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ON π-MORPHIC MODULES

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Abstract

Let R be an arbitrary ring with identity and M be a right R-module with $S = \text{End}(M_R)$. Let $f \in S$. f is called π-morphic if $M/f^n(M) \cong r_M(f^n)$ for some positive integer n. A module M is called π-morphic if every $f \in S$ is π-morphic. It is proved that M is π-morphic and image-projective if and only if S is right π-morphic and M generates its kernel. S is unit-π-regular if and only if M is π-morphic and π-Rickart if and only if M is π-morphic and dual π-Rickart. M is π-morphic and image-injective if and only if S is left π-morphic and M cogenerates its cokernel.

Keywords: Endomorphism rings; π-morphic rings; π-morphic modules; unit π-regular rings.

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1. Introduction

Throughout this paper all rings have an identity, all modules considered are unital right modules and all ring homomorphisms are unital (unless explicitly stated otherwise).

A ring R is said to be *strongly π-regular* (*π-regular, right weakly π-regular*) if for every element $x \in R$ there exists an integer $n > 0$ such that $x^n \in x^{n+1}R$ (respectively $x^n \in x^nRx^n$, $x^n \in x^nRx^nR$). It is called *unit-π-regular* if for every $a \in R$, there exist a unit element $x \in R$ and a positive integer n such that $a^n = a^nxa^n$. In the case of n = 1 there exists a unit x such that $a = axa$ for all $a \in R$, then R is *unit regular*. Clearly, a strongly π-regular ring is a π-regular ring.

We say also that the ring R is (von Neumann) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$ for some element x in R, that is, a is regular.

A module M is said to satisfy Fitting's lemma if, for all $f \in S$, there exists an integer $n \geq 1$, depending on f, such that $M = f^nM \oplus \text{Ker}(f^n)$. Hence a module satisfies

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Fitting's lemma if and only if its endomorphism ring is strongly π-regular (see for detail [4]).

Let M be a module. It is a well-known theorem of Erlich [2] that a map $\alpha \in S$ is unit regular if and only if it is regular and $M/\alpha(M) \cong ker(\alpha)$. We say that the ring R is *left morphic* if every element a satisfies $R/Ra \cong l(a)$.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n.
We also denote $r_M(I) = \{m \in M \mid Im = 0\}$ where I is any subset of S; $r_R(N) = \{r \in R \mid Nr = 0\}$ and $l_S(N) = \{f \in S \mid fN = 0\}$ where N is any subset of M. The maps between modules are assumed to be homomorphisms unless otherwise stated in the context.

2. Morphic Modules and π-Morphic Modules

Let M be a module with $S = \text{End}(M_R)$, the ring of endomorphisms of the right R-module M and $\mathbf{1}$ be the identity endomorphism of M. Let $f \in S$. f is called *morphic* if $M/f(M) \cong r_M(f)$. The module M is called *morphic* if every $f \in S$ is morphic. Morphic modules are studied in [5]. An endomorphism $f \in S$ is called *π-morphic if* $M/f^n(M) \cong r_M(f^n)$ for some positive integer n. The module M is called *π-morphic* if every $f \in S$ is π-morphic. In the sequel S will stand for $\text{End}(M_R)$ for the right R-module M is considered.

It is clear that every morphic module is π-morphic.

2.1. Example. There exists a π-morphic module which is not morphic.

Let e_{ij} denote 3×3 matrix units. Consider the ring
$R = \{(e_{11} + e_{22} + e_{33})a + e_{12}b + e_{13}c + e_{23}d \mid a, b, c, d \in \mathbb{Z}_2\}$ and the right R-module $M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ where right R-module operation is given by

$$(x, y, z)((e_{11} + e_{22} + e_{33})a + e_{12}b + e_{13}c + e_{23}d) = (xa, xb + ya, xc + yd + za)$$

