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Helix preserving mappings

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In this study, we define two types of mappings that preserve the constant angle between the tangent vector field and the axis of a given helix in Euclidean spaces. The first type generates helices in the n-dimensional Euclidean space from helices in the same space. The second type generates helices in the (n+1)-dimensional Euclidean space from helices in the n-dimensional Euclidean space. In addition, we give invariants of these mappings and study polynomial, rational, conical, ellipsoidal, and hyperboloidal helices supported by examples.

KEYWORDS
helix, map, n-helix mapping, (n+1)-helix mapping

MSC CLASSIFICATION
53A04; 53A05; 58C25

1 | INTRODUCTION

Helices are one of the most interesting curves that have been studied researchers from varying fields like mathematics, architecture, engineering, and biology. We can find helices from micro scales to macro scales in nature. One of the famous examples for micro scales is DNA. According to the latest researches, the shape of DNA is a helix because it takes the least amount of energy and takes up the least space.[1] In addition to DNA, bacterial flagella, seashells, and horns can be given as examples of helices in nature.[1,2] In computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion, or the design of highways.[3] Also, we can find helical structures in industrial products like screws and polyaniline-type polymers.[1,2]

A curve is called a helix if its tangent vector field makes a constant angle with a fixed direction. In the Euclidean three-dimensional space, a necessary and sufficient condition for a curve to be a helix is that the ratio of curvature to torsion $(\frac{k_1}{k_2})$ be constant. This result known as Lancret's theorem.[4] If a helix lies on a surface, it is named by the name of the surface, eg, spherical helix and ellipsoidal helix.

The notion of a helix in Euclidean three-dimensional space can be generalized to higher dimensions in many ways.[5,6] However, we use the classical definition, ie, a curve is called a helix if its tangent vector field makes a constant angle with a fixed direction like in the papers.[7-9]

Pythagorean-hodograph (PH) curves were introduced by Farouki and Sakkalis.[10] We can use PH curve concept to define helices. By means of this concept, Altunkaya and Kula[7] studied polynomial helices in the Euclidean n-space. They showed methods to construct rational helices from polynomial helices.

As we know, Klein[11] described geometry as the study of invariants under certain allowed transformations, ie, special kind of mappings. Therefore, finding invariants under mappings can be considered one of the main goals of geometry. The methods in the Altunkaya and Kula's paper[7] led us to study for finding mappings that preserve general helices in the Euclidean n-space.

In this work, we introduce two mappings that generate new helices from related helices by preserving the constant angle between the tangent vector field and the axis of a helix. Furthermore, we give characterizations of these mappings and find invariants of them. Also, we generate polynomial, rational, conical, ellipsoidal, and hyperboloidal helices by using these mappings. To the best of our knowledge, there is no paper discussing mappings that preserve helices in Euclidean spaces in the literature.

2 | BASIC CONCEPTS

Let $\mathbb{E}^n$ denote the Euclidean n-space with the standart metric

$$g = dx_1^2 + dx_2^2 + \dots + dx_n^2$$

where $(x_1, x_2, \dots, x_n)$ is a rectangular coordinate system of $\mathbb{E}^n$.[4,12]

The standard inner product of real vector space R^n with the standart orthonormal basis $\{e_1, e_2, \dots, e_n\}$ is given by

$$< X, Y >= \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_2, ..., x_n), Y = (y_1, y_2, ..., y_n) \in R^n$. In particular, the norm of a vector $X \in R^n$ is given by $||X||^2 =< X, X >$.

Let $\alpha : I \subset R \to \mathbb{E}^n$ be an arbitrary curve in $\mathbb{E}^n$ and $\{V_1, V_2, ..., V_n\}$ be the moving Frenet frame along the curve α where V_i $\{(i = 1, 2, ..., n)\}$ denotes the ith Frenet vector field. Then, the Frenet formulae are given by

$$\begin{cases} V_1'(t) = v(t)k_1(t) \ V_2 \ (t) \\ V_i'(t) = v(t)(-k_{i-1}(t) \ V_{i-1} \ (t) + \ k_i(t) \ V_{i+1} \ (t)), \ i = 2, 3, ..., n-1 \\ V_n'(t) = -v(t)k_{n-1}(t) \ V_{n-1} \ (t) \end{cases}$$

where $v(t) = ||\alpha'(t)||$ and $k_i \ (i = 1, 2, ..., n-1)$ denote the ith curvature function of the curve α.[8,13]

We call α a regular curve of order m (where $m \leqslant n$) if and only if for any $t \in I$

$$\{\alpha'(t), \alpha''(t), ..., \alpha^{(m)}(t)\}$$

is a linearly independent subset of R^n. For $m = n$, we call that the curve α is a regular curve. In this study, we work with regular curves.

