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# A NEW TYPE OF CONTRACTION VIA MEASURE OF NON-COMPACTNESS WITH AN APPLICATION TO VOLTERRA INTEGRAL EQUATION

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Abstract. Darbo fixed point theorem is a powerful tool which is used in many fields in mathematics. Because of this feature, many generalizations of this theorem and its relations with other subjects have been investigated. Here we introduce a generalization of an $F$-contraction of Darbo type mapping and define a new contraction by using both function classes and uniformly convergent sequences of functions and examine some of its properties. Afterward, we show that the new type of contraction, which we call $F$-Darbo type contraction, has more general results than many already studied in the literature. Furthermore, we explain the results of $F$-Darbo type contraction mapping with an interesting example. Finally, we give an application to solve the Volterra-type integral equation with the new type contraction.

## 1. Introduction

The Kuratowski, Istratescu, and Hausdorff measures of noncompactness are the main MNCs (see [**5**, **8**, **9**, **16**]) while the axiomatic definition given by Banaś and Goebel [**16**] is the most widely used. Darbo [**7**] used the Kuratowski MNC to establish a fixed point theorem which is widely known as Darbo fixed point theorem. With the help of the Darbo fixed point theorem, the existence and uniqueness of the fixed point of a set-valued mapping has been proved. These results regarding fixed point theory have achieved a wide application area for the solution of integral, integro-differential and functional equations. The Darbo fixed point theorem has been generalized by several researchers (see [**1**–**3**, **10**, **11**, **13**, **17**, **19**]). Another important concept applied to the Darbo fixed point theorem is the $F$-contraction defined by Wardowski [**21**]. Cosentino and Vetro [**6**] proved that the Darbo contraction mapping satisfies the $F$-contraction conditions and showed that $F$-contraction mapping has a solution for integral equations.

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In this study, we define a new $F$-Darbo type contraction under functions having certain conditions. Also, we investigate how a new $F$-Darbo type contraction behaves under the sequences of functions used by Kirk [**14**], Kirk and Xu [**15**], Karakaya et al. [**12**]. Besides, we examin existence of fixed point according to the conditions of this new $F$-Darbo type contraction mapping. Considering the hypothesis of theorems that we proved, we construct an interesting example using the sequences of functions. Finally, we show that the new contraction has a solution for the Volterra-type integral equation.

## 2. Preliminaries

We will now give notations and preliminaries used in the sequel of this article. Let $A$ be nonempty subset of the Banach space. We define $\bar{A}$ and $\overline{co}(A)$ the closure and closed convex hull of $A$, respectively. Also, we denote $B(X)$ and $RC(X)$ the family of all nonempty bounded subset of $X$ and the subfamily consisting all relatively compact subset of $X$, respectively. We denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers, by $\mathbb{R}_0^+$ the set of all nonnegative real numbers and by $\mathbb{N}$ the set of all positive integers. Furthermore, let $\ker\mu = \{A \in B(X) : \mu(A) = 0\}$ denotes the kernel of the mapping $\mu\colon B(X) \to \mathbb{R}^+$ (see [**4**]).

Definition 2.1 (see [**4**]). Let $X$ be a Banach space and $B(X)$ the family of bounded subset of $X$. A map $\mu\colon B(X) \to \mathbb{R}^+$ which satisfies the following:

- (M1) The family $\ker\mu$ is nonempty and $\ker\mu \subset RC(X)$,
- (M2) $A \subset B$ implies $\mu(A) \leqslant \mu(B)$,
- (M3) $\mu(\bar{A}) = \mu(A)$,
- (M4) $\mu(\overline{co}(A)) = \mu(A)$,
- (M5) $\mu(\lambda A + (1-\lambda)B) \leqslant \lambda\mu(A) + (1-\lambda)\mu(B)$ for all $\lambda \in [0,1]$,
- (M6) Let $(A_n)$ be a sequence of closed sets in $B(X)$ such that $A_{k+1} \subset A_k$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty}\mu(A_k) = 0$, then intersection set $A_\infty = \bigcap_{k=1}^{\infty} A_k$ is nonempty and $A_\infty \subset \ker\mu$.

