On developing an optimal Jarratt-like class for solving nonlinear equations

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Abstract. It is attempted to derive an optimal class of methods without memory from Ozban's method [A. Y. Ozban, Some New Variants of Newton's Method, Appl. Math. Lett. 17 (2004) 677-682]. To this end, we try to introduce a weight function in the second step of the method and to find some suitable conditions, so that the modified method is optimal in the sense of Kung and Traub's conjecture. Also, convergence analysis along with numerical implementations are included to verify both theoretical and practical aspects of the proposed optimal class of methods without memory.
Keywords: nonlinear equations, Kung and Traub's conjecture, iterative method, optimal method, convergence analysis.

1. Introduction

The main objective of this work is to derive an optimal class of methods without memory for approximating a simple root of a nonlinear equation. For this purpose, we consider a non-optimal method without memory developed by Ozban in [8]. Although this method is one of the most cited works in the literature, it is not optimal in the sense of Kung and Traub's conjecture. Based on this conjecture, any two-step method without memory is optimal if it has convergence order four using three functional evaluations per iteration [4, 12], while

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the pointed method uses three functional evaluations per iteration and has convergence order three, see Theorem 4.1 in [8].

There are so many two-step optimal methods without memory which we recall some of them here. To the best of our knowledge, there are three general kinds of optimal methods without memory: Jarratt-, Ostrowski- and Steffensen-type methods. Jarratt's method [2] is given by:

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.1}$$

where its error equation is $e_{n+1} = (c_2^3 - c_2c_3 + \frac{c_4}{9}c)e_n^4 + O(e_n^5)$, with $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, $k = 2, 3, ...$, and α is a simple zero of $f(x) = 0$, i.e., $f'(\alpha) \neq 0 = f(\alpha)$. Jarratt's method (1.1) uses three functional evaluations per iteration and has convergence order four so it is optimal. In other words, it uses functional evaluation of its derivation in two points, say $f'(x_n)$ and $f'(y_n)$, and one functional evaluation of the given function, says $f(x_n)$, in each iteration. Such methods in which, one uses two evaluations of the derivatives of the given functions and one evaluation of the given function are called Jarratt-type methods. Soleymani et al. [10] suggested the following optimization of Jarratt-type method:

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (1+(\frac{f(x_n)}{f'(x_n)})^3)(2-\frac{7}{4}s+\frac{3}{4}s^2)\frac{2f(x_n)}{f'(x_n)+f'(y_n)}. \end{cases} \tag{1.2}$$

Another optimal method of this type is considered by Lotfi [5]

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (2-\frac{7}{4}s+\frac{3}{4}s^2)\frac{2f(x_n)}{f'(x_n)+f'(y_n)}. \end{cases} \tag{1.3}$$

Some other optimal Jarratt-type methods and different anomalies in a Jarratt family can be found in the literature [5, 10]. Similar to Jarratt-type methods, there is another set of methods in which they use derivative of the function in each iteration. However, these kinds use two function evaluations and one derivative evaluation, say $f(x_n)$, $f(y_n)$ and $f'(x_n)$. We call these kinds of iterative methods Ostrowski-type methods. Indeed, Ostrowski's method is given by [3]

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)}, \end{cases} \tag{1.4}$$

with the following error equation $e_{n+1} = (c_2^3 - c_2c_3)e_n^4 + O(e_n^5)$.