Definition 2.1. The curve $\alpha : I \subset R \to \mathbb{E}^n$ is called a helix if its tangent vector V_1 makes a constant angle with a fixed direction U, called the axis.[6]

Remark 2.1. All the mappings in this work are built for helices with the axis e_n in $\mathbb{E}^n$. Moreover, mappings for the helices that have different axes can also be built similarly.

3 | n-HELIX MAPPING

Now, we define a mapping which maps a helix in $\mathbb{E}^n$ to another helix in $\mathbb{E}^n$.

Definition 3.1. Let $\mathcal{H} : \mathbb{E}^n \backslash M \to \mathbb{E}^n \backslash M$ be the mapping defined by

$$\mathcal{H}\,(x_1, x_2, ..., x_n) = \frac{c}{x_1^2 + x_2^2 + ... + (1-a)x_n^2}\,(x_1, x_2, ..., x_n)$$

where $M = \left\{(x_1, x_2, \dots, x_n) \mid \ x_1^2 + x_2^2 + \dots + x_{n-1}^2 \neq (a-1)\,x_n^2\right\}$, $c \neq 0$ and $a > 1$. We call $\mathcal{H}$ as $n-$*helix mapping*.

We note that an involution is a mapping whose inverse is itself. Thus, we have the corollary below.

Corollary 3.1. *The mapping $\mathcal{H}$ is an involution.*

In addition, we can give the following theorem for the mapping of $\mathcal{H}$.

Theorem 3.1. *The mapping $\mathcal{H}$ leaves the hypercone*

$$K = \left\{(x_1, x_2, \dots, x_n) \mid \quad x_1^2 + x_2^2 + \dots + x_{n-1}^2 = b\, x_n^2, \quad b \neq a - 1 > 0\right\}$$

invariant.

Proof. Let

$$\mathcal{H}\,(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$$

for $(x_1, x_2, \dots, x_n) \in \mathbb{E}^n \backslash M$. Therefore, we can write

Then,

$$y_i = \frac{c}{x_1^2 + x_2^2 + \dots + (1-a)x_n^2}\, x_i, \quad 1 \leq i \leq n,$$

and

$$\begin{aligned} y_1^2 + y_2^2 + \dots + y_{n-1}^2 &= \left(\frac{c}{x_1^2 + x_2^2 + \dots + (1-a)\,x_n^2}\right)^2 \left(x_1^2 + x_2^2 + \dots + x_{n-1}^2\right) \\ &= b\left(\frac{c}{x_1^2 + x_2^2 + \dots + (1-a)\,x_n^2}\right)^2 x_n^2 \\ &= b\, y_n^2. \end{aligned}$$

This completes the proof. □

Now, we give fairly easy, but an important result in the Lemma 3.1, which characterizes helices with the axis e_n in E^n. By Lemma 3.1, we show that the mapping $\mathcal{H}$ generates a new helix from the related helix and preserves the constant angle between the tangent vector field and the axis of the helix in Theorem 3.2.

Lemma 3.1. $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \,:\, I \subset R \to \mathbb{E}^n$ *is a helix whose tangent vector field makes a constant angle* $\theta = \arccos(\frac{1}{\sqrt{a}})$ *with the axis e_n if and only if* $\|\alpha'\|^2 = a\left(\alpha_n'\right)^2$ *where* $a > 1$.

Proof. If α is a helix with the conditions above, then

$$\langle V_1, e_n \rangle = \frac{{\alpha_n}'}{\|\alpha'\|} = \frac{1}{\sqrt{a}}.$$

Therefore,

$$\|\alpha'\|^2 = a\left(\alpha_n'\right)^2.$$

Conversely, if $\|\alpha'\|^2 = a\left(\alpha_n'\right)^2$, then

$$V_1 = \frac{1}{\sqrt{a}\alpha_n'}\left({\alpha_1}', {\alpha_2}', \dots, {\alpha_n}'\right).$$

Hence,

$$\langle V_1, e_n \rangle = \frac{1}{\sqrt{a}}.$$

Therefore, α is a helix in $\mathbb{E}^n$ whose tangent vector field makes a constant angle $\theta = \arccos(\frac{1}{\sqrt{a}})$ with the axis e_n. □

Theorem 3.2. *Consider the curve* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \,:\, I \subset R \to \mathbb{E}^n$. *Then, α is a helix whose tangent vector field makes a constant angle* $\theta = \arccos(\frac{1}{\sqrt{a}})$ *with the axis e_n if and only if*

$$\beta = \mathcal{H}\,(\alpha) = \frac{c}{\alpha_1^2 + \alpha_2^2 + \dots + (1-a)\alpha_n^2}\,(\alpha_1, \alpha_2, \dots, \alpha_n) \tag{3.1}$$

is a helix in $\mathbb{E}^n$ whose tangent vector field makes the constant angle $\theta = \arccos(\frac{1}{\sqrt{a}})$ *with the axis e_n where* $c \neq 0$ *and* $a > 1$.