Theorem 2.1 (see [**18**]). *Let $A$ be a nonempty, bounded, closed and convex subset of a Banach space $X$. Let $T$ be a compact and continuous self mapping. Then $T$ has a fixed point in $A$.*

Theorem 2.2 (see [**7**]). *Let $A$ be a nonempty, bounded, closed and convex subset of a Banach space $X$. Let $T$ be a continuous self mapping on $A$. Assume that there exists a constant $\alpha \in [0,1]$ such that $\mu(TB) \leqslant \alpha\mu(B)$, for any subset $B$ of $A$, then $T$ has a fixed point.*

Now, let $F\colon \mathbb{R}^+ \to \mathbb{R}$ be a function that verifies the following conditions:

- $(F1)$ $F$ is non-decreasing,
- $(F2)$ For each sequence $\{\beta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+$ of positive numbers $\lim_{k\to\infty}\beta_k = 0$ if and only if $\lim_{k\to\infty}F(\beta_k) = -\infty$,
- $(F3)$ There exists $\alpha \in (0,1)$ such that $\lim_{\beta\to 0^+}\beta^\alpha F(\beta) = 0$.

For the function defined above, we denote with $\Lambda$ the family of functions $F$ that satisfy the conditions $(F1)$–$(F3)$ and with $\digamma$ the family of all functions $F$ that satisfy the condition $(F1)$–$(F2)$.

Note that the function $F\colon \mathbb{R}^+ \to \mathbb{R}$ defined by $F(x) = \ln x$ for all $x \in \mathbb{R}^+$ satisfies the conditions $(F1)$–$(F3)$ and hence $F \in \Lambda$. On the other hand, the function $F\colon \mathbb{R}^+ \to \mathbb{R}$ defined by $F(x) = -\frac{1}{x}$ for all $x \in \mathbb{R}^+$ satisfies the properties $(F1)$–$(F2)$, but it does not satisfy the property $(F3)$ and hence $F \in \digamma$ but $F \notin \Lambda$.

Definition 2.2 (see [**21**]). Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an $F$-contraction if there exist $F \in \Lambda$ and $\tau \in \mathbb{R}^+$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)),$$

for all $x, y \in X$.

We denote with $\Gamma$ the family of function $\tau\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition:

$$(i) \quad \lim \inf_{x \to a^+} \tau(x) > 0 \text{ for all } a \in \mathbb{R}_0^+. \tag{2.1}$$

Moreover, let $\tau_n \to \tau$ be a uniform convergence in $n$. We also denote with $\Gamma'$ the family of uniformly convergent sequences of functions $\tau_n\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition:

$$(ii) \quad \sup_n \lim \inf_{x \to a^+} \tau_n(x) > 0 \text{ for all } a \in \mathbb{R}_0^+.$$

Note that the $\tau\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\tau(x) = -\frac{1}{2x}$ for all $x \in \mathbb{R}^+$ satisfies condition $(i)$ and hence $\tau \in \Gamma$. Again, we pay attention to $\tau_n\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\tau_n(x) = \frac{n}{1+2nx}$ for all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. So, we can denote $(\tau_n) \in \Gamma'$.

Definition 2.3 (see [**20**]). Let $A$ be a non-empty, bounded, closed and convex subset of a Banach space $X$. A self-operator $T$ on $A$ is called an $F$-contraction of Darbo-type mapping if there exist $F \in \Lambda$ and $\tau \in \Gamma$ such that

$$\tau(\mu(B)) + F(\mu(TB)) \leqslant F(\mu(B)),$$

for any $B \subset A$ with $\mu(B), \mu(TB) > 0$ where $\mu$ is the measure of noncompactness defined in $X$.

Let us introduce some properties of the sequences of functions that we will use in the generalization of the Darbo contraction mapping throughout the work as follows.

Definition 2.4. Let $(\psi_n)$ be a sequence of functions from $\mathbb{R}_0^+$ into $\mathbb{R}_0^+$. This sequence converges uniformly to a function $\psi$ if for every $\varepsilon > 0$, there is an integer $n_0$ such that $|\psi_n(x) - \psi(x)| < \varepsilon$, for all $x \in \mathbb{R}_0^+$ and $n \geqslant n_0$, $n \in \mathbb{N}$.

## 3. A new *F*-Darbo type contraction defined by functions classes

In this section, firstly we define a new $F$-Darbo type contraction defined by functions classes, and then we introduce $F$-Darbo type contraction defined by sequences of functions.

