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Some Special Legendre Mates of Spherical Legendre Curves

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Abstract.: In this study, we consider some special mates of spherical Legendre curves by using Legendre frame along spherical front or frontal on Euclidean unit sphere. In this sense, we define orthogonal-type and parallel-type spherical Legendre mates. After, we get some characterizations between Legendre curvatures of orthogonal-type Legendre mates. In particularly, when spherical front is a regular curve, we give relationships between Legendre and Frenet frame or Legendre and Sabban frame. Moreover, we obtain that the evolute and the involute of spherical front correspond to a second and third orthogonal-type Legendre mate of spherical front, respectively. After, we give examples for first, second or third orthogonal-type spherical Legendre mates of some special spherical fronts or frontals. Finally, we show that there is no parallel-type Legendre mates of spherical frontal.

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1. Introduction

In the field of mathematics, constructing of algebraic and geometric structures which have similar or certain common features has been always a matter of curiosity. In particular, in classic theory of curves of differential geometry, ”examining to partner curves that have a certain common feature and obtaining characterizations between partner curves” has remained popular to this day.

In this sense, in the category of partner curves which are defined according to whether any two Frenet vector fields are parallel (specifically linearly dependent or coincident) or orthogonal at corresponding points of regular curves in Euclidean 3-space, as well-known classic example: ”The pair of regular curves whose tangent line at each point of the reference curve coincides with the normal line of the other curve and whose tangents are orthogonal at corresponding points, are called an involute-evolute curves. Then, the reference curve is called the evolute of the other curve and also, the other curve is called involute of the reference curve. So, evolute of involute of a curve is itself. Also, the locus of the centers of osculating spheres (circles) of a regular spatial (planar) curve is called evolute [16]."

A parametrized differentiable curve whose first derivative (i.e. velocity vector) is non-zero at each point is called regular. If the first derivative of a curve is zero at any point, then this point is called singular point of the curve. Also, if the first and second derivative of a curve is linear dependent (or equivalently, its curvature is zero) at any point, then this point is called inflection point of the curve. Moreover, in singular points, curvature and torsion are not defined since the tangent vectors of the curve are not continuous. So, Frenet frame of the curve is not well defined at the singular points. However, a Frenet-like moving frame along the curve is defined, which is also well-defined at singular points.

Now, let's give about some studies about singular curves from perspective of Legendre singularity. In [1], by using Legendre singularity theory, Arnold introduce spherical geometry of fronts as singular curves in unit sphere. Also, in [19], Uribe-Vargas give the singularity theory of caustics and fronts in unit sphere. Especially, in [17], Takahashi gives more explicitly theory of the spherical front (or frontal) as singular curves in the unit sphere. He define a moving frame along spherical front (or frontal) of Legendre curve and so, give the curvature of Legendre curve as the geodesic curvature of regular spherical curve in the unit sphere. Moreover, as a generalization of evolute of regular spherical curves, he define evolute of spherical front (or frontal) by using moving frame of the front (or frontal) and the curvature of spherical Legendre curve. Some remarkable articles about front or frontal are also [3-, 11-, 18, 20].

In this study, by using the concept in [17] and as inspired from partner curves of regular curves in classic theory, we consider orthogonal-type and parallel-type spherical Legendre mates of spherical Legendre curves with respect to moving frame (Legendre frame) along a front or frontal in the unit sphere. This study is designed as follows. In section 2, we give brief theory of regular and singular spherical curves in Euclidean 3-space $\mathbb{R}^3$. After, we give the concept about spherical Legendre curves (or Legendre immersions) and its spherical frontal (or spherical front) as singular spherical curves. Moreover, when a spherical front is especially regular, we obtain relationships between Legendre frame and Frenet frame or Sabban frame. In section 3, we introduce three kinds of orthogonal type Legendre mates of spherical Legendre immersion by using Legendre frame along its spherical front or frontal on unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. We give some characterizations between Legendre curvatures of orthogonal-type Legendre mates. Especially, we obtain that evolute and involute of spherical front curve correspond to second and third orthogonal-type Legendre mate of spherical front, respectively. After, we give examples for first, second or third orthogonal-type spherical Legendre mates of some special spherical fronts or frontals on $\mathbb{S}^2$. In section 4, we define three kinds of parallel type Legendre mates of spherical Legendre curve by using Legendre frame along its spherical frontal on $\mathbb{S}^2$. Frontal partner curves on unit sphere which is defined in [9] are correspond to parallel-type Legendre mates. But, the author did not taken into account the condition that the frontal partner curve must be spherical Legendre curve on unit sphere and the Proof of Theorems is stated with respect to Euclidean 3-space instead of $\mathbb{S}^2$. Consequently, in the sense of frontal partner curve on unit sphere, we proof that there is no parallel-type spherical Legendre mate of a spherical frontal on $\mathbb{S}^2$.

