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On application of matrix summability to Fourier series

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Bor has recently obtained a main theorem dealing with absolute weighted mean summability of Fourier series. In this paper, we generalized that theorem for $|A, \theta_n|_k$ summability method. Also, some new and known results are obtained dealing with some basic summability methods.

KEYWORDS
absolute matrix summability, infinite series, Fourier series, Hölder inequality, Minkowski inequality, summability factors

1 | INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n^α and t_n^α, we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is,[1]

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if[2,3]

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{3}$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability.Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v, \quad P_n \neq 0, \tag{5}$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see Hardy[4]).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if[5]

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{6}$$

In the special case, when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability.

1.1 | An application of absolute matrix summability to Fourier series

For any sequence (λ_n), we write that

$$\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1} \quad \text{and} \quad \Delta \lambda_n = \lambda_n - \lambda_{n+1}.$$

A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if

$$\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty.$$

Let f be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality the constant term in the Fourier series of f can be taken to be 0, so that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t), \tag{7}$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt.$$

We write

$$\phi(t) = \frac{1}{2} \left\{ f(x+t) + f(x-t) \right\}, \tag{8}$$

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \quad (\alpha > 0). \tag{9}$$

It is well known that if $\phi(t) \in \mathcal{BV}(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C, 1)$ mean of the sequence $(nC_n(x))$. [6]Given a normal matrix $A = (a_{nv})$, we associate 2 lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \dots \quad \bar{\Delta} a_{nv} = a_{nv} - a_{n-1}, v \quad a_{-1,0} = 0 \tag{10}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta} \bar{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots . \tag{11}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{12}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{13}$$

Let $A = (a_{nv})$ be a normal matrix, ie, a lower triangular matrix of nonzero diagonal entries. Then, A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \dots . \tag{14}$$

Let (θ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if[7]

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|\bar{\Delta}A_n(s)\right|^k < \infty, \tag{15}$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s). \tag{16}$$

If we take $\theta_n = \frac{P_n}{p_n}$, then $|A, \theta_n|_k$ summability, then we have $|A, p_n|_k$ summability,[8] and if we take $\theta_n = n$, then we have $|A|_k$ summability.[9] And also, if we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we have $|\bar{N}, p_n|_k$ summability. Furthermore, if we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $|A, \theta_n|_k$ summability reduces to $|C, 1|_k$ summability.[2] Finally, if we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we obtain $|R, p_n|_k$ summability.[10]

2 | THE KNOWN RESULTS

Recently, many papers have been done for absolute summability factors of infinite series and Fourier series.[11-13] Bor[11] has proved the following theorem dealing with the Fourier series.

Theorem 2.1. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{17}$$

If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, (X_n) is a positive monotonic nondecreasing sequence, the sequences (X_n), (λ_n) satisfy the following conditions and

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{18}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{19}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n(x)|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{20}$$

then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3 | THE MAIN RESULT

The aim of this paper is to generalize Theorem 2.1 for $|A, \theta_n|_k$ summability method for Fourier series in the following form.

Theorem 3.1. *Let $k \geq 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{no} = 1, \quad n = 0, 1, \dots , \tag{21}$$

$$a_{n-1,v} \geq a_{nv}, \quad for \qquad n \geq v+1, \tag{22}$$

$$\hat{a}_{n,v+1} = O(v|\bar{\Delta} a_{nv}|). \tag{23}$$

Let $(\theta_n a_{nn})$ be a nonincreasing sequence. If the conditions (17) to (19) in Theorem 2.1 and (θ_n) holds for the following condition,

$$\sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} a_{nn} \frac{|t_n(x)|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{24}$$

are satisfied, then the series $\sum C_n(x)\lambda_n$ is summable $|A, \theta_n|_k$, $k \geq 1$.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.1. *From the conditions (21) and (22) in Theorem 3.1, we have*[14]

$$\sum_{v=0}^{n-1} |\bar{\Delta} a_{nv}| = a_{nn}, \tag{25}$$

$$\hat{a}_{n,v+1} \geq 0, \tag{26}$$

$$\sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} = O(1). \tag{27}$$

Lemma 3.2. *Under the conditions (18) and (19) in Theorem 2.1, we have the following*[15]*:*

$$nX_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty, \tag{28}$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty. \tag{29}$$

Proof of Theorem. Let $(I_n(x))$ denotes the A transform of the series $\sum_{n=1}^{\infty} C_n(x)\lambda_n$. Then, by (10) and (11), we have

$$\bar{\Delta} I_n(x) = \sum_{v=1}^{n} \hat{a}_{nv} C_v(x) \lambda_v.$$

Applying Abel transformation to this sum, we have that

$$\begin{aligned}
\bar{\Delta} I_n(x) &= \sum_{v=1}^{n} \hat{a}_{nv} C_v(x) \lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta \left(\frac{\hat{a}_{nv} \lambda_v}{v} \right) \sum_{r=1}^{v} r C_r(x) + \frac{\hat{a}_{nn} \lambda_n}{n} \sum_{r=1}^{n} r C_r(x) \\
&= \sum_{v=1}^{n-1} \Delta \left(\frac{\hat{a}_{nv} \lambda_v}{v} \right) (v+1) t_v(x) + \hat{a}_{nn} \lambda_n \frac{n+1}{n} t_n(x) \\
&= \sum_{v=1}^{n-1} \bar{\Delta} a_{nv} \lambda_v t_v(x) \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta \lambda_v t_v(x) \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{t_v(x)}{v} + a_{nn} \lambda_n t_n(x) \frac{n+1}{n} \\
&= I_{n,1}(x) + I_{n,2}(x) + I_{n,3}(x) + I_{n,4}(x).
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |I_{n,r}(x)|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{30}$$