Definition 3.1. Let $A$ be a non-empty, bounded, closed and convex subset of a Banach space $X$. Assume that the mapping $\psi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and satisfies $\psi(x) = 0 \Leftrightarrow x = 0$. Then, a self-mapping $T$ on $A$ is called an $F$-Darbo type contraction mapping if there exist $F \in \digamma$ and $\tau \in \Gamma$ such that

$$\tau(\mu(A)) + F(\psi(\mu(TA))) \leqslant F(\psi(\mu(A))),$$

for any $A \subset X$ with $\mu(A), \psi(\mu(A)), \psi(\mu(TA)) > 0$ where $\mu$ is measure of noncompactness defined in $X$.

Let the functions $\psi, \psi_n\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be continuous. We shall assume that $(\psi_n)$ is non-decreasing that is, $\psi_n \leqslant \psi_{n+1}$. Also it satisfies the condition

$$\psi_n(x) \leqslant \psi(x) \leqslant x, \tag{3.1}$$

for all $n \in \mathbb{N}$ and for every $x \in \mathbb{R}_0^+$.

Definition 3.2. Let $A$ be a non-empty, bounded, closed and convex subset of a Banach space $X$. Assume that $(\psi_n)$ and $(\tau_n)$ are two uniformly convergent sequences of functions such that $\psi_n \to \psi$ and $\tau_n \to \tau$. Also, let the sequence of functions $\psi_n\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be continuous. Then, a self-mapping $T$ on $A$ is called an $F$-Darbo type contraction mapping if there exist $F \in \digamma$ and $(\tau_n) \in \Gamma'$ such that

$$\tau_n(\mu(A)) + F(\psi_n(\mu(TA))) \leqslant F(\psi_n(\mu(A))), \tag{3.2}$$

for any $A \subset X$ with $\mu(A), \psi_n(\mu(A)), \psi_n(\mu(TA)) > 0$ for all $n \in \mathbb{N}$ where $\mu$ is measure of noncompactness defined in $X$.

Theorem 3.1. *Let $A$ be a nonempty, bounded, closed and convex subset of a Banach space $X$. Assume that $T$ is a compact and continuous self-mapping on $A$. Suppose that there exist $F \in \digamma$ and $\tau \in \Gamma$ such that $T$ is an $F$-Darbo type contraction mapping under the conditions of Definition* 3.1. *Then $T$ has a fixed point in $A$.*

Proof. At the first step of proof, we assume that there exists a sequence $(A_k)$ which is nonempty, closed and convex subset of $A$ such that

$$TA_k \subset A_k \subset A_{k-1} \text{ for all } k \in \mathbb{N}. \tag{3.3}$$

Now, let $A_0 = A$ and let $(A_k)$ be sequence with initial element $A_0$ such that $A_k = \overline{co}(TA_{k-1})$ for all $k \in \mathbb{N}$. From (3.3), it is easy to see that $TA_0 \subset A_0$. Again from the condition (3.3) and the definition of $(A_k)$, we have

$$TA_k \subset A_k \text{ imply } A_{k+1} = \overline{co}(TA_k) \subset A_k,$$

and after one step, we can write that $TA_{k+1} \subset TA_k \subset A_k$. From the definition of the measure of noncompactness, if there exists a number $k$ such that $\mu(A_k) = 0$ then $A_k$ is a compact set. Under the conditions of Theorem 2.1, since $T$ is compact and continuous self mapping on $A_k$, we get that $T$ has a fixed point in $A_k$ and so in $A$.

On the other hand, we suppose that $\mu(A_k) > 0$ for all $k \in \mathbb{N}$ and prove $\mu(A_k) \to 0$ as $k \to \infty$. By considering (3.3), it can be seen that $\mu(A_k)$ is decreasing and hence it converges to a real number $r \geqslant 0$.

From the property of the function $\tau$ given in (2.1), there exist $r > 0$ and $k_0 \in \mathbb{N}$ such that $\tau(\mu(A_k)) \geqslant r$ for all $k > k_0$. We consider together with (M4) of Definition 2.1, then we can write that

$$\begin{aligned}\tau(\mu(A_k)) + F(\psi(\mu(A_{k+1}))) &= \tau(\mu(A_k)) + F(\psi(\mu(\overline{co}(TA_k))))\\ &= \tau(\mu(A_k)) + F(\psi(\mu(TA_k)))\\ &\leqslant F(\psi(\mu(A_k))).\end{aligned}$$