where $(x, y, z) \in M, (e_{11} + e_{22} + e_{33})a + e_{12}b + e_{13}c + e_{23}d \in R$. Let $f \in S = \text{End}(M)$. It is a routine check that there exist x, $z \in \mathbb{Z}_2$ such that
$f(1, 0, 0) = (x, 0, z)$, $f(0, 1, 0) = (0, x, 0)$, $f(0, 0, 1) = (0, 0, x)$. For any $(a, b, c) \in M$, $f(a, b, c) = (xa, ya + xb, za + xc)$.
(i) Let $x = 0$, $y = 0$, $z = 1$. Then $f_1(a, b, c) = (0, 0, a)$ implies $f_1^2 = 0$ which gives $r_M(f_1^2) = M$. Hence $M/f_1^2(M) \cong r_M(f_1^2)$.
(ii) Let $x = 1$, $y = 0$, $z = 1$. Then $f_2(a, b, c) = (a, b, a + c)$ implies $r_M(f_2) = 0$ and $f_2(M) = M$. Hence $M/f_2(M) \cong r_M(f_2)$.
(iii) Let $x = 1$, $y = 0$, $z = 0$. Then $f_3(a, b, c) = (a, b, c)$ and f_3 is the identity endomorphism of M.
(iv) Let $x = 0$, $y = 1$, $z = 0$. Then $f_4(a, b, c) = (0, a, 0)$ and $f_4^2 = 0$.
(v) Let $x = 0$, $y = 1$, $z = 1$. Then $f_5(a, b, c) = (0, a, a)$ and so $f_5^2 = 0$.
(vi) Let $x = 1$, $y = 1$, $z = 0$. Then $f_6(a, b, c) = (a, a + b, c)$. Hence f_6 is an isomorphism.
(vii) Let $x = 1$, $y = 1$, $z = 1$. Then $f_7(a, b, c) = (a, a + b, a + c)$. Hence f_7 is an isomorphism.
(viii) The last one f_8 is the zero endomorphism.
It follows that M is π-morphic. However $r_M(f_1) = (0) \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $f_1(M) = (0) \times (0) \times \mathbb{Z}_2$ shows that M is not morphic since, otherwise, $M/f_1(M) \cong r_M(f_1)$, contrary to the fact that $e_{12}1 + e_{13}1 \in R$ would annihilate $r_M(f_1)$ from the right but not $M/\big((0) \times (0) \times \mathbb{Z}_2\big) = M/f_1(M) = r_M(f_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times (0)$.

2.2. Lemma. *Let $f \in S$. If $M/f^n(M) \cong r_M(f^n)$, there exists $g \in S$ such that $f^nM = r_M(g)$ and $g(M) = r_M(f^n)$.*

Proof. Assume that $M/f^nM \cong r_M(f^n)$. Let $M \xrightarrow{\pi} M/f^nM \xrightarrow{h} r_M(f^n)$ where π is the coset map and h is the isomorphism. Set $g = h\pi$. Then $g(M) = r_M(f^n)$ and $r_M(g) = f^n(M)$. □

2.3. Proposition. *Let M be a module, and let $f \in S$ be π-morphic. Then the following conditions are equivalent:*

(1) $r_M(f) = 0$.
(2) *f is an automorphism.*

Proof. Assume that f in S is π-morphic. Then there exists a positive integer n such that $M/f^n(M) \cong r_M(f^n)$. By Lemma 2.2 there exists $g \in S$ such that $f^nM = r_M(g)$ and $g(M) = r_M(f^n)$. Assume (1) holds. Then $r_M(f) = 0$ and so $r_M(f^n) = 0$. This shows that $f^n(M) = M$. Hence $f(M) = M$ and f is an automorphism and (2) holds. (2) $\Rightarrow$ (1) always holds. □

2.4. Theorem. *Let M be a π-morphic module. Then the following holds.*
(1) *For any $f \in S$, if $r_M(f) = 0$ then f^n is an automorphism of M for some positive integer n.*
(2) *For any $f \in S$, if $f(M) = M$ then f^n is an automorphism of M for some positive integer n.*