It is worth noting that Ostrowski's method (1.4) is a special case, b=0, of King's family [3] which is defined as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n) + bf(y_n)}{f(x_n) + (b-2)f(y_n)} \dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{1.5}$$

Also, we consider the first two-step iterative method by Kung and Traub [4] as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)^2 f(y_n)}{f'(x_n)(f(x_n) - f(y_n))^2}. \end{cases}$$

There is another kind of the Ostrowski-type method which can be obtained via Hermit-interpolation as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, x_n, x_n](y_n - x_n)}. \end{cases} \tag{1.6}$$

Finally, there is another kind of optimal two-step methods without memory in which, one does not use derivatives. We call them Steffensen-type method. In what follows, we recall two of them. First, we consider the first two-step derivative-free version of Kung and Traub [4] given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, \omega_n]}, \qquad \omega_n = x_n + f(x_n), \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f[x_n, y_n]} \dfrac{f(\omega_n)}{(f(\omega_n) - f(y_n))}. \end{cases} \tag{1.7}$$

Bi et al. [9] and Zheng et al. [14] simultaneously derived the following method based on Newton interpolation

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, \omega_n]}, \qquad \omega_n = x_n + \gamma f(x_n), \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, x_n, \omega_n](y_n - x_n)}. \end{cases} \tag{1.8}$$

All of the mentioned methods can be considered as a special case of the optimal class of two-step methods without memory. Detailed description, convergence and analysis of these methods may be found in [1, 3, 5, 6, 7, 12] and references therein .

This work is organized as follows: Section 2 is devoted to extracting optimal method from non-optimal method by Ozban [8]. Furthermore, we discuss the convergence analysis of the developed method in this section, and also some concrete functions are given based on the developed method. Section 3 represents numerical implimentations and comparisons. Finally, Section 4 concludes this work. For some given methods in this work, we append their Mathematica codes, too.

2. Method and result

In this section, we deal with developing a new optimal class of Jarratt-type methods to approximate a simple zero of $f(x) = 0$. Also, we discuss a theoretical aspect of the developed class, namely convergence analysis. We recall the following method by Ozban [8]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{(f'(x_n) + f'(y_n))}{2f'(y_n)} \dfrac{f(x_n)}{f'(x_n)}, \qquad (n = 0, 1, ...). \end{cases} \tag{2.1}$$

Theorem 4.1. in [8] considers the error analysis of this method. The following self-explanatory Mathematica code decodes and deciphers the same results quickly. We introduce the following abbreviations used in this program.

$\mathtt{c_k} = \mathtt{f^{(k)}}(\alpha)/(\mathtt{k!f'}(\alpha))$, $\mathtt{e} = \mathtt{x_n} - \alpha$, $\mathtt{ey1} = \mathtt{y_n} - \alpha$, $\mathtt{ey} = \mathtt{x_{n+1}} - \alpha$, $\mathtt{f[e]} = \mathtt{f(e)}$, $\mathtt{f1a} = \mathtt{f'}(\alpha)$.

Program 1. Mathematica code:
$f[\mathrm{e}] = \mathtt{f1a}(\mathrm{e} + \mathrm{c}_2\mathrm{e}^2 + \mathrm{c}_3\mathrm{e}^3 + \mathrm{c}_4\mathrm{e}^4)$;
$\mathtt{ey1} = e - \mathtt{Series}\left[\frac{f[e]}{f'[e]}, \{e, 0, 3\}\right]$ //Simplify;
$\mathtt{ey} = e - \mathtt{Series}\left[\frac{f[e](f'[e]+f'[\mathrm{ey1}])}{2f'[e]f'[\mathrm{ey1}]}, \{e, 0, 3\}\right]$ //Simplify
$\mathtt{Out[ey]} = \frac{c_3 e^3}{2} + \mathtt{O[e]}^4$

Remark 1. As can be seen, this method is not optimal based on Kung and Traub's conjecture. It uses three functional evaluations per iteration while it has convergence order three. Here, our aim is to modify method (2.1) in such a way that it becomes optimal. More details are given in what follows.