Proof. If α is a helix with the conditions above, then

$$\|\alpha'\|^2 = a\left(\alpha_n'\right)^2.$$

By taking

$$u = \frac{c}{\alpha_1^2 + \alpha_2^2 + \dots + (1-a)\alpha_n^2},$$

we have $\beta = u\alpha$. With straightforward calculations, we have

$$\begin{aligned}\|\beta'\|^2 &= \left(u'\right)^2\|\alpha\|^2 + 2uu'\langle\alpha, \alpha'\rangle + au^2(\alpha'_n)^2 \\ &= a\left[(u\alpha_n)'\right]^2.\end{aligned}$$

Therefore, by using Lemma 3.1; β is a helix in $\mathbb{E}^n$ whose tangent vector field makes a constant angle $\theta = \arccos(\frac{1}{\sqrt{a}})$ with the axis e_n.

Conversely, if β is a helix with the conditions above, then

$$\|\beta'\|^2 = a\left[(u\alpha_n)'\right]^2.$$

With straightforward calculations, we have

$$u^2\left(\left(\alpha_1'\right)^2 + \left(\alpha_2'\right)^2 + \dots + (1-a)\left(\alpha_n'\right)^2\right) = 0.$$

Therefore,

$$\|\alpha'\|^2 = a\left(\alpha_n'\right)^2.$$

This completes the proof. □

The following results in Remark 3.1 and Remark 3.2 are important properties of the mapping $\mathcal{H}$.

Remark 3.1. By Corollary 3.1, we also have

$$\mathcal{H}(\beta) = \alpha. \tag{3.2}$$

Remark 3.2. If α is a helix in $\mathbb{E}^3$ whose tangent vector field V_1 makes a constant angle θ with a fixed direction d. Then, $< V_1, d >= \cos(\theta)$. As we know $\frac{k_1}{k_2} = \tan(\theta)$ (see[12], pp. 160). As a result of Theorem 3.2, we easily see the mapping $\mathcal{H}$ preserves the ratio $\frac{k_1}{k_2}$; since the curves α and $\beta = \mathcal{H}(\alpha)$ make the same angle θ with the same fixed direction d.

Corollary 3.2. *If α is a polynomial helix, then the curve $\beta = \mathcal{H}(\alpha)$ is a rational helix.*

Example 3.1. Consider the polynomial helix

$$\alpha(t) = \left(-t^3 + 3t, 3t^2, t^3 + 3t\right)$$

whose tangent vector field

$$V_1(t) = \left(-\frac{t^2-1}{\sqrt{2}\left(t^2+1\right)}, \frac{\sqrt{2}t}{t^2+1}, \frac{1}{\sqrt{2}}\right)$$

makes constant angle $\theta = \arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$ with e_3.

Then, by taking $c = 1$ in (3.1), we have the rational helix

$$\beta(t) = \left(\frac{3-t^2}{t^3}, \frac{3}{t^2}, \frac{t^2+3}{t^3}\right)$$

with the tangent vector field

$$V_1(t) = \left(-\frac{t^2 - 9}{\sqrt{2}\left(t^2 + 9\right)}, \frac{3\sqrt{2}t}{t^2 + 9}, \frac{1}{\sqrt{2}} \right).$$

We can see these curves in Figure 1.

$$\begin{aligned} \mathcal{H}\left(\beta(t)\right) &= \mathcal{H}\left(\frac{3 - t^2}{t^3}, \frac{3}{t^2}, \frac{t^2 + 3}{t^3} \right) \\ &= \alpha(t). \end{aligned}$$

Example 3.2. Consider the conical helix

$$\alpha(t) = (e^t \cos 3t, e^t \sin 3t, \frac{4}{5} e^t)$$

whose tangent vector field

$$V_1(t) = \frac{1}{\sqrt{266}} \left(5(\cos 3t - 3 \sin 3t), 5(\sin 3t + 3 \cos 3t), 4\right).$$

makes the constant angle $\theta = \arccos(2\sqrt{\frac{2}{133}})$ with e_3.