Proof. (1) Let $f \in S$ with $r_M(f) = 0$. By hypothesis there exists a positive integer n such that $M/f^nM \cong r_M(f^n)$ and $r_M(f) = 0$ implies $r_M(f^n) = 0$. So $M = f^nM$. Hence f^n is an automorphism.
(2) Assume that $f(M) = M$. Then $f^i(M) = M$ for all $i \geq 1$. By hypothesis there exists a positive integer n such that $M/f^nM \cong r_M(f^n)$. Then $r_M(f^n) = 0$. Hence f^n is an automorphism. □

Recall that the ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for any a, $b \in R$. A module M is called *directly finite* if its endomorphism ring is directly finite, equivalently for any endomorphisms f and g of M, $fg = 1$ implies $gf = 1$ where 1 is the identity endomorphism of M.

2.5. Corollary. *Let M be a π-morphic module. Then it is directly finite.*

Proof. Let f, $g \in S$ with $fg = 1$. By Proposition 2.3, g is an automorphism. Hence $gf = 1$. □

2.6. Lemma. *Let f be a π-morphic element. If $h : M \to M$ is an automorphism, then there exists a positive integer n such that f^nh and hf^n are both morphic. In particular, every π-unit regular endomorphism is morphic.*

Proof. By Lemma 2.2, there exist $g \in S$ and a positive integer n such that $g(M) = r_M(f^n)$ and $r_M(g) = f^n(M)$. Then $(f^nh)(M) = f^n(h(M)) = f^n(M) = r_M(g) = r_M(h^{-1}g)$. Next we show $r_M(f^nh) = (h^{-1}g)(M)$. For if $m \in r_M(f^nh)$, then $(f^nh)(m) = 0$ or $h(m) \in r_M(f^n)$. Hence $m \in (h^{-1}g)(M)$ since $r_M(f^n) = g(M)$. So $r_M(f^nh) \leq (h^{-1}g)(M)$. For the converse inclusion, let $m \in (h^{-1}g)(M)$. Then $h(m) \in g(M)$. So $h(m) \in r_M(f^n)$ since $r_M(f^n) = g(M)$. Hence $(f^nh)(m) = 0$ or $m \in r_M(f^nh)$. Thus $(h^{-1}g)(M) \leq r_M(f^nh)$. It follows that $r_M(f^nh) = (h^{-1}g)(M)$, and so f^nh is morphic. Similarly hf^n is morphic. □

2.7. Examples. (1) Every strongly π-regular ring is π-morphic as a right module over itself.
(2) Every module satisfying Fitting's lemma is π-morphic.
(3) Let R be an Artinian ring. Then every finitely generated R module is π-morphic.

Proof. (1) and (2) are clear. (3) Let R be an Artinian ring and M be a finitely generated module. Then M is both Artinian and Noetherian. By Proposition 11.7 in [1], M satisfies Fitting's lemma. Therefore M is π-morphic. $\square$

2.8. Theorem. *Every direct summand of a π-morphic module is π-morphic.*

Proof. Let $M = N \oplus K$ and $S_N = \mathrm{End}_R(N)$ and $f \in S_N$. Define $M \stackrel{g}{\to} M$ by $g(m) = f(n) + k$ where $m = n + k$ and $n \in N$, $k \in K$. Clearly $g \in S$ and $g(M) = f(N) \oplus K$ and $r_M(g) = r_N(f)$. By hypothesis there exists a positive integer n such that $M/g^n(M) \cong r_M(g^n)$. It is apparent that $g^n(M) = f^n(N) \oplus K$. Hence $N/f^n(N) \cong (N \oplus K)/(f^n(N) \oplus K) = M/g^n(M) \cong r_M(g^n) = r_N(f^n)$. $\square$

2.9. Remark. One may suspect that for π-morphic modules M_1 and M_2,
$M = M_1 \oplus M_2$ is π-morphic module provided $\mathrm{Hom}(M_i, M_j) = 0$ for $1 \leq i \neq j \leq 2$. But we cannot prove it.