Let us consider the following changes to (2.1). The first step of Ozban's method, namely Newton's method, is exchanged with the first step of Jarratt's method, namely weighted Newton's method. Then in the second step of Ozban's method, we introduce a weight function, say $h(t)$, as follows

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - h(t_n)\dfrac{f(x_n)}{f'(x_n)}\dfrac{(f'(x_n)+f'(y_n))}{2f'(y_n)}, \end{cases} \tag{2.2}$$

where $t_n = \frac{f'(y_n)}{f'(x_n)}$. Now, it is tried to optimize this new method. To this end, we impose some conditions on $h(t)$ so that we achieve an optimal class of Jarratt-type methods. Instead of using pencil-paper method to discuss the mentioned aim, we prefer to use the Mathematica approach. We think this technique has several advantages: it is fast, it saves space of the paper, and it avoids involving tedious and cumbersome calculations with using many terms of Taylor's series.

We reuse the symbols introduced before in giving the error equation for the method (2.1), also the rest of the abbreviations used are introduced as follows
$\mathtt{h} = h(0)$, $\mathtt{h1} = h'(0)$, $\mathtt{h2} = h''(0)$.

Program 2. Mathematica code:
$f[\mathrm{e}] = \mathtt{f1a}(\mathrm{e} + \mathrm{c}_2\mathrm{e}^2 + \mathrm{c}_3\mathrm{e}^3 + \mathrm{c}_4\mathrm{e}^4)$;
$\mathtt{ey1} = e - \mathtt{Series}\left[\frac{2*f[e]}{3f'[e]}, \{e,0,8\}\right]$;
$\mathtt{t} = \frac{\mathtt{f'[ey1]}}{f'[e]}$;
$h[\mathtt{t}] = \mathtt{h} + \mathtt{h1}t + \frac{\mathtt{h2}t^2}{2}$;
$\mathtt{ey} = \mathtt{e} - \frac{\mathtt{f[e](f'[e]+f'[ey1])*h[t]}}{\mathtt{2f'[e]f'[ey1]}}$//FullSimplify;
$\mathtt{a1} = \mathtt{Coefficient}[\mathtt{ey}, e]$//Simplify
$\mathtt{a2} = \mathtt{Coefficient}[\mathtt{ey}, e^2]$//Simplify
$\mathtt{a3} = \mathtt{Coefficient}[\mathtt{ey}, e^3]$//Simplify
$\mathtt{a4} = \mathtt{Coefficient}[\mathtt{ey}, e^4]$//Simplify

Out[a1]=1-h $-$ h1 $- \frac{\mathtt{h2}}{2}$
Out[a2] $= \frac{1}{6}(2\ \mathtt{h+10h1} + \mathtt{9h2})c_2$
Out[a3] $= \frac{1}{9}(-(2\ \mathtt{h+42h1} + \mathtt{49h2})c_2^2 + 3\ (2\ \mathtt{h+10h1} + \mathtt{9h2})c_3)$
Out[a4] $= \frac{1}{54}((20\ \mathtt{h+684h1} + \mathtt{978h2})c_2^3 - 3(22\ \mathtt{h+294h1} + \mathtt{347h2})c_2c_3$
$+ (58\ \mathtt{h+266h1} + \mathtt{237h2})c_4))$

To obtain an optimal class, the coefficients of e, e^2, and e^3 in the error equation of the new class (2.1) need to vanish. By solving the above system of equations simultaneously, equations $Out[a1]$, $Out[a2]$ and $Out[a3]$, the desired results are obtained and we have $\{\{\mathtt{h} \to \frac{7}{4}, \mathtt{h1} \to -\frac{5}{4}, \mathtt{h2} \to 1\}\}$. In other words, to provide the fourth order of convergence of the proposed method, it is necessary to choose $h = \frac{7}{4}$, $h1 = \frac{-5}{4}$ and $h2 = 1$. Therefore, we have established the following theorem about the convergence order of the optimal class (2.2).