Then, by taking $c = -9$ in (3.1), we have the conical helix

$$\beta(t) = \left(-e^{-t} \cos 3t, -e^{-t} \sin 3t, -\frac{4}{5} e^{-t} \right)$$

with the tangent vector field

$$V_1(t) = \frac{1}{\sqrt{266}} \left(5(\cos 3t + 3 \sin 3t), -5(3 \cos 3t - \sin 3t), 4\right).$$

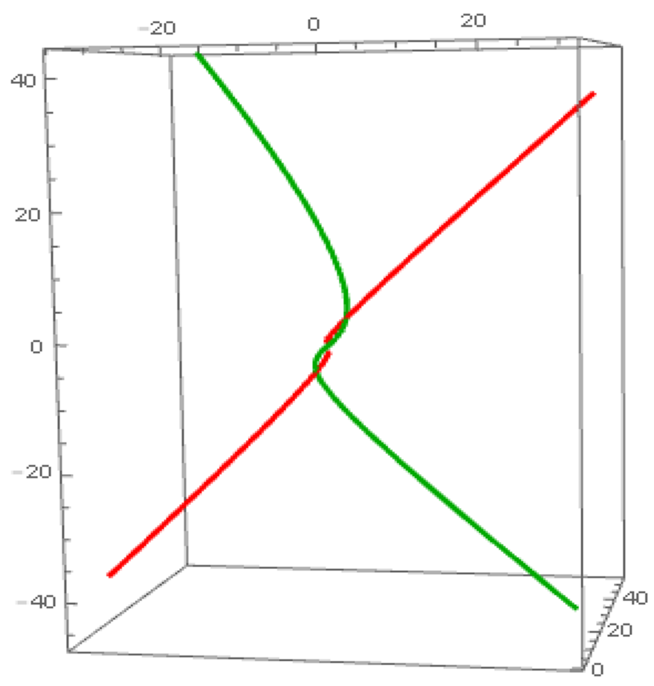

FIGURE 1 The helices α (green) and β (red) [Colour figure can be viewed at wileyonlinelibrary.com]

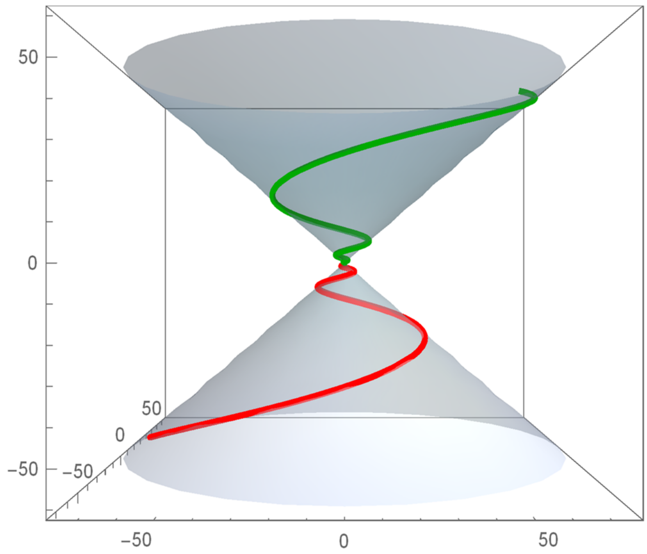

FIGURE 2 The helices α (green) and β (red) on the cone K [Colour figure can be viewed at wileyonlinelibrary.com]

These curves both lie on the cone

$$K = \left\{ (x_1, x_2, x_3) \mid \quad x_1^2 + x_2^2 = \frac{25}{16} x_3^2 \right\}$$

as shown in Figure 2.

By means of the mapping $\mathcal{H}$, we can find the parametric equations of ellipsoidal and hyperboloidal helices which are new for researchers.

Corollary 3.3. *Let us consider the helix* $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ *where*

$$\alpha_1(t) = \frac{d\sqrt{t^2+1}\left(t\cos\left(f\tan^{-1}(t)\right) - f\sin\left(f\tan^{-1}(t)\right)\right)}{f\left(\left(f^2-2\right)t^2-1\right)},$$
$$\alpha_2(t) = \frac{d\sqrt{t^2+1}\left(t\sin\left(f\tan^{-1}(t)\right) + f\cos\left(f\tan^{-1}(t)\right)\right)}{f\left(\left(f^2-2\right)t^2-1\right)},$$
$$\alpha_3(t) = \frac{e\sqrt{f^2-1}\,t\sqrt{t^2+1}}{f\left(\left(f^2-2\right)t^2-1\right)},$$