After the calculation done above, we have

$$\begin{aligned}\tau(\mu(A_k)) + F(\psi(\mu(A_{k+1}))) &\leqslant F(\psi(\mu(A_k)))\\ F(\psi(\mu(A_{k+1}))) &\leqslant F(\psi(\mu(A_k))) - \tau(\mu(A_k))\\ F(\psi(\mu(A_{k+1}))) &\leqslant F(\psi(\mu(A_k))) - r\\ F(\psi(\mu(A_k))) &\leqslant F(\psi(\mu(A_{k-1}))) - r\\ &\vdots\end{aligned}$$

for all $k > k_0$. By with the same idea, we get

$$\begin{aligned}F(\psi(\mu(TA_k))) &\leqslant F(\psi(\mu(A_k))) \leqslant F(\psi(\mu(A_{k-1}))) - r\\ &\leqslant \ldots \leqslant F(\psi(\mu(A_{k_0}))) - (k-k_0)r,\end{aligned}$$

for all $k > k_0$ and so $\lim_{k\to\infty} F(\psi(\mu(A_k))) = -\infty$.

From (F2), we have $\lim_{k\to\infty} \psi(\mu(A_k)) = 0$, from the property of $\psi$, we get $\mu(A_k) \to 0$ as $k \to \infty$. Moreover, we obtain that since $\lim_{k\to\infty} \mu(A_k) = 0$, then the intersection set $A_\infty = \bigcap_{k=1}^{\infty} A_k$ is nonempty and $A_\infty \subset \ker \mu$. We can consider Theorem 2.1 again, hence we conclude that $T$ has a fixed point in $A_\infty$ and then in $A$. □

THEOREM 3.2. *Let $A$ be a nonempty, bounded, closed and convex subset of a Banach space $X$. Assume that $T$ is a compact and continuous self-mapping on $A$. Suppose that there exist $F \in \digamma$ and $(\tau_n) \in \Gamma'$ such that $T$ is an $F$-Darbo type contraction mapping under the conditions of Definition* 3.2. *Then $T$ has a fixed point in $A$.*

PROOF. Since the set iteration, which is the first part of this theorem is similar to the proof given in Theorem 3.1, we omit it. Now, we assume that $\sup_n \tau_n(\mu(A_k)) \geqslant r$ for all $k > k_0$. For all $n, k \in \mathbb{N}$, we have

$$\begin{aligned}\tau_n(\mu(A_k)) + F(\psi_n(\mu(A_{k+1}))) &= \tau_n(\mu(A_k)) + F(\psi_n(\mu(\overline{co}(TA_k))))\\ &= \tau_n(\mu(A_k)) + F(\psi_n(\mu(TA_k)))\\ &\leqslant F(\psi_n(\mu(A_k))).\end{aligned}$$

After this step, we get

$$\begin{aligned}\tau_n(\mu(A_k)) + F(\psi_n(\mu(A_{k+1}))) &\leqslant F(\psi_n(\mu(A_k)))\\ F(\psi_n(\mu(A_{k+1}))) &\leqslant F(\psi_n(\mu(A_k))) - \tau_n(\mu(A_k))\\ F(\psi_n(\mu(A_{k+1}))) &\leqslant F(\psi_n(\mu(A_k))) - r\end{aligned}$$

$$F(\psi_n(\mu(A_k))) \leqslant F(\psi_n(\mu(A_{k-1}))) - r$$
$$F(\psi_n(\mu(A_{k-1}))) \leqslant F(\psi_n(\mu(A_{k-2}))) - r$$
$$\vdots$$

Therefore, we obtain that $F(\psi_n(\mu(TA_k))) \leqslant F(\psi_n(\mu(A_{k_0}))) - (k-k_0)r$, for all $k > k_0$. As a result, we get $\lim_{k\to\infty} F(\psi_n(\mu(A_k))) = -\infty$. From $(F2)$, we have $\lim_{k\to\infty} \psi_n(\mu(A_k)) = 0$. Also, since $\lim_{k\to\infty} \psi_n(\mu(A_k)) = 0$ and $\psi_n \to \psi$ uniformly in $n$, we obtain $\lim_{k\to\infty} \psi_n(\mu(A_k)) = 0$ as $n \to \infty$. Here, by using property of $\psi$ in Definition 3.1, we get $\mu(A_k) \to 0$ as $k \to \infty$. Moreover, we obtain that since $lim_{k\to\infty}\mu(A_k) = 0$, then the intersection set $A_\infty = \bigcap_{k=1}^{\infty} A_k$ is nonempty and $A_\infty \subset \ker\mu$. Hence, we obtain that $T$ has a fixed point in $A_\infty$ and $A_\infty \subset A$. □