Example 2.10 reveals that direct sum of π-morphic modules need not depend on the condition $\mathrm{Hom}(M_i, M_j) = 0$.

2.10. Example. Consider the ring $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \right\}$ and the right R-module $M = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$, and the submodules $N = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$ and $K = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid c \in \mathbb{Z}_2 \right\}$.
Then $M = N \oplus K$. Clearly N and K are π-morphic right R-modules. Let e_{ij} denote the 3×3 matrix units in M and for $e_{23}c \in K$ define $K \stackrel{h}{\to} N$ by $h(e_{23}c) = e_{13}c \in N$. Then $0 \neq h \in \mathrm{Hom}(K, N)$. For any $f \in S$, there exist a, b, c, u, $v \in \mathbb{Z}_2$ such that f is given by $f \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax & bx + ay + cz \\ 0 & 0 & ux + vz \\ 0 & 0 & 0 \end{pmatrix}$. It is easily checked that all f's are morphic endomorphisms. $\square$

2.11. Proposition. *Let $M = K \oplus N$ be a π-morphic module and $K \stackrel{f}{\to} N$ be a homomorphism. Then K is isomorphic to a direct summand of N.*

Proof. For $k+n \in M$ where $k \in K$, $n \in N$, define $g(k+n) = f(k)+n$. Then g is a right R-module homomorphism of M and $g^2 = g$. So $M = g(M) \oplus (1-g)(M) = (f(K)+N) \oplus \{k - f(k) \mid k \in K\}$. Clearly $r_M(g) = (1-g)(M) = \{k - f(k) \mid k \in K\}$ is a direct summand of N. By hypothesis there exists a positive integer n such that $M/g^n(M) \cong r_M(g^n)$. Since $g^2 = g$, so $K \cong K \oplus (N/f(K) + N) \cong (K \oplus N)/(f(K) + N) \cong M/g(M) = r_M(g)$ is a direct summand of N. $\square$

A right R-module M is called *generalized right principally injective* (briefly *right GP-injective*) if, for any nonzero $a \in R$, there exists a positive integer n depending on a such that $a^n \neq 0$ and any right homomorphism from a^nR to M extends to one of R_R into M, equivalently, $lr(a^n) = Ra^n$ (see, [6, Lemma 5.1]). Similarly, M is *left GP-injective* S-module means that for any $f \in S$ there exists a positive integer n such that $f^n \neq 0$ and any map α from Sf^n to M extends to one of ${}_SS$ into M, equivalently, if for any $f \in S$, there exists a positive integer n with $f^n \neq 0$ such that $f^nM = r_Ml_S(f^n)$.

A module M is called *image-projective* if, whenever $gM \le fM$ where $f, g \in S$, then $g \in fS$, that is $g = fh$ for some $h \in S$.

2.12. Lemma. *Let M be a module with $S = End_R(M)$.*
(1) *If M is π-morphic, then M is left GP-injective S-module.*
(2) *If M is π-morphic and image-projective, then S is right π-morphic.*
(3) *If S is right π-morphic and M generates its kernel, then M is π-morphic.*