$d, e \neq 0$ *and* $f > 1$*. Then,* $\beta(t) = \mathcal{H}(\alpha(t)) = (\beta_1(t), \beta_2(t), \beta_3(t))$ *where*

$$\beta_1(t) = \frac{cd\left(t\cos\left(f\tan^{-1}(t)\right) - f\sin\left(f\tan^{-1}(t)\right)\right)}{f\sqrt{t^2+1}},$$
$$\beta_2(t) = \frac{cd\left(t\sin\left(f\tan^{-1}(t)\right) + f\cos\left(f\tan^{-1}(t)\right)\right)}{f\sqrt{t^2+1}},$$
$$\beta_3(t) = \frac{ce\sqrt{f^2-1}\,t}{f\sqrt{t^2+1}}$$

is an ellipsoidal helix which lies on the ellipsoid

$$\frac{x^2}{c^2d^2} + \frac{y^2}{c^2d^2} + \frac{z^2}{c^2e^2} = 1.$$

Example 3.3. If we take $c = 1, f, d = 2, e = 3$ in Corollary 3.3 and simplify the trigonometric expressions we have the helix

$$\alpha(t) = \left(-\frac{t\left(t^2+3\right)}{\sqrt{t^2+1}\left(2t^2-1\right)}, \frac{2}{\sqrt{t^2+1}\left(2t^2-1\right)}, \frac{3\sqrt{3}\,t\sqrt{t^2+1}}{4t^2-2} \right)$$

and the ellipsoidal helix

$$\beta(t) = \left(-\frac{t\left(t^2+3\right)}{\left(t^2+1\right)^{3/2}}, \frac{2}{\left(t^2+1\right)^{3/2}}, \frac{3\sqrt{3}\,t}{2\sqrt{t^2+1}} \right).$$

The helix β lies on the ellipsoid (See Figure 3)

$$\frac{1}{4}\left(x^2+y^2\right) + \frac{z^2}{9} = 1.$$

Corollary 3.4. *Let us consider the helix* $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ *where*

$$\alpha_1(t) = \frac{d\sqrt{t^2+1}\left(t\cos\left(f\tan^{-1}(t)\right) - f\sin\left(f\tan^{-1}(t)\right)\right)}{f\left(\left(f^2-2\right)t^2-1\right)},$$

$$\alpha_2(t) = \frac{d\sqrt{t^2+1}\left(t\sin\left(f\tan^{-1}(t)\right) + f\cos\left(f\tan^{-1}(t)\right)\right)}{f\left(\left(f^2-2\right)t^2-1\right)},$$

$$\alpha_3(t) = \frac{e\sqrt{1-f^2}\,t\sqrt{t^2+1}}{f\left(\left(f^2-2\right)t^2-1\right)},$$

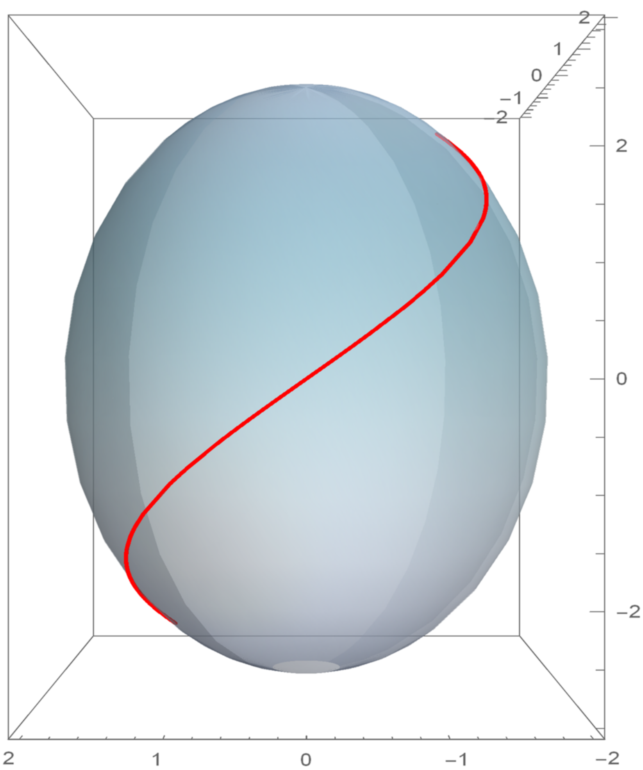

FIGURE 3 The helix β on the ellipsoid [Colour figure can be viewed at wileyonlinelibrary.com]

