5.6}$$

For the sake of completeness, we give the proof of the estimate (5.6). Taking into account (2.1) and Remark 2.3, it follows that

$$\begin{aligned}
\int_B \widetilde{\Phi}\left(\frac{|b(x) - b_B|w^{-1}(x)}{\Phi^{-1}\big(w(B)^{-1}\big)|B|}\right)w(x)\,dx &\lesssim \int_B \widetilde{\Phi}\left(\frac{|b(x) - b_B|\widetilde{\Phi}^{-1}\big(w(B)^{-1}\big)w(B)}{w(x)|B|}\right)w(x)\,dx \\
&\lesssim \frac{1}{w(B)}\int_B \left\{\sum_{i=0}^{1}\left[\frac{|b(x) - b_B|}{w(x)}\right]^{p_i'}\left[\frac{w(B)}{|B|}\right]^{p_i'}\right\}w(x)\,dx.
\end{aligned}$$

Since $w \in A_{p_0} \subset A_{p_1}$, we know that $w^{1-p_i'} \in A_{p_i'}$ for $i \in \{0,1\}$ (see, for example, [10, p. 136]). By this and Lemma 5.4, we conclude that, for $i \in \{0,1\}$,

$$\begin{aligned}
&\frac{1}{w(B)}\int_B |b(x) - b_B|^{p_i'}\left[\frac{w(B)}{|B|}\right]^{p_i'} \frac{1}{w^{p_i'}(x)}w(x)\,dx \\
&\approx \left[\frac{1}{|B|}\int_B w(x)\,dx\right]^{p_i'-1}\left[\frac{1}{|B|}\int_B w^{1-p_i'}(x)\,dx\right]\left\{\frac{1}{w^{1-p_i'}(B)}\int_B |b(x) - b_B|^{p_i'}w^{1-p_i'}(x)\,dx\right\} \lesssim 1,
\end{aligned}$$

which yields to (5.6).

Let us estimate J_1 now.

$$\begin{aligned}
J_1 &= \int_{\complement_{(2B)}} \frac{|b(y) - b_B|}{|x_0-y|^{n-\alpha}}|f(y)|\,dy \\
&\approx \int_{\complement_{(2B)}} |b(y) - b_B||f(y)| \int_{|x_0-y|}^\infty \frac{dt}{t^{n+1-\alpha}}\,dy
\end{aligned}$$

$$
\begin{aligned}
&\approx \int_{2r}^{\infty}\int_{2r\le|x_0-y|\le t}|b(y)-b_B||f(y)|\,dy\,\frac{dt}{t^{n+1-\alpha}}\\
&\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}|b(y)-b_B||f(y)|\,dy\,\frac{dt}{t^{n+1-\alpha}}\\
&\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}\left|b(y)-b_{B(x_0,t)}\right||f(y)|\,dy\frac{dt}{t^{n+1-\alpha}}\\
&\qquad+\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\int_{B(x_0,t)}|f(y)|\,dy\,\frac{dt}{t^{n+1-\alpha}}\\
&\lesssim \int_{2r}^{\infty}\left\||b(\cdot)-b_{B(x_0,t)}|w^{-1}(\cdot)\right\|_{L_w^{\widetilde{\Phi}}(B(x_0,t))}\|f\|_{L_w^{\Phi}(B(x_0,t))}\frac{dt}{t^{n+1-\alpha}}\quad\text{(by (2.4))}\\
&\quad+\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\|f\|_{L_w^{\Phi}(B(x_0,t))}\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\,\frac{dt}{t^{1-\alpha}}\quad\text{(by Lemma 2.9)}\\
&\lesssim \|b\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_w^{\Phi}(B(x_0,t))}\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\,\frac{dt}{t^{1-\alpha}}\quad\text{(by (5.1) and (5.6))}\\
&\lesssim \|b\|_*\,\|f\|_{M_w^{\Phi,\varphi}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\varphi(x_0,t)\,\frac{dt}{t^{1-\alpha}}.
\end{aligned}
$$

Also using (5.4), we get

$$
J_0(x)+J_1\lesssim\|b\|_* r^{\alpha}M_b f(x)+\|b\|_*\|f\|_{M_w^{\Phi,\varphi}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\varphi(x_0,t)\,\frac{dt}{t^{1-\alpha}},
$$

where $J_0(x):=|[b,I_\alpha]f_1(x)|$.