Example 3.1. We will establish an example under condition (3.1). Now, we consider the following sequences of functions

$$\psi_n(x) = \frac{nx}{1+2n}, \quad \tau_n(x) = \frac{1+2nx}{n},$$

where $\psi_n \to \psi$ and $\tau_n \to \tau$. Also, it is easy to see $\psi_n(x) \leqslant \psi(x) \leqslant x$ for all $n \in \mathbb{N}$. Let $F\colon \mathbb{R}^+ \to \mathbb{R}$ be a mapping given by $F(x) = \ln x$. By using (3.2), we have

$$\frac{1+2n\mu(A)}{n} + \ln\Big(\frac{n\mu(TA)}{1+2n}\Big) \leqslant \ln\Big(\frac{n\mu(A)}{1+2n}\Big)$$
$$\ln e^{\frac{1+2n\mu(A)}{n}} + \ln\Big(\frac{n\mu(TA)}{1+2n}\Big) \leqslant \ln\Big(\frac{n\mu(A)}{1+2n}\Big).$$

Then, we can write the following inequality:

$$e^{\frac{1+2n\mu(A)}{n}}\Big(\frac{n\mu(TA)}{1+2n}\Big) \leqslant \frac{n\mu(A)}{1+2n}.$$

It is clear that the inequality verifies for every $n \in \mathbb{N}$. If we take limit over $n$, we get Darbo contraction [**7**] as follows:

$$\mu(TA) \leqslant \frac{1}{e^{2\mu(A)}}\mu(A).$$

Corollary 3.1. *In Definition 3.2, if we take $\psi_n \to \psi$, $\tau_n \to \tau$ uniformly in $n$ and $\psi(x) = x$, then we obtain the results given in* [**20**].

## 4. An application of the new $F$-Darbo type contraction to Volterra-type integral equation

We first consider a Volterra-type integral. After, we define that a mapping $T$ on $BC(\mathbb{R}_0^+)$ has an $F$-Darbo type contraction with the aid of uniformly convergent sequences of functions. At the same time, we show that the Volterra-type integral equation has a solution with this new $F$-Darbo type contraction. Hence, we obtain that $T$ has a fixed point.

There are many Volterra-type integral equations in the literature. We use the following form

$$\nu(x) = f(x, \nu(x)) + \int_0^x M(x, s, \nu(s))\, ds, \tag{4.1}$$

where $M\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ and $f\colon \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $BC(\mathbb{R}_0^+)$ denote the space of all bounded and continuous functions on $\mathbb{R}_0^+$. Also, we consider the norm on space $BC(\mathbb{R}_0^+)$

$$\|\nu\| = \sup\{|\nu(x)|,\ x \in \mathbb{R}_0^+\}.$$

Now, let us establish the modulus of continuity of the functional $\nu$ on $[0, H]$.

Let $A$ be a nonempty bounded subset on $BC(\mathbb{R}_0^+)$ and $H \in \mathbb{R}_0^+$. Therefore, for $\nu \in A$, we define the modulus of continuity as follows:

$$\gamma^H(\nu, \varepsilon) = \sup\{|\nu(x) - \nu(s)| : x,\ s \in [0, H],\ |x - s| \leqslant \varepsilon\}.$$

However, we can show

$$\gamma^H(A, \varepsilon) = \sup\{\gamma^H(\nu, \varepsilon) : \nu \in A\} \text{ and } \gamma_0^H(A) = \lim_{\varepsilon \to 0} \gamma_0^H(A, \varepsilon).$$

Hence, we obtain that $\gamma_0(A) = \lim_{H \to +\infty} \gamma_0^H(A)$. Also, for fixed $x \in \mathbb{R}_0^+$, we can write $A(x) = \{\nu(x) : \nu \in A\}$. Now, we can define the measure of noncompactness on the family of all nonempty bounded subset of $BC(\mathbb{R}_0^+)$ as follows:

$$\mu(A) = \gamma_0(A) + \limsup_{x \to \infty} \operatorname{diam} A(x), \tag{4.2}$$

where $\operatorname{diam} A(x) = \sup\{|\nu(x) - \eta(x)| : \eta, \nu \in A\}$.