Proof. (1) Let $f \in S$. By hypothesis there exist a positive integer n and $g \in S$ such that $f^nM = r_M(g)$ and $r_M(f^n) = gM$. Since $l_S(f^n) = l_S(f^nM)$, $r_Ml_S(f^n) = r_Ml_S(f^nM) = r_Ml_S(r_M(g)) = r_M(g) = f^nM$.
(2) Let $f \in S$. By hypothesis there exist $g \in S$ and a positive integer n such that $f^n(M) = r_M(g)$ and $r_M(f^n) = g(M)$. Then $gf^n = 0$. Hence $f^n \in r_S(g)$ and so $f^nS \le r_S(g)$. Let $h \in r_S(g)$. Then $gh(M) = 0$ and $h(M) \le r_M(g) = f^n(M)$. By image-projectivity of M there exists $h' \in S$ such that $f^nh' = h \in f^nS$ or $r_S(g) \le f^nS$. Thus $r_S(g) = f^nS$. Next we prove $r_S(f^n) = gS$. If $h \in r_S(f^n)$, then $f^nh = 0$ and $f^nh(M) = 0$ and $h(M) \le r_M(f^n) = g(M)$. By image-projectivity of M there exists an $h' \in S$ such that $h = gh' \in gS$. So $r_S(f^n) \le gS$. Let $h \in gS$. There exists an $h' \in S$ such that $h = gh'$. $r_M(f^n) = g(M)$ implies $f^ng = 0$. Hence $g \in r_S(f^n)$. Thus $gS \le r_S(f^n)$ and so $gS = r_S(f^n)$.
(3) Let $f \in S$. There exist $g \in S$ and a positive integer n such that $f^nS = r_S(g)$ and $r_S(f^n) = gS$. We prove $f^n(M) = r_M(g)$ and $r_M(f^n) = g(M)$. $f^nS = r_S(g)$ implies $gf^n = 0$ and so $f^n(M) \le r_M(g)$. Let $h \in S$ such that $h(M) \le r_M(g)$. So $gh = 0$ and $h \in f^nS$. There exists $h' \in S$ such that $h = f^nh'$. Hence $h(M) \le f^nh'(M) \le f^n(M)$. Since M generates $r_M(g)$, $r_M(g) \le f^n(M)$, $r_M(g) = f^n(M)$. Next we prove $r_M(f^n) = g(M)$. $r_S(f^n) = gS$ implies $f^ng = 0$. Then $g(M) \le r_M(f^n)$. Let $h(M) \le r_M(f^n)$. Then $f^nh(M) = 0$ and so $f^nh = 0$ and $h \in r_S(f^n) = gS$. There exists $h' \in S$ such that $h = gh'$. Hence $h(M) \le gh'(M) \le g(M)$ and $r_M(f^n) \le g(M)$ since M generates $r_M(f^n)$. Thus $r_M(f^n) = g(M)$. □

The following theorem generalizes Theorem 32 in [5] to π-morphic modules.

2.13. Theorem. *Let M be a module. Then the following are equivalent:*

(1) *M is π-morphic and image-projective.*
(2) *S is right π-morphic and M generates its kernel.*

Proof. Clear by Lemma 2.12. □

Let M be a module. In [7], the module M is called *π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $r_M(f^n) = eM$, while in [3], M is said to be *Rickart* if for any $f \in S$, there exists $e^2 = e \in S$ such that $r_M(f) = eM$. Rickart module is named as kernel-direct in [5]. In [8], M is called *dual π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $f^n(M) = eM$, while in [3], M is said to be *dual Rickart* if for any $f \in S$, there exists $e^2 = e \in S$ such that $f(M) = eM$. Dual-Rickart module is named as image-direct in [5]. Erlich [2] proved that a map $f \in S$ is unit-regular if and only if f is regular and morphic. We state and prove this theorem for π-regular rings.

2.14. Theorem. *Let $f \in S$. Then the following are equivalent:*
(1) *f is unit-π-regular.*
(2) *f is π-regular and morphic.*

Proof. (1) ⇒ (2) Every unit-π-regular ring is π-regular. There exist a unit g and a positive integer n such that $f^n = f^ngf^n$. Then gf^n is an idempotent, $r_M(f^n) = (1 - gf^n)M$ and