Thus, by (5.5) we obtain

$$
\begin{aligned}
J_0(x)+J_1&\lesssim\|b\|_*\min\left\{\varphi(x_0,r)^{\beta-1}M_b f(x),\varphi(x_0,r)^{\beta}\|f\|_{M_w^{\Phi,\varphi}}\right\}\\
&\lesssim\|b\|_*\sup_{s>0}\min\left\{s^{\beta-1}M_b f(x),s^{\beta}\|f\|_{M_w^{\Phi,\varphi}}\right\}\\
&=\|b\|_*(M_b f(x))^{\beta}\,\|f\|_{M_w^{\Phi,\varphi}}^{1-\beta}.
\end{aligned}
$$

Consequently for every $x\in B$ we have

$$
J_0(x)+J_1\lesssim\|b\|_*(M_b f(x))^{\beta}\,\|f\|_{M_w^{\Phi,\varphi}}^{1-\beta}. \tag{5.7}
$$

By using the inequality (5.7) we have

$$
\|J_0(\cdot)+J_1\|_{L_w^{\Psi}(B)}\lesssim\|b\|_*\left\|(M_b f)^{\beta}\right\|_{L_w^{\Psi}(B)}\|f\|_{M_w^{\Phi,\varphi}}^{1-\beta}.
$$

Note that from (2.3) we get

$$
\int_B\Psi\left(\frac{(M_b f(x))^{\beta}}{\|M_b f\|_{L_w^{\Phi}(B)}^{\beta}}\right)w(x)\,dx=\int_B\Phi\left(\frac{M_b f(x)}{\|M_b f\|_{L_w^{\Phi}(B)}}\right)w(x)\,dx\le 1.
$$

Thus $\|(M_b f)^\beta\|_{L^\Psi_w(B)} \le \|M_b f\|^\beta_{L^\Phi_w(B)}$. Therefore, we have

$$\|J_0(\cdot)+J_1\|_{L^\Psi_w(B)} \lesssim \|b\|_* \|M_b f\|^\beta_{L^\Phi_w(B)} \, \|f\|^{1-\beta}_{M^{\Phi,\varphi}_w}.$$

We will now focus on the estimation of J_2.

$$\begin{aligned}
\|J_2\|_{L^\Psi_w(B)} &= \left\| \int_{\complement(2B)} \frac{|b(\cdot)-b_B|}{|x_0-y|^{n-\alpha}} |f(y)|\,dy \right\|_{L^\Psi_w(B)} \\
&\approx \|b(\cdot)-b_B\|_{L^\Psi_w(B)} \int_{\complement(2B)} \frac{|f(y)|}{|x_0-y|^{n-\alpha}}\,dy \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(w(B)^{-1}\big)} \int_{\complement(2B)} \frac{|f(y)|}{|x_0-y|^{n-\alpha}}\,dy \quad \text{(by (5.2))} \\
&\approx \frac{\|b\|_*}{\Psi^{-1}\big(w(B)^{-1}\big)} \int_{\complement(2B)} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}}\,dy \\
&\approx \frac{\|b\|_*}{\Psi^{-1}\big(w(B)^{-1}\big)} \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |f(y)|\,dy \,\frac{dt}{t^{n+1-\alpha}} \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(w(B)^{-1}\big)} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\,dy \,\frac{dt}{t^{n+1-\alpha}} \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(w(B)^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi_w(B(x_0,t))} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) t^{\alpha-1}\,dt \quad \text{(by Lemma 2.9)} \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(w(B)^{-1}\big)} \|f\|_{M^{\Phi,\varphi}_w} \int_{2r}^{\infty} \varphi(x_0,t) t^{\alpha-1}\,dt \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(w(B)^{-1}\big)} \|f\|_{M^{\Phi,\varphi}_w} \varphi(x_0,r)^\beta. \quad \text{(by (5.5))}
\end{aligned}$$