$d, e \neq 0$ and $|f| < 1$. Then, $\beta(t) = \mathcal{H}(\alpha(t)) = (\beta_1(t), \beta_2(t), \beta_3(t))$ where

$$\beta_1(t) = \frac{cd\left(t\cos\left(f\tan^{-1}(t)\right) - f\sin\left(f\tan^{-1}(t)\right)\right)}{f\sqrt{t^2+1}},$$

$$\beta_2(t) = \frac{cd\left(t\sin\left(f\tan^{-1}(t)\right) + f\cos\left(f\tan^{-1}(t)\right)\right)}{f\sqrt{t^2+1}},$$

$$\beta_3(t) = \frac{ce\sqrt{1-f^2}\,t}{f\sqrt{t^2+1}}$$

is a hyperboloidal helix which lies on the one sheeted hyperboloid

$$\frac{x^2}{c^2d^2} + \frac{y^2}{c^2d^2} - \frac{z^2}{c^2e^2} = 1.$$

Example 3.4. If we take $c, d = 1, e = 3, f = 1/3$ in Corollary 3.4, we have the helix $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ where

$$\alpha_1(t) = \frac{9\sqrt{t^2+1}\left(\sin\left(\frac{1}{3}\tan^{-1}(t)\right) - 3t\cos\left(\frac{1}{3}\tan^{-1}(t)\right)\right)}{17t^2+9},$$

$$\alpha_2(t) = -\frac{9\sqrt{t^2+1}\left(3t\sin\left(\frac{1}{3}\tan^{-1}(t)\right) + \cos\left(\frac{1}{3}\tan^{-1}(t)\right)\right)}{17t^2+9},$$

$$\alpha_3(t) = -\frac{54\sqrt{2}\,t\sqrt{t^2+1}}{17t^2+9}$$

and the hyperboloidal helix $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$ where

$$\beta_1(t) = \frac{3t\cos\left(\frac{1}{3}\tan^{-1}(t)\right) - \sin\left(\frac{1}{3}\tan^{-1}(t)\right)}{\sqrt{t^2+1}},$$

$$\beta_2(t) = \frac{3t\sin\left(\frac{1}{3}\tan^{-1}(t)\right) + \cos\left(\frac{1}{3}\tan^{-1}(t)\right)}{\sqrt{t^2+1}},$$

$$\beta_3(t) = \frac{6\sqrt{2}\,t}{\sqrt{t^2+1}}.$$

The helix β lies on the one sheeted hyperboloid (See Figure 4)

$$x^2 + y^2 - \frac{z^2}{9} = 1.$$

Corollary 3.5. *Let us consider the helix $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ where*

$$\alpha_1(t) = \frac{d\sqrt{t^2-1}\left(t\cos\left(f\coth^{-1}(t)\right) + f\sin\left(f\coth^{-1}(t)\right)\right)}{f\left(\left(f^2+2\right)t^2-1\right)},$$

$$\alpha_2(t) = \frac{d\sqrt{t^2-1}\left(t\sin\left(f\coth^{-1}(t)\right) - f\cos\left(f\coth^{-1}(t)\right)\right)}{f\left(\left(f^2+2\right)t^2-1\right)},$$

$$\alpha_3(t) = \frac{e\sqrt{f^2+1}\,t\sqrt{t^2-1}}{f\left(\left(f^2+2\right)t^2-1\right)},$$

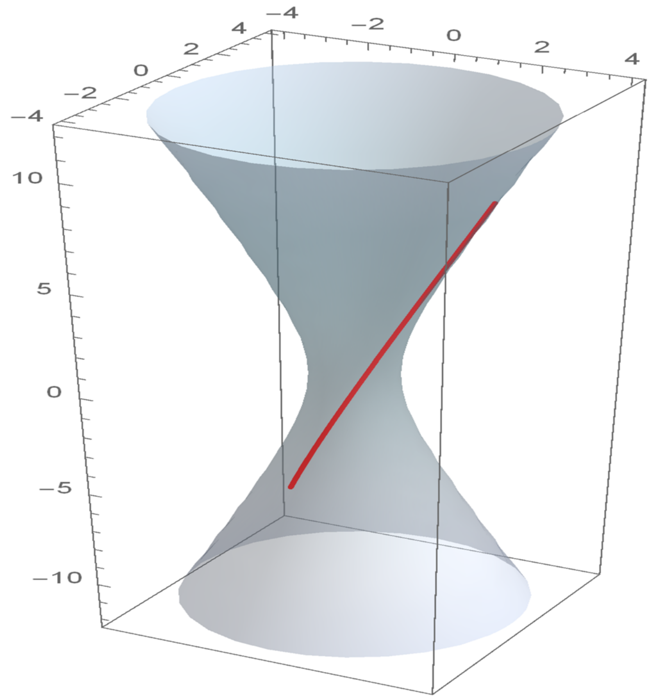

FIGURE 4 The helix β on the one sheeted hyperboloid [Colour figure can be viewed at wileyonlinelibrary.com]