Let us consider the operator $T$ on $BC(\mathbb{R}_0^+)$ defined by

$$(T\nu)(x) = f(x, \nu(x)) + \int_0^x M(x, s, \nu(s))\, ds, \tag{4.3}$$

for all $x \in \mathbb{R}_0^+$.

After these preliminaries, we have to show that existence of a solution of (4.1) is equivalent to the problem of existence of a fixed point of (4.3).

Using the explanation mentioned above, we can give the following theorem.

THEOREM 4.1. *Let $T$ be an operator on $BC(\mathbb{R}_0^+)$ defined by* (4.3) *and assume that the following conditions are satisfied:*

(i) *the function $x \to f(x, 0)$ is an element of the space $BC(\mathbb{R}_0^+)$.*

(ii) *let $\psi_n \to \psi$ be a sequence of functions on $\mathbb{R}_0^+$ and uniform convergence in $n$ and $\psi_n(x) \leqslant \psi(x) \leqslant x$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}_0^+$. Also $\tau_n\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a sequence of functions and $\tau_n \to \tau$ uniform convergence in $n$. Therefore, there exists $(\tau_n) \in \mathbb{R}_0^+$ for all $n \in \mathbb{N}$ such that for each $x \in \mathbb{R}_0^+$ and for all $\nu, \eta \in \mathbb{R}$, hence we have*

$$|f(x, \nu) - f(x, \eta)| \leqslant e^{-\tau_n(x)} \psi_n(|\nu - \eta|).$$

(iii) *there exist continuous functions* $\vartheta, \xi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *such that*

$$\lim_{x\to+\infty} \vartheta(x) \int_0^x \xi(s)\, ds = 0, \quad \text{and} \quad |M(x,s,\nu(s))| \leqslant \vartheta(x)\xi(s),$$

*for all* $x, s \in \mathbb{R}_0^+$ *such that* $s \leqslant x$ *and for all* $\nu \in \mathbb{R}$.

(iv) *there exists a positive* $r_0 > 0$ *and* $\ell$ *where*

$$\ell = \sup_{x\geqslant 0} \left\{ |f(x,0)| + \vartheta(x) \int_0^x \xi(s)\, ds \right\}.$$

*Then* $T$ *has a fixed point in* $BC(\mathbb{R}_0^+)$.

Proof. In the first step, we have to show that the operator $T$ is well-defined and continuous on $D(r_0) = \{\nu \in BC(\mathbb{R}_0^+) : \|\nu\| \leqslant r_0\}$. From (4.3) and by the conditions on $f$ and $M$, we infer that $(T\nu)$ is continuous for $\nu \in BC(\mathbb{R}_0^+)$. So, we have

$$\begin{aligned}
|(T\nu)(x)| &= \left| f(x,\nu(x)) - f(x,0) + f(x,0) + \int_0^x M(x,s,\nu(s))\, ds \right| \\
&\leqslant |f(x,\nu(x)) - f(x,0)| + |f(x,0)| + \left| \int_0^x M(x,s,\nu(s))\, ds \right| \\
&\leqslant e^{-\tau_n(x)} \psi_n(|\nu(x)|) + |f(x,0)| + \vartheta(x) \int_0^x \xi(s)\, ds \\
&\leqslant e^{-\tau_n(x)} \psi_n(|\nu(x)|) + \ell
\end{aligned}$$

where $\ell$ is given by condition (iv). To show that $T(D(r_0)) \subset D(r_0)$, we take $J = \sup_n e^{-\tau_n(x)}$ such that $J < 1$, $\ell = r_0(1-J)$ and $\psi_n(\|\nu\|) \leqslant \|\nu\|$ for all $n \in \mathbb{N}$. Also, taking supremum according to $x$ and considering continuity $\psi_n$ and $\|\nu\| \leqslant r_0$, we have

$$\begin{aligned}
\|T\nu\| &\leqslant e^{-\tau_n(x)} \psi_n(\|\nu\|) + \ell \leqslant e^{-\tau_n(x)} \|\nu\| + \ell \\
&\leqslant \sup_n e^{-\tau_n(x)} \|\nu\| + \ell \leqslant J\|\nu\| + \ell \leqslant Jr_0 + \ell \leqslant r_0.
\end{aligned}$$