Consequently by using Theorem 5.5, we get

$$\begin{aligned}
\|[b,I_\alpha]f\|_{M^{\Psi,\eta}} &= \sup_{x_0\in\mathbb{R}^n, r>0} \eta(x_0,r)^{-1} \Psi^{-1}\big(w(B)^{-1}\big) \big\|[b,I_\alpha]f\big\|_{L^\Psi_w(B)} \\
&\lesssim \|b\|_* \|f\|^{1-\beta}_{M^{\Phi,\varphi}_w} \left(\sup_{x_0\in\mathbb{R}^n, r>0} \varphi(x_0,r)^{-1} \Phi^{-1}\big(w(B)^{-1}\big) \|M_b f\|_{L^\Phi_w(B)} \right)^\beta + \|b\|_* \|f\|_{M^{\Phi,\varphi}_w} \\
&\lesssim \|b\|_* \|f\|_{M^{\Phi,\varphi}_w}.
\end{aligned}$$

□

The following theorem is one of our main results.

Theorem 5.10. *Let $0<\alpha<n$, Φ be a Young function, $w \in A_\infty$, $b \in BMO(\mathbb{R}^n) \setminus \{const\}$, $\beta \in (0,1)$ and $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.*

1. *If Φ is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$ and $\varphi(t)$ satisfies (5.3), then the condition (5.5) is sufficient for the boundedness of the operator $|b, I_\alpha|$ from $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ to $M^{\Psi,\eta}_w(\mathbb{R}^n)$.*
2. *If $\Phi \in \Delta_2$ and $\varphi \in \mathcal{G}^w_\Phi$, then the condition (4.3) is necessary for the boundedness of the operator $|b, I_\alpha|$ from $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ to $M^{\Psi,\eta}_w(\mathbb{R}^n)$.*

3. *Let Φ is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$. If $\varphi \in \mathcal{G}_\Phi^w$ satisfies the conditions*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big)\varphi(x,t) \le C\,\varphi(x,r), \tag{5.8}$$

and

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi(x,t)t^\alpha \frac{dt}{t} \le Cr^\alpha \varphi(x,r) \tag{5.9}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $C > 0$ does not depend on x and r, then the condition (4.3) is necessary and sufficient for the boundedness of the operator $|b, I_\alpha|$ from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.

Proof. From the proof of Theorem 5.9, we know that the boundedness result is still true if one has $|b, I_\alpha|$ instead of $[b, I_\alpha]$, see, for example, [9, Remark 3]. Hence, the first part of the theorem is a corollary of Theorem 5.9.

We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 5.8 we have $r_0^\alpha |b(x) - b_{B_0}| \lesssim |b, I_\alpha|\chi_{B_0}(x)$. Therefore, by (5.2) and Lemma 3.3

$$\begin{aligned} r_0^\alpha &\le \frac{\| |b, I_\alpha|\chi_{B_0}\|_{L_w^\Psi(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L_w^\Psi(B_0)}} \lesssim \frac{1}{\|b\|_*} \| |b, I_\alpha|\chi_{B_0}\|_{L_w^\Psi(B_0)} \Psi^{-1}\big(w(B_0)^{-1}\big) \\ &\le \frac{1}{\|b\|_*}\eta(B_0)\| |b, I_\alpha|\chi_{B_0}\|_{M_w^{\Psi,\eta}} \lesssim \eta(B_0)\|\chi_{B_0}\|_{M_w^{\Phi,\varphi}} \le \frac{\eta(B_0)}{\varphi(B_0)} \le \varphi(B_0)^{\beta-1}. \end{aligned}$$

Since this is true for every B_0, we are done.

The third statement of the theorem follows from the first and second parts of the theorem. □

If we take $\Phi(t) = t^p$, $p \in [1, \infty)$ and $\beta = \frac{p}{q}$ with $p < q < \infty$ at Theorem 5.10 we get the following new result for generalized Morrey spaces.