$M \cong f^n(M) \oplus (1 - gf^n)M$. Hence $M/f^n(M) \cong r_M(f^n)$.
(2) $\Rightarrow$ (1) Let $f^n = f^n g f^n$ where $g \in S$. Then
$M = f^n M \oplus (1 - f^n g)M = r_M(f^n) \oplus (gf^n)M$.
Let $h : f^n M \to gf^n(M)$ be defined by $hf^n(m) = gf^n(m)$ where $f^n(m) \in f^n(M)$. Then h and f^n are isomorphisms and inverse each other. Now
$M = f^n(M) \oplus (1 - f^n g)(M)$ and $M/r_M(f^n) \cong f^n(M)$. By morphic condition we have $M/f^n(M) \cong r_M(f^n)$. Then $M/f^n(M) \cong (1 - (f^n g))(M)$ gives rise to an isomorphism $(1 - (f^n g))(M) \xrightarrow{h'} r_M(f^n)$. Set $\alpha = h \oplus h'$. Let $m = x + y$ with $x \in f^n(M)$ and $y \in (1 - f^n g)(M)$. Then $(f^n \alpha f^n)(x+y) = (f^n h f^n)(x) + (f^n h' f^n)(y) = (f^n g f^n)(y) + 0 = f^n(y) + f^n(x) = f^n(x + y)$. Hence $f^n \alpha f^n = f^n$. □

2.15. Theorem. *Let M be a module with $S = End_R(M)$. The following are equivalent:*
(1) *S is unit-π-regular.*
(2) *M is π-morphic and π-Rickart.*
(3) *M is π-morphic and dual π-Rickart.*

Proof. (1) $\Rightarrow$ (2) Let S be unit-π-regular and $f \in S$. There exist a unit $g \in S$ and a positive integer n such that $f^n = f^n g f^n$. By virtue of Theorem 2.14, M is π-morphic. M is π-Rickart since $1 - gf^n$ is an idempotent and $r_M(f^n) = (1 - gf^n)M$.
(2) $\Rightarrow$ (3) Let $f \in S$. There exists a positive integer n such that $M/(f^n M) \cong r_M(f^n)$. By Lemma 2.2 there exists a $g \in S$ such that $g(M) = r_M(f^n)$ and $r_M(g) = f^n(M)$. By (2), $r_M(g)$ is π-Rickart, therefore $f^n(M)$ is direct summand.
(3) $\Rightarrow$ (1) Let $f \in S$. By (3), there exist a positive integer n and $g \in S$ such that $f^n M = r_M(g)$ and $r_M(f^n) = g(M)$. By (3), $f^n M$ and $g(M)$ are direct summand and so is $r_M(f^n)$. Hence S is π-regular ring by [9, Corollary 3.2]. By Theorem 2.14, S is unit-π-regular. □

Example 2.16 shows that there exists a π-Rickart module which is not π-morphic.

2.16. Example. Consider $M = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}$-module. It can be easily determined that $S = \text{End}_{\mathbb{Z}}(M)$ is $\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. For any $f = \begin{bmatrix} a & 0 \\ \bar{b} & \bar{c} \end{bmatrix} \in S$, we have the following cases.

Case 1. Assume that $a = 0$, $\bar{b} = \bar{0}$, $\bar{c} = \bar{1}$ or $a = 0$, $\bar{b} = \bar{c} = \bar{1}$. In both cases f is an idempotent, and so $r_M(f) = (1 - f)M$.

Case 2. If $a \neq 0$, $\bar{b} = \bar{0}$, $\bar{c} = \bar{1}$ or $a \neq 0$, $\bar{b} = \bar{c} = \bar{1}$, then $r_M(f) = 0$.

Case 3. If $a \neq 0$, $\bar{b} = \bar{c} = \bar{0}$ or $a \neq 0$, $\bar{b} = \bar{1}$, $\bar{c} = \bar{0}$, then $r_M(f) = 0 \oplus \mathbb{Z}/2\mathbb{Z}$.

Case 4. If $a = 0$, $\bar{b} = \bar{1}$, $\bar{c} = \bar{0}$, then $f^2 = 0$. Hence $r_M(f^2) = M$.

Therefore M is a π-Rickart module. Now we prove it is not π-morphic. Let
$f = \begin{bmatrix} 2 & 0 \\ \bar{0} & \bar{1} \end{bmatrix} \in S$. For each positive integer n, $r_M(f^n) = 0$ and
$f^n(M) = 2^n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. Then $M/f^n(M) \cong (\mathbb{Z}/2\mathbb{Z})^n$. But $(\mathbb{Z}/2\mathbb{Z})^n$ can not be isomorphic to $r_M(f^n) = 0$.