$d, e \neq 0$. Then, $\beta(t) = \mathcal{H}(\alpha(t)) = (\beta_1(t), \beta_2(t), \beta_3(t))$ where

$$\beta_1(t) = \frac{cd\left(t\cos\left(f\coth^{-1}(t)\right) + f\sin\left(f\coth^{-1}(t)\right)\right)}{f\sqrt{t^2-1}},$$
$$\beta_2(t) = \frac{cd\left(t\sin\left(f\coth^{-1}(t)\right) - f\cos\left(f\coth^{-1}(t)\right)\right)}{f\sqrt{t^2-1}},$$
$$\beta_3(t) = \frac{ce\sqrt{f^2+1}\,t}{f\sqrt{t^2-1}}$$

is a hyperbolodial helix which lies on the two sheeted hyperboloid

$$\frac{x^2}{c^2d^2} + \frac{y^2}{c^2d^2} - \frac{z^2}{c^2e^2} = -1.$$

Example 3.5. If we take $c = 1, d = 3, e = 2, f = 5$ in Corollary 3.5, we have the helix $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ where

$$\alpha_1(t) = \frac{3\sqrt{t^2-1}\left(t\cos\left(5\coth^{-1}(t)\right) + 5\sin\left(5\coth^{-1}(t)\right)\right)}{135t^2-5},$$
$$\alpha_2(t) = \frac{3\sqrt{t^2-1}\left(t\sin\left(5\coth^{-1}(t)\right) - 5\cos\left(5\coth^{-1}(t)\right)\right)}{135t^2-5},$$
$$\alpha_3(t) = \frac{2\sqrt{26}\,t\sqrt{t^2-1}}{135t^2-5}$$

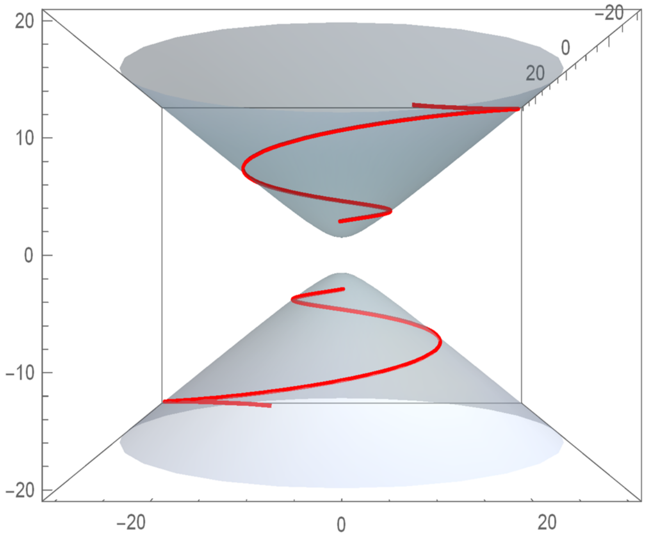

FIGURE 5 The helix β on the two sheeted hyperboloid [Colour figure can be viewed at wileyonlinelibrary.com]

and the hyperboloidal helix $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$ where

$$\beta_1(t) = \frac{3\left(t\cos\left(5\coth^{-1}(t)\right) + 5\sin\left(5\coth^{-1}(t)\right)\right)}{5\sqrt{t^2-1}},$$
$$\beta_2(t) = \frac{3\left(t\sin\left(5\coth^{-1}(t)\right) - 5\cos\left(5\coth^{-1}(t)\right)\right)}{5\sqrt{t^2-1}},$$
$$\beta_3(t) = \frac{2\sqrt{26}\,t}{5\sqrt{t^2-1}}.$$

The helix β lies on the two sheeted hyperboloid (See Figure 5)

$$\frac{1}{9}\left(x^2+y^2\right) - \frac{z^2}{4} = -1.$$

4 | (n+1)-HELIX MAPPING

In this section, we define a mapping which maps a helix in $\mathbb{E}^n$ to another helix in $\mathbb{E}^{n+1}$.

Definition 4.1. Let $\mathcal{G} : \mathbb{E}^n \backslash N \to \mathbb{E}^{n+1}$ be the mappping defined by

$$\mathcal{G}(x_1, x_2, ..., x_n) = \frac{c}{d^2 + x_1^2 + x_2^2 + ... + (1-a)x_n^2}(d, x_1, x_2, ..., x_n)$$

where $N = \left\{(x_1, x_2, \dots, x_n) \mid \; d^2 + x_1^2 + x_2^2 + \dots + x_{n-1}^2 \neq (a-1)x_n^2\right\}, c \neq 0, d \neq 0$ and $a > 1$. We call $\mathcal{G}$ as $(n+1)-$*helix mapping*.