Corollary 5.11. *Let $0 < \alpha < n$, $w \in A_\infty$, $1 < p < q < \infty$ and $b \in BMO(\mathbb{R}^n) \setminus \{const\}$.*

1. *If $w \in A_p$ and $\varphi(x,r)$ satisfies*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \le C\varphi(x,r),$$

then the condition

$$r^\alpha \varphi(x,r) + \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi(x,r)t^\alpha \frac{dt}{t} \le C\varphi(x,r)^{\frac{p}{q}}$$

for all $x \in \mathbb{R}^n$ and $r > 0$ and $C > 0$ does not depend on x and r, is sufficient for the boundedness of the operator $|b, I_\alpha|$ from $M_w^{p,\varphi}(\mathbb{R}^n)$ to $M_w^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_p^w$, then the condition*

$$r^\alpha \varphi(x,r) \le C\varphi(x,r)^{\frac{p}{q}} \tag{5.10}$$

for all $x \in \mathbb{R}^n$ and $r > 0$ and $C > 0$ does not depend on x and r, is necessary for the boundedness of the operator $|b, I_\alpha|$ from $M_w^{p,\varphi}(\mathbb{R}^n)$ to $M_w^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

3. *Let $w \in A_p$. If $\varphi \in \mathcal{G}_p^w$ satisfies the conditions (5.8) and (5.9), then the condition (5.10) is necessary and sufficient for the boundedness of the operator $|b, I_\alpha|$ from $M_w^{p,\varphi}(\mathbb{R}^n)$ to $M_w^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.*

The following theorem characterizes the BMO space via the boundedness of the operator $[b, I_\alpha]$.

Theorem 5.12. *Let $0 < \alpha < n$, Φ be a Young function, $w \in A_\infty$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\beta \in (0,1)$ and $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi\big(t^{1/\beta}\big)$.*

1. *If Φ is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$ and $\varphi(x,t)$ satisfies (5.3) and*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\varphi(x,t)t^\alpha \frac{dt}{t} \le C\varphi(x,r)^\beta, \tag{5.11}$$

$$t^\alpha \varphi(x,t) \le C\varphi(x,t)^\beta \tag{5.12}$$

hold for all $x \in \mathbb{R}^n$ and $t > 0$, where $C > 0$ does not depend on x and t, then the condition $b \in BMO(\mathbb{R}^n)$ is sufficient for the boundedness of the operator $[b, I_\alpha]$ from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_\Phi^w$ and the condition*

$$\varphi(x,t)^\beta \le Ct^\alpha \varphi(x,t) \tag{5.13}$$

hold for all $x \in \mathbb{R}^n$ and $t > 0$, where $C > 0$ does not depend on x and t, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary for the boundedness of the operator $[b, I_\alpha]$ from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.

3. *If $\Phi \in \nabla_2$, $\varphi \in \mathcal{G}_\Phi^w$, condition (5.11) holds and $\varphi(x,t)^\beta \approx t^\alpha \varphi(x,t)$, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of the operator $[b, I_\alpha]$ from $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $M_w^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof.

(1) The first statement of the theorem follows from Theorem 5.9.

(2) We shall now prove the second part. We use the idea given in [35] (see also [8, 48 51]). Choose $z_0 \in \mathbb{R}^n$ and $\delta > 0$ such that in the neighborhood $\{z : |z - z_0| < \sqrt{n}\delta\}$, the function $|z|^{n-\alpha}$ can be represented as a Fourier series which converges absolutely. That is

$$|z|^{n-\alpha} = \sum_{n=0}^\infty a_n e^{i v_n \cdot z}.$$

Let $z_1 = \frac{z_0}{\delta}$. For any ball $B = B(x_0, r)$, let $y_0 = x_0 - 2rz_1$ and $B' = B(y_0, r)$. Then for $x \in B$ and $y \in B'$, we have that

$$\left|\frac{x-y}{2r} - z_1\right| \le \left|\frac{x - x_0}{2r}\right| + \left|\frac{y - y_0}{2r}\right| \le 1.$$