In [5], M is called *an image-injective module* if for each $f \in S$, every R-module homomorphisms from $f(M)$ to M extends to M. By this definition we state and prove dual versions of Lemma 2.12.

2.17. Lemma. *Let M be a module with $S = End_R(M)$.*
(1) *If S is left π-morphic, then M is image-injective.*
(2) *If M is π-morphic and image-injective, then S is left π-morphic.*
(3) *If S is left π-morphic and M cogenerates its cokernel, then M is π-morphic.*

Proof. (1) By Lemma 2.12, S is right GP-injective. Let f, $g \in S$. There exists a positive integer n depending on f such that $f^n \neq 0$ and any map $f^nS \xrightarrow{g'} S$ extends to an endomorphism of S. Let $f^n(M) \xrightarrow{g} M$ be a right R-module homomorphism and set $h = gf^n$. Then $r_S(f^n) \leq r_S(h)$. The map $f^nS \xrightarrow{t} hS$ defined by $t(f^ns) = hs$ where $s \in S$ is well defined right S-module homomorphism. By the GP-injectivity of S, t extends to an endomorphism g' of S so that $g'f^n = h$. Let $m \in M$. $g'f^n(m) = h(m) = gf^n(m)$. Hence g extends to $g' \in S$. Thus M is image-injective.
(2) Let $f \in S$. There exist $g \in S$ and a positive integer n such that $f^n(M) = r_M(g)$ and $r_M(f^n) = g(M)$. We prove $Sf^n = l_S(g)$ and $l_S(f^n) = Sg$. $r_M(f^n) = g(M)$ implies $f^ng = 0$. Then $f^n \in l_S(g)$ and so $Sf^n \leq l_S(g)$. Let $h \in l_S(g)$. Then $hg = 0$ or $f^n(M) = g(M) \leq r_M(h)$. Since $f^n(M) = g(M)$, the map defined t by $f^n(M) \xrightarrow{t} h(M)$ extends to an endomorphism α of M. Then $\alpha f^n = h \in Sf^n$. Hence $l_S(g) \leq Sf^n$ and so $l_S(g) = Sf^n$.

$f^n(M) = r_M(g)$ implies $gf^n = 0$. So $g \in l_S(f^n)$ and $Sg \leq l_S(f^n)$. Let $h \in l_S(f^n)$. Then $hf^n = 0$. Hence $r_M(g) = f^n(M) \leq r_M(h)$. So the map defined by $g(M) \xrightarrow{t} h(M)$ is a module homomorphism and, by image-injectivity of M it extends to an endomorphism α of M. Hence $h = \alpha g \in Sg$. Thus $l_S(f^n) \leq Sg$ and so $l_S(f^n) = Sg$ and S is left π-morphic.
(3) Let $f \in S$. We prove that there exist $g \in S$ and a positive integer n such that $f^n(M) = r_M(g)$ and $r_M(f^n) = g(M)$. By hypothesis S is left π-morphic, there exist $g \in S$ and a positive integer n such that $Sf^n = l_S(g)$ and $l_S(f^n) = Sg$. $Sf^n = l_S(g)$ implies $f^ng = 0$ and $g(M) \leq r_M(f^n)$. Let $m \in r_M(f^n) - g(M)$. Then $\overline{0} \neq \overline{m} \in M/g(M)$. By hypothesis, M cogenerates $M/g(M)$. There exists a map $M/g(M) \xrightarrow{t} M$ such that $t(\overline{m}) \neq 0$. Now define $M \xrightarrow{\alpha} M$ by $\alpha(x) = t(\overline{x})$. Then $tg(x) = 0$ for all $x \in M$. Hence $\alpha g = 0$. So $\alpha \in l_S(g) = Sf^n$. There exists $s \in S$ such that $\alpha = sf^n$. This leads us a contradiction since $0 \neq \alpha(m) = sf^n(m) = 0$. Thus $r_M(f^n) = g(M)$.
On the other hand $l_S(f^n) = Sg$ implies $gf^n = 0$ and $f^n(M) \leq r_M(g)$. Let $m \in r_M(g) - f^n(M)$. As in the preceding paragraph there exist s, $\alpha \in S$ such that $\alpha = sg$ and $\alpha(m) \neq 0$. Since $g(m) = 0$, this would lead us to a contradiction again. Thus $f^n(M) = r_M(g)$. □