Hence, it can be seen that $T$ defines from $D(r_0)$ into $D(r_0)$. Now, let us take $\varepsilon > 0$ so that, $\psi_n(\|\nu - \eta\|) \leqslant \|\nu - \eta\| \leqslant \varepsilon$, we get

$$\begin{aligned}
|(T\nu)(x) - (T\eta)(x)| &\leqslant e^{-\tau_n(x)} \psi_n(|\nu(x) - \eta(x)|) \\
&\qquad + \int_0^x |M(x,s,\nu - \eta(s)) - M(x,s,\nu - \eta(s))|\, ds \\
&\leqslant e^{-\tau_n(x)} \psi_n(|\nu(x) - \eta(x)|) \\
&\qquad + \int_0^x |M(x,s,\nu(s))|\, ds + \int_0^x |M(x,s,\eta(s))|\, ds.
\end{aligned}$$

By using condition (ii) and for all $x \in \mathbb{R}_0^+$, we also have

$$|(T\nu)(x) - (T\eta)(x)| \leqslant e^{-\tau_n(x)} \psi_n(|\nu(x) - \eta(x)|) + 2\vartheta(x) \int_0^x \xi(s)\, ds. \tag{4.4}$$

Besides, by condition (iii), there exists a positive number $H$ such that

$$2\vartheta(x)\int_0^x \xi(s)\,ds < \varepsilon \quad \text{for all } x \geqslant H. \tag{4.5}$$

As a result, from (4.4) and (4.5), we make inference that

$$|(T\nu)(x) - (T\eta)(x)| \leqslant e^{-\tau_n(x)}\psi_n(|\nu(x) - \eta(x)|) + 2\vartheta(x)\int_0^x \xi(s)\,ds < 2\varepsilon, \tag{4.6}$$

for all $x \geqslant H$. Then, by using the modulus of continuity mentioned above, we can write

$$\begin{aligned}\gamma^H(M,\varepsilon) = \sup\{&|M(x,s,\nu(s)) - M(x,s,\eta(s))| : x,s \in [0,H],\\ &\nu,\eta \in [-r_0,r_0], |v-\eta| \leqslant \varepsilon\}.\end{aligned}$$

Since $M(x,s,\nu(s))$ is a uniformly continuous function on $[0,H]\times[0,H]\times[-r_0,r_0]$, we conclude that $\lim_{\varepsilon\to 0}\gamma^H(M,\varepsilon) = 0$. Again from (4.4), for an arbitrarily $x \in [0,H]$, we have

$$\begin{aligned}|(T\nu)(x) - (T\eta)(x)| &\leqslant e^{-\tau_n(x)}\psi_n(|\nu(x) - \eta(x)|)\\ &\quad + \int_0^x |M(x,s,\nu(s)) - M(x,s,\eta(s))|\,ds\\ &\leqslant \varepsilon + \int_0^x \gamma^H(M,\varepsilon)\,ds = H\gamma^H(M,\varepsilon) + \varepsilon.\end{aligned}$$

By considering property of $\gamma^H(M,\varepsilon)$ and from (4.6), we get that the operator $T$ is continuous on $D(r_0)$.

At this step of the proof, we prove that $T$ has a fixed point in $D(r_0)$. Now as the begining the proof, let $A$ be a nonempty subset $D(r_0)$, fixed $\varepsilon > 0$ and $H > 0$, and taking $x,s \in [0,H]$ such that $|x-s| \leqslant \varepsilon$. Also, we have

$$\begin{aligned}|(T\nu)(x) - (T\nu)(s)| &\leqslant |f(x,\nu(x)) - f(s,\nu(s))|\\ &\quad + \left|\int_0^x M(x,r,\nu(r))\,dr - \int_0^s M(s,r,\nu(r))\,dr\right|\\ &\leqslant |f(x,\nu(x)) - f(s,\nu(x))| + |f(s,\nu(x)) - f(s,\nu(s))|\\ &\quad + \int_0^x |M(x,r,\nu(r)) - M(s,r,\nu(r))|\,dr\\ &\quad + \int_s^r M(s,r,\nu(r))\,dr\\ &\leqslant \gamma_1^H(f,\varepsilon) + e^{-\tau_n(x)}\psi_n(\gamma^H(\nu,\varepsilon)) + \int_0^x \gamma_1^H(M,\varepsilon)\,dr\\ &\quad + \vartheta(x)\int_s^x \xi(r)\,dr \leqslant \gamma_1^H(f,\varepsilon) + e^{-\tau_n(x)}\psi_n(\gamma^H(\nu,\varepsilon))\\ &\quad + H\gamma_1^H(f,\varepsilon) + \varepsilon\sup\{\vartheta(s)\xi(x) : x,s \in [0,H]\},\end{aligned} \tag{4.7}$$