Now set $s(x) = sgn(b(x) - b_{B'})$, then

$$\begin{aligned}\int_B |b(x) - b_{B'}|\,dx &= \int_B (b(x) - b_{B'})s(x)\,dx = \frac{1}{|B'|}\int_B\int_{B'}(b(x) - b(y))s(x)\,dy\,dx \\ &\approx 2^{n-\alpha}\delta^{\alpha-n}r^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}\left|\frac{\delta(x-y)}{2r}\right|^{n-\alpha} s(x)\chi_B(x)\chi_{B'}(y)\,dy\,dx \\ &\approx r^{-\alpha}\sum_{n=0}^\infty a_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}} e^{i v_n\cdot\frac{\delta}{2r}(x-y)} s(x)\chi_B(x)\chi_{B'}(y)\,dy\,dx.\end{aligned}$$

Taking

$$g_n(y) = e^{-i(\delta/2r)v_n\cdot y}\chi_{B'}(y) \quad \text{and} \quad h_n(x) = e^{i(\delta/2r)v_n\cdot x}s(x)\chi_B(x),$$

we obtain

$$\begin{aligned}\int_B |b(x) - b_{B'}|\,dx &\approx r^{-\alpha}\sum_{n=0}^{\infty} a_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} g_n(y)h_n(x)\,dy\,dx \\ &\le Cr^{-\alpha}\sum_{n=0}^{\infty}|a_n|\int_{\mathbb{R}^n}\big|[b, I_\alpha]g_n(x)\big||h_n(x)|\,dx \\ &= Cr^{-\alpha}\sum_{n=0}^{\infty}|a_n|\int_B \big|[b, I_\alpha]g_n(x)\big|\,dx.\end{aligned}$$

Applying Lemmas 2.9 and 3.3, we have

$$\begin{aligned}\int_B |[b, I_\alpha]g_n(x)|\,dx &\le 2|B|\Psi^{-1}\big(w(B)^{-1}\big)\big\|[b, I_\alpha]g_n\big\|_{L^\Psi_w(B)} \\ &\lesssim |B|\eta(B)\big\|[b, I_\alpha]g_n\big\|_{M^{\Psi,\eta}_w(\mathbb{R}^n)} \lesssim r^n\eta(B)\|g_n\|_{M^{\Phi,\varphi}_w(\mathbb{R}^n)} \lesssim r^n\eta(B)\varphi(B')^{-1} \\ &\lesssim r^n\varphi(B)^{\beta-1} \lesssim r^{n+\alpha}.\end{aligned}$$

Thus we have obtained

$$\frac{1}{|B|}\int_B |b(x) - b_B|\,dx \le \frac{2}{|B|}\int_B |b(x) - b_{B'}|\,dx \lesssim 1,$$

which completes the proof of the theorem.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

ACKNOWLEDGEMENTS

The authors thank the referees for careful reading of the paper and useful comments. The research of V. S. Guliyev was partially supported by the grant of 1st Azerbaijan–Russia Joint Grant Competition (Agreement number no. EIF-BGM-4-RFTF-1/2017-21/01/1-M-08) and by the RUDN University Strategic Academic Leadership Program.

REFERENCES

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), 765–778.
[2] M. Bramanti, *Commutators of integral operators with positive kernels*, Matematiche (Catania) **49** (1994), no. 1, 149–168.
[3] F. Deringoz, *Parametric Marcinkiewicz integral operator and its higher order commutators on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **37** (2017), no. 4, 24–32.
[4] F. Deringoz, V. S. Guliyev, and S. Hasanov, *Characterizations for the Riesz potential and its commutators on generalized Orlicz–Morrey spaces*, J. Inequal. Appl. **2016**, Paper No. 248, 22 pp.
[5] F. Deringoz, V. S. Guliyev, and S. Hasanov, *Maximal operator and its commutators on generalized weighted Orlicz–Morrey spaces*, Tokyo J. Math. **41** (2018), no. 2, 347–369.
[6] F. Deringoz, V. S. Guliyev, E. Nakai, Y. Sawano, and M. Shi, *Generalized fractional maximal and integral operators on Orlicz and generalized Orlicz–Morrey spaces of the third kind*, Positivity **23** (2019), no. 3, 727–757.
[7] F. Deringoz, V. S. Guliyev, and S. Samko, *Boundedness of maximal and singular operators on generalized Orlicz–Morrey spaces*, Oper. Theory Adv. Appl. **242** (2014), 139–158.
[8] Y. Ding, *A characterization of BMO via commutators for some operators*, Northeast. Math. J. **13** (1997), no. 4, 422–432.
[9] Y. Ding and S. Lu, *Higher order commutators for a class of rough operators*, Ark. Mat. **37** (1999), 33–44.
[10] J. Duoandikoetxea, *Fourier analysis*, Grad. Stud. Math., vol. 29, Amer. Math. Soc., Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.
[11] J. Duoandikoetxea and M. Rosenthal, *Boundedness of operators on certain weighted Morrey spaces beyond Muckenhoupt range*, Potential Anal., available at https://doi.org/10.1007/s11118-019-09805-8.