2.18. Theorem. *Let M be a module. Then the following are equivalent:*
(1) *M is π-morphic and image injective.*
(2) *S is left π- morphic and M cogenerates its cokernel.*

Proof. Clear from Lemma 2.17. □

A ring R is said to be *right Kasch* if every simple right R-module embeds in R, equivalently, if $l(I) \neq 0$ for every proper (maximal) right ideal I of R (see also [6, page 51]). Let M be a module. M is called *Kasch module* if any simple module in $\sigma[M]$ embeds in M, where $\sigma[M]$ is the category consisting of all M-subgenerated right R-modules, while M is *strongly Kasch* if any simple right R-module embeds in M. It is easy to see that a ring R is right Kasch if and only if the right R-module R is Kasch if and only if the right R-module R is strongly Kasch since $\sigma[R]$ is just the category of all right R-modules for details see [10].

2.19. Proposition. *Let M be a π-morphic module. If every maximal right ideal of S is principal, then S is a right Kasch ring.*

Proof. Let I be maximal right ideal of S. Then $I = fS$ for some $f \in S$. There exists a positive integer n such that $M/f^nM \cong r_M(f^n)$. Assume that $r_M(f^n) = 0$. Then $f^nM = M = fM$. Hence f^n is an isomorphism. Thus $I = S$. It is a contradiction.

It follows that for any nonzero $0 \neq f \in I$ there exists a positive integer n such that $M/f^nM \cong r_M(f^n) \neq 0$. Consider the diagram $M \overset{\pi}{\to} M/f^nM \overset{\varphi}{\to} r_M(f^n)$ where π is coset map and φ is the isomorphism. Then $\varphi\pi f^n = 0$. Hence $0 \neq \varphi\pi f^{n-1} \in l_S(f)$. □

2.20. Corollary. *Let R be a right π-morphic ring and every maximal right ideal be principal. Then R is right Kasch.*

Proof. Clear from Lemma 2.19 by considering $M = R_R$ and $S = \mathrm{End}_R(R) \cong R$. □

2.21. Proposition. *Let S be a right π-morphic ring. Then the following conditions are equivalent:*

(1) *S is a right Kasch ring.*
(2) *Every maximal right ideal of S is an annihilator.*
(3) *Every maximal right ideal of S is principal.*

Proof. Note that every π-morphic ring is directly finite by Corollary 2.5. In [6] it is noted that (1) ⇒ (2) always holds.
(2) ⇒ (3) Let I be a maximal right ideal of S. Then there exists a nonzero right ideal A of S such that $I = l(A)$. Let $0 \neq a \in A$, there exist $b \in S$ and a positive integer n such that such that $a^nS = r(b)$ and $r(a^n) = bS$. Hence $I \subseteq l(a^n) \neq S$. Therefore, $I = r(a^n)$.
(3) ⇒ (1) To complete the proof we show that $l(I) \neq 0$ for every maximal right ideal I of S. Let I be a maximal right ideal. By (3), $I = aS$ for some $a \in S$. We invoke hypothesis here to find $b \in S$ and a positive integer n such that $a^nS = r(b)$ and $r(a^n) = bS$. Then $a^nb = 0$ and $ba^n = 0$. If $b = 0$, then $a^nS = S$. By Corollary 2.5, a is invertible and so $I = S$. This contradicts being I maximal. It follows that $b \neq 0$. Let t be a nonzero positive integer such that $ba^t = 0$ and $ba^{t-1} \neq 0$. Hence $ba^t = 0$ implies $0 \neq ba^{t-1} \in l(I)$. So S is right Kasch. □

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