First, by applying Hölder inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,1}(x)|^k &\le \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\left|\frac{v+1}{v}\right|\left|\bar{\Delta}a_{nv}\right||\lambda_v||t_v(x)|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\sum_{v=1}^{n-1}\left|\bar{\Delta}a_{nv}\right||\lambda_v|^k|t_v(x)|^k\times\left\{\sum_{v=1}^{n-1}\left|\bar{\Delta}a_{nv}\right|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\left\{\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}||\lambda_v|^k|t_v(x)|^k\right\}\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v(x)|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\bar{\Delta}a_{nv}|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||t_v(x)|^k a_{vv}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}(\theta_r a_{rr})^{k-1}a_{rr}\frac{|t_r(x)|^k}{X_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}a_{vv}\frac{|t_v(x)|^k}{X_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\quad \text{as}\quad m\to\infty
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1, Lemma 3.1, and Lemma 3.2. Now, using Hölder inequality, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,2}(x)|^k &\le \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\left|\frac{v+1}{v}\right||\hat{a}_{n,v+1}||\Delta\lambda_v||t_v(x)|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v||t_v(x)|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\bar{\Delta}a_{nv}||t_v(x)|^k\times\left\{\sum_{v=1}^{n-1}\left|\bar{\Delta}a_{nv}\right|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\bar{\Delta}a_{nv}||t_v(x)|^k\\
&= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}(v|\Delta\lambda_v|)|t_v(x)|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\bar{\Delta}a_{nv}|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}a_{vv}\frac{1}{X_v^{k-1}}|t_v(x)|^k(v|\Delta\lambda_v|)\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}(\theta_r a_{rr})^{k-1}a_{rr}\frac{1}{X_r^{k-1}}|t_r(x)|^k+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}a_{vv}\frac{1}{X_v^{k-1}}|t_v(x)|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v|+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad \text{as}\quad m\to\infty
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1, Lemma 3.1, and Lemma 3.2. Again, as in $I_{n,1}$, we have that

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,3}(x)|^k = \sum_{n=2}^{m+1}\theta_n^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v(x)}{v}\right|^k \leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\lambda_{v+1}|\frac{|t_v(x)|}{v}\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}||\lambda_{v+1}||t_v(x)|\right\}^k = O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}||\lambda_{v+1}|^k|t_v(x)|^k \times \left\{\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}||\lambda_{v+1}|^k|t_v(x)|^k = O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^k|t_v(x)|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\bar{\Delta}a_{nv}|$$
$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}a_{vv}|t_v(x)|^k|\lambda_{v+1}|^{k-1}|\lambda_{v+1}| = O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_{v+1}||t_v(x)|^k a_{vv}$$
$$= O(1) \quad \text{as} \quad m \to \infty$$

by virtue of the hypotheses of Theorem 3.1, Lemma 3.1, and Lemma 3.2. Finally, as in $I_{n,1}$, we have that

$$\sum_{n=1}^{m}\theta_n^{k-1}|I_{n,4}(x)|^k = O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k|\lambda_n|^k|t_n(x)|^k = O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||t_n(x)|^k$$
$$= O(1)\sum_{n=1}^{m}(\theta_n a_{nn})^{k-1}a_{nn}\frac{1}{X_n^{k-1}}|\lambda_n||t_n(x)|^k = O(1) \quad \text{as} \quad m \to \infty$$

by virtue of hypotheses of Theorem 3.1, Lemma 3.1, and Lemma 3.2. This completes the proof of Theorem 3.1. □

4 | APPLICATIONS

We can apply Theorem 3.1 to weighted mean $A = (a_{nv})$ is defined as $a_{nv} = \frac{p_v}{P_n}$ when $0 \leq v \leq n$, where $P_n = p_0 + p_1 + \ldots + p_n$. We have that

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \quad \text{and} \quad \hat{a}_{n,v+1} = \frac{(p_n P_v)}{(P_n P_{n-1})}$$

The following results can be easily verified.

1. If we take $\theta_n = \frac{P_n}{p_n}$ in Theorem 3.1, then we have a result dealing with $|A, p_n|_k$ summability.[16]
2. If we take $\theta_n = n$ in Theorem 3.1, then we have a result dealing with $|A|_k$ summability.
3. If we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we have Theorem 2.1.
4. If we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3.1, then we have a new result concerning $|C, 1|_k$ summability.
5. If we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we get $|R, p_n|_k$ summability.

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