Theorem 1. Let α be a simple zero of $f(x) = 0$ and function $h(t)$ is chosen so that the conditions $h(0) = \frac{7}{4}$, $h'(0) = \frac{-5}{4}$, and $h''(0) = 1$ hold. If an initial approximation is sufficiently close to α, then the equation (2.2) has the order of convergence four with the following error equation

$$e_{n+1} = (\frac{79}{27}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5). \tag{2.3}$$

The function $h(t)$ can take many forms satisfying the conditions of Theorem 1, examples of which are:

$$h_1(t) = \frac{7}{4} - \frac{5}{4}t + \frac{1}{2}t^2, \qquad h_2(t) = \frac{1}{\frac{4}{7} + \frac{20}{49}t + \frac{44}{343}t^2}, \qquad h_3(t) = \frac{\frac{7}{4} - \frac{1}{2}t - \frac{3}{4}t^2}{1+t}.$$

Accordingly, we can consider the following optimal method as a typical example of our proposed class

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (\frac{7}{4} - \frac{5}{4}t_n + \frac{1}{2}t_n^2)\frac{f(x_n)}{f'(x_n)}\frac{(f'(x_n) + f'(y_n))}{2f'(y_n)}. \end{cases} \tag{2.4}$$

3. Numerical implimentations

To verify the applicability of the derived method (2.2) of the optimal class of Jarratt-type methods we give two examples. Also, we report the results of the other methods given in this work for comparison. The implementations were ran in Mathematica. In Tables 1 and 2, the values of the computational order of convergence are computed by the following approximate formula (see Weerakoon and Fernando [13])

$$coc = \frac{ln(|x_{n+1} - \alpha|/|x_n - \alpha|)}{ln(|x_n - \alpha|/|x_{n-1} - \alpha|)},$$

where $|x_n - \alpha|$ denots absolute errors of approximations and $a(-b)$ means $a \times 10^{-b}$.

Example 1. Consider the following nonlinear equation

$$f(x) = e^{2+x-x^2} - \cos(1+x) + x^3 + 1, \qquad \alpha = -1,$$

with the initial approximation $x_0 = -0.7$.

Example 2. Consider the following nonlinear equation

$$f(x) = ln(1+x^2) + e^{-3x+x^2}\sin(x), \qquad \alpha = 0,$$

with the initial approximation $x_0 = 0.35$.

We have reported the obtained numerical results in Table 1 and 2. These results

Table 1: Numerical results of Example 1 in the first three iterations

	Absolute Error			
Two-point methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	coc
New Method (2.4)	0.2189(−3)	0.1566(−15)	0.4104(−64)	4
Method (1.4)	0.4557(−3)	0.2790(−14)	0.3925(−59)	4
Method (1.7)	0.4357(−1)	0.1170(−5)	0.5534(−23)	4
Method (1.1)	0.6543(−3)	0.1411(−13)	0.3056(−56)	4
Method (1.2)	0.6113(−2)	0.1490(−8)	0.5245(−35)	4

Table 2: Numerical results of Example 2 in the first three iterations

	Absolute Error			
Two-point methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	coc
New Method (2.4)	0.1965(−2)	0.2035(−9)	0.2326(−37)	4
Method (1.4)	0.5733(−2)	0.2999(−8)	0.2158(−33)	4
Method (1.7)	0.8517(−2)	0.1282(−6)	0.5757(−26)	4
Method (1.3)	0.1990(−2)	0.3071(−9)	0.1734(−36)	4
Method (1.2)	0.1948(−1)	0.3038(−5)	0.1747(−20)	4

confirm the theoretical prediction, which has been proved in the previous section. Moreover, it can be concluded that the proposed method (2.4) generates slightly better results in comparison with the other numerical methods mentioned in this paper.

4. Conclusion

In this research, a new optimal fourth order method based on Ozban's method has been developed for solving simple roots of nonlinear equations. The presented method has the convergence order four. It supports the Kung and Traub's conjecture requiring only three function evaluations per iteration and it has the efficiency index $4^{1/3} \approx 1.587$, which is better than Ozban's method $3^{1/3} \approx 1.390$ (for the definition of efficiency index see [11]).

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