where

$$\gamma_1^H(f,\varepsilon) = \sup\{|f(x,\nu) - f(s,\nu)| : x, s \in [0,H],\ \nu \in [-r_0, r_0],\ |x-s| \leqslant \varepsilon\},$$

$$\gamma_1^H(M,\varepsilon) = \sup\{|M(x,r,\nu) - M(s,r,\nu)| : x, s, r \in [0,H],\ \nu \in [-r_0, r_0],\ |x-s| \leqslant \varepsilon\},$$

since $f$ and $M$ are uniform continuity, we can write $\lim_{\varepsilon\to 0}\gamma_1^H(f,\varepsilon) = 0$ and $\lim_{\varepsilon\to 0}\gamma_1^H(M,\varepsilon) = 0$. Furthermore, since $\vartheta$ and $\xi$ are two continuous functions on $\mathbb{R}_0^+$, we conclude that $\sup\{\vartheta(s)\xi(x) : x, s \in [0,H]\}$ is finite. With these facts, the inequality in (4.7) implies that $\psi_n(\gamma_0^H(TA)) \leqslant \lim_{\varepsilon\to 0} e^{-\tau_n(x)}\psi_n(\gamma^H(A,\varepsilon))$. It follows that $\psi_n(\gamma_0^H(TA)) \leqslant e^{-\tau_n(x)}\psi_n(\gamma_0^H(A))$ and hence

$$\psi_n(\gamma_0(TA)) \leqslant e^{-\tau_n(x)}\psi_n(\gamma_0(A)). \tag{4.8}$$

From property (3.1), we have

$$\begin{aligned}\psi_n(|(T\nu)(x) - (T\eta)(x)|) &\leqslant |(T\nu)(x) - (T\eta)(x)| \leqslant |f(x,\nu(x)) - f(x,\eta(x))| \\ &\quad + \int_0^x |M(x,s,\nu(s))|\,ds + \int_0^x |M(x,s,\eta(s))|\,ds \\ &\leqslant e^{-\tau_n(x)}\psi_n(|\nu(x) - \eta(x)|) + 2\vartheta(x)\int_0^x \xi(s)\,ds.\end{aligned}$$

Using the notation of diameter of a set, we deduce that

$$\psi_n(\operatorname{diam}(TA)(x)) \leqslant e^{-\tau_n(x)}\psi_n(\operatorname{diam} A(x)) + 2\vartheta(x)\int_0^x \xi(s)\,ds,$$

and so we get

$$\psi_n\Big(\limsup_{x\to\infty}\operatorname{diam}(TA)(x)\Big) \leqslant e^{-\tau_n(x)}\psi_n\Big(\limsup_{x\to\infty}\operatorname{diam}(A)(x)\Big). \tag{4.9}$$

Let us take $e^{-\tau_n(\mu(A))}$ as $x = \mu(A)$. Also, combining (4.2), (4.8) and (4.9) together with condition (3.1), we have $\psi_n(\mu(TA)) \leqslant e^{-\tau_n(\mu(A))}\psi_n(\mu(A))$. By applying to logarithms, we can write this inequality as

$$\ln(\psi_n(\mu(TA))) \leqslant \ln(e^{-\tau_n(\mu(A))}\psi_n(\mu(A))),$$

for all $A \subset BC(\mathbb{R}_0^+)$ with $\mu(A)$, $\mu(TA) > 0$, and after calculations, we obtain that $\tau_n(\mu(A)) + \ln(\psi_n(\mu(TA))) \leqslant \ln(\psi_n(\mu(A)))$, for all $A \subset BC(\mathbb{R}_0^+)$ with $\mu(A)$, $\mu(TA) > 0$.

As a result, it can be seen that $T$ is an $F$-Darbo type contraction mapping with sequences of functions. Hence $T$ has a fixed point in $D(r_0)$ which solves the Volterra integral equation given by (4.1) on $BC(\mathbb{R}_0^+)$. □

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