[12] Eridani, H. Gunawan, E. Nakai, and Y. Sawano, *Characterizations for the generalized fractional integral operators on Morrey spaces*, Math. Inequal. Appl. **17** (2014), no. 2, 761–777.

[13] Eridani, M. I. Utoyo, and H. Gunawan, *A characterization for fractional integrals on generalized Morrey spaces*, Anal. Theory Appl. **28** (2012), no. 3, 263–268.

[14] S. Gala, Y. Sawano, and H. Tanaka, *A remark on two generalized Orlicz–Morrey spaces*, J. Approx. Theory **98** (2015), 1–9.

[15] J. Garcia-Cuerva, E. Harboure, C. Segovia, and J. L. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J. **40** (1991), no. 4, 1397–1420.

[16] A. Gogatishvili and V. Kokilashvili, *Criteria of weighted inequalities in Orlicz classes for maximal functions defined on homogeneous type spaces*, Georgian Math. J. **1** (1994), no. 6, 641–673.

[17] V. S. Guliyev, Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{G}$, Doctor's degree dissertation, Moscow, Mat. Inst. Steklov, 1994. (Russian).

[18] V. S. Guliyev, *Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications*, Baku, 1999. (Russian).

[19] V. S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. **2009**, Art. ID 503948, 20 pp.

[20] V. S. Guliyev, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (2012), no. 3, 33–61.

[21] V. S. Guliyev, *Characterizations for the fractional maximal operator and its commutators in generalized weighted Morrey spaces on Carnot groups*, Anal. Math. Phys. **10** (2020), no. 2, 1–20.

[22] V. S. Guliyev and F. Deringoz, *On the Riesz potential and its commutators on generalized Orlicz–Morrey spaces*, J. Funct. Spaces **2014**, Art. ID 617414, 11 pp.

[23] V. S. Guliyev, S. G. Hasanov, Y. Sawano, and T. Noi, *Non-smooth atomic decompositions for generalized Orlicz–Morrey spaces of the third kind*, Acta Appl. Math. **145** (2016), 133–174.

[24] V. S. Guliyev, T. Karaman, R. Ch. Mustafayev, and A. Serbetci, *Commutators of sublinear operators generated by Calderón–Zygmund operator on generalized weighted Morrey spaces*, Czechoslovak Math. J. **64(139)** (2014), no. 2, 365–386.

[25] V. S. Guliyev and M. N. Omarova, *Multilinear singular and fractional integral operators on generalized weighted Morrey spaces*, Azerb. J. Math. **5** (2015), no. 1, 104–132.

[26] V. S. Guliyev, M. N. Omarova, and Y. Sawano, *Boundedness of intrinsic square functions and their commutators on generalized weighted Orlicz–Morrey spaces*, Banach J. Math. Anal. **9** (2015), no. 2, 44–62.

[27] H. Gunawan, *A note on the generalized fractional integral operators*, J. Indones. Math. Soc. **9** (2003), 39–43.

[28] V. H. Hamzayev, *Sublinear operators with rough kernel generated by Calderon–Zygmund operators and their commutators on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **38** (2018), no. 1, 79–94.

[29] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[30] K.-P. Ho, *Characterizations of BMO by A_p weights and p-convexity*, Hiroshima Math. J. **41** (2011), no. 2, 153–165.

[31] K.-P. Ho, *Vector-valued maximal inequalities on weighted Orlicz–Morrey spaces*, Tokyo J. Math. **36** (2013), no. 2, 499–512.