Theorem 4.1. *Let us take the cone*

$$K = \left\{(x_1, x_2, \dots, x_n) \mid \; x_1^2 + x_2^2 + \dots + x_{n-1}^2 = b x_n^2\right\}$$

in $\mathbb{E}^n$. *Then,*

$$\mathcal{G}(K) = N_1 \cap N_2$$

where

$$N_1 = \left\{(y_0, y_1, y_2, \dots, y_n) \mid \; \left(y_0 - \frac{c}{2d}\right)^2 + y_1^2 + y_2^2 + \dots + y_{n-1}^2 + (1-a)\,y_n^2 = \frac{c^2}{4d^2}\right\}$$

and

$$N_2 = \left\{ (y_0, y_1, y_2, \dots, y_n) \mid \quad y_1^2 + y_2^2 + \dots + y_{n-1}^2 = b y_n^2 \right\}$$

are two hypersurfaces in $\mathbb{E}^{n+1}$.

Proof. Let,

$$\mathcal{G}(x_1, x_2, ..., x_n) = (y_0, y_1, y_2, ..., y_n)$$

for $(x_1, x_2, ..., x_n) \in \mathbb{E}^n \backslash N$. Similar to Theorem 3.1, we can write

$$y_0 = \frac{c\,d}{d^2 + \sum_{i=1}^{n-1} x_i^2 + (1-a)\,x_n^2},$$

$$y_j = \frac{c}{d^2 + \sum_{i=1}^{n-1} x_i^2 + (1-a)\,x_n^2}\, x_j, \quad 1 \le j \le n,$$

Therefore, we have

$$\sum_{j=0}^{n-1} y_j^2 + (1-a)\, y_n^2 = \frac{c}{d}\, y_0$$

and also

$$\sum_{j=1}^{n-1} y_j^2 - b\, y_n^2 = 0.$$

This completes the proof. □

Similar as Theorem 3.2, we may prove that Theorem 4.2 holds.

Theorem 4.2. *Consider the curve* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) : I \subset R \to \mathbb{E}^n$. *Then,* α *is a helix in* $\mathbb{E}^n$ *whose tangent vector field makes a constant angle* $\theta = \arccos(\frac{1}{\sqrt{a}})$ *with the axis* e_n *if and only if*

$$\beta = \mathcal{G}(\alpha) = \frac{c}{d^2 + \alpha_1^2 + \alpha_2^2 + ... + (1-a)\alpha_n^2}\,(d, \alpha_1, \alpha_2, ..., \alpha_n) \tag{4.1}$$

is a helix in $\mathbb{E}^{n+1}$ *whose tangent vector field makes the constant angle* $\theta = \arccos(\frac{1}{\sqrt{a}})$ *with the axis* $(0, e_n)$ *where* $c \neq 0$, $d \neq 0$, $a > 1$ *and* $d^2 + \alpha_1^2 + \alpha_2^2 + ... + (1-a)\alpha_n^2 \neq 0$.

Corollary 4.1. *If* α *is a polynomial helix, then the curve* $\beta = \mathcal{G}(\alpha)$ *is a rational helix.*

Example 4.1. Consider the polynomial helix

$$\alpha(t) = \left(-\frac{t\left(t^2-3\right)}{3\sqrt{2}}, \frac{t^2}{\sqrt{2}}, \frac{t\left(t^2+3\right)}{3\sqrt{2}} \right)$$

which makes constant angle $\theta = \frac{\pi}{4}$ with the tangent vector field

$$V_1(t) = \left(-\frac{t^2-1}{\sqrt{2}\left(t^2+1\right)}, \frac{\sqrt{2}t}{t^2+1}, \frac{1}{\sqrt{2}} \right).$$

Then, by taking $c = -1/6$, $d = 1$ in (4.1), we have the rational helix

$$\beta(t) = \mathcal{G}(\alpha(t)) = \left(\frac{1}{t^4-6}, \frac{3\sqrt{2}t - \sqrt{2}t^3}{6\left(t^4-6\right)}, \frac{t^2}{\sqrt{2}\left(t^4-6\right)}, \frac{\sqrt{2}t^3 + 3\sqrt{2}t}{6\left(t^4-6\right)} \right)$$

with the tangent vector field

$$V_1(t) = \left(\frac{12t^3}{t^6 + 9t^4 + 18t^2 + 18}, \frac{-t^6 + 9t^4 - 18t^2 + 18}{\sqrt{2}\left(t^6 + 9t^4 + 18t^2 + 18\right)}, \frac{3\sqrt{2}t\left(t^4 + 6\right)}{t^6 + 9t^4 + 18t^2 + 18}, \frac{1}{\sqrt{2}} \right).$$

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CONFLICTS OF INTEREST

There are no conflicts of interest to this work.

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