[32] K.-P. Ho, *Two-weight norm, Poincaré, Sobolev and Stein–Weiss inequalities on Morrey spaces*, Publ. Res. Inst. Math. Sci. **53** (2017), 119–139.

[33] K.-P. Ho, *Fractional integral operators with homogeneous kernels on Morrey spaces with variable exponents*, J. Math. Soc. Japan **69** (2017), 1059–1077.

[34] K.-P. Ho, *Weak type estimates of the fractional integral operators on Morrey spaces with variable exponents*, Acta Appl. Math. **159** (2019), 1–10.

[35] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16** (1978), 263–270.

[36] Y. Komori and S. Shirai, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282** (2009), no. 2, 219–231.

[37] Y. Liang, E. Nakai, D. Yang, and J. Zhang, *Boundedness of intrinsic Littlewood–Paley functions on Musielak–Orlicz Morrey and Campanato spaces*, Banach J. Math. Anal. **8** (2014), no. 1, 221–268.

[38] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, ed.), ICM 90 Satellite Proceedings, Springer-Verlag, Tokyo, 1991, 183–189.

[39] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[40] E. Nakai, *Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.

[41] E. Nakai, *Generalized fractional integrals on Orlicz–Morrey spaces*, Banach and Function Spaces (Kitakyushu, 2003), Yokohama Publishers, Yokohama, 2004, pp. 323–333.

[42] E. Nakai, *Recent topics of fractional integrals*, Sugaku Expositions **20** (2007), no. 2, 215–235

[43] E. Nakai, *Orlicz–Morrey spaces and the Hardy-Littlewood maximal function*, Studia Math. **188** (2008), no. 3, 193–221.

[44] S. Nakamura, *Generalized weighted Morrey spaces and classical operators*, Math. Nachr. **289** (2016), no. 17–18, 2235–2262.

[45] S. Nakamura and Y. Sawano, *The singular integral operator and its commutator on weighted Morrey spaces*, Collect. Math. **68** (2017), no. 2, 145–174.

[46] S. Nakamura, Y. Sawano, and H. Tanaka, *The fractional operators on weighted Morrey spaces*, J. Geom. Anal. **28** (2018), no. 2, 1502–1524.

[47] S. Nakamura, Y. Sawano, and H. Tanaka, *Weighted local Morrey spaces*, Ann. Acad. Sci. Fenn. Math. **45** (2020), 67–93.

[48] T. Nogayama, *Boundedness of commutators of fractional integral operators on mixed Morrey spaces*, Integral Transforms Spec. Funct. **30** (2019), no. 10, 790–816.

[49] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Acad. Polon. A (1932), 207–220; reprinted in: Collected Papers, PWN, Warszawa (1988), 217–230. (German).

[50] W. Orlicz, *Über Räume* (L^M), Bull. Acad. Polon. A (1936), 93–107; reprinted in: Collected Papers, PWN, Warszawa (1988), 345–359. (German).

[51] M. Paluszyński, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J. **44** (1995), no. 1, 1–17.

[52] J. Peetre, *On the theory of* $\mathcal{L}_{p,\lambda}$, J. Funct. Anal. **4** (1969), 71–87.

[53] J. Poelhuis and A. Torchinsky, *Weighted local estimates for singular integral operators*, Trans. Amer. Math. Soc. **367** (2015), no. 11, 7957–7998.

[54] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, M. Dekker, Inc., New York, 1991.

[55] Y. Sawano, *A thought on generalized Morrey spaces*, J. Indones. Math. Soc. **25** (2019), no. 3, 210–281.

[56] Y. Sawano, S. Sugano, and H. Tanaka, *Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6481–6503.

[57] Y. Sawano, S. Sugano, and H. Tanaka, *Orlicz–Morrey spaces and fractional operators*, Potential Anal. **36** (2012), no. 4, 517–556.

How to cite this article: Guliyev V S, Deringoz F. *Riesz potential and its commutators on generalized weighted Orlicz–Morrey spaces*. Mathematische Nachrichten. 2022;1–19. https://doi.org/10.1002/mana.201900559