2. PRELIMINARY

Let $\mathbb{R}^3$ denote the 3-dimensional Euclidean space, that is, the 3-dimensional real vector space with the standard inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{3} x_i y_i$ for all $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Also, the norm of a vector $\boldsymbol{x} \in \mathbb{R}^3$ is defined by $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the standard orthonormal basis where $\boldsymbol{e}_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$ is an unit vector in $\mathbb{R}^3$ for $i, j = 1, 2, 3$ such that δ_{ij} is the Kronecker delta function. Then the cross product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ on $\mathbb{R}^3$ is given by $\boldsymbol{x} \times \boldsymbol{y} = (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1)$.

2.1. **Regular Spherical Curves.** Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a regular curve for open interval $I \subset \mathbb{R}$. Namely, we have $\dot{\boldsymbol{\alpha}}(t) \neq 0$ for all $t \in I$, where $\dot{\boldsymbol{\alpha}}(t) = \frac{d\boldsymbol{\alpha}}{dt}(t)$. Then $\boldsymbol{\alpha}$ is called unit speed curve with arc-length parameter s, if $\|\dot{\boldsymbol{\alpha}}(s)\| = 1$ for all s. Now, if $\boldsymbol{\alpha}$ is a non-unit speed curve with arbitrary parameter t, then the tangent vector, the principal normal vector and the binormal vector of $\boldsymbol{\alpha}$ is defined by $\boldsymbol{t}(t) = \frac{\dot{\boldsymbol{\alpha}}(t)}{\|\dot{\boldsymbol{\alpha}}(t)\|}$, $\boldsymbol{n}(t) = \boldsymbol{b}(t) \times \boldsymbol{t}(t)$, $\boldsymbol{b}(t) = \frac{\dot{\boldsymbol{\alpha}}(t) \times \ddot{\boldsymbol{\alpha}}(t)}{\|\dot{\boldsymbol{\alpha}}(t) \times \ddot{\boldsymbol{\alpha}}(t)\|}$, respectively. Then $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is called Frenet frame of $\boldsymbol{\alpha}$ and well known Frenet formulas are given by

$$\begin{pmatrix} \dot{\boldsymbol{t}} \\ \dot{\boldsymbol{n}} \\ \dot{\boldsymbol{b}} \end{pmatrix} = \|\dot{\boldsymbol{\alpha}}\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}, \tag{2.1}$$

where the curvature κ and the torsion τ of the curve $\boldsymbol{\alpha}$ are given by $\kappa = \frac{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|}{\|\dot{\boldsymbol{\alpha}}\|^3}$, $\tau = \frac{\det(\dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}})}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|^2}$.

Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{S}^2$ be a non-unit speed regular spherical curve on the unit sphere $\mathbb{S}^2 = \left\{ x \in \mathbb{R}^3 \mid \|x\| = 1 \right\}$. Since $\|\boldsymbol{\alpha}(t)\| = 1$ and $\|\boldsymbol{t}(t)\| = 1$ for each $t \in I$, we have the unit vector $\boldsymbol{e}(t) = \boldsymbol{\alpha}(t) \times \boldsymbol{t}(t)$ on $\mathbb{S}^2$, which are orthogonal to both $\boldsymbol{\alpha}$(t) and $\boldsymbol{t}(t)$. So, $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{e}\}$ is an orthonormal frame of $\mathbb{R}^3$ and it is called Sabban (or curve-surface) frame along $\boldsymbol{\alpha}$ on $\mathbb{S}^2$. Then, we get the following Frenet type formulas are given by

$$\begin{pmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{t}} \\ \dot{\boldsymbol{e}} \end{pmatrix} = \|\dot{\boldsymbol{\alpha}}\| \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{t} \\ \boldsymbol{e} \end{pmatrix}, \tag{2.2}$$

where the geodesic curvature κ_g of the curve $\boldsymbol{\alpha}$ are given by $\kappa_g = \frac{\det(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}})}{\|\dot{\boldsymbol{\alpha}}\|^3}$.

Remark 2.1. *Let's give that the following well-known characterizations for regular curves in $\mathbb{R}^3$, whose curvatures κ and torsion τ or regular spherical curves whose geodesic curvature κ_g: It is an unit speed spherical curve which is lying on a sphere with radius ρ if and only if $\left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \left(\frac{1}{\tau}\right)\right)^2 = \rho^2$. In particular, it is a great circle (geodesic) on a sphere with radius ρ if and only if $\kappa = \frac{1}{\rho}$ and $\tau = 0$ (or equivalently, $\kappa_g = 0$) [16]. It is a general helix if and only if the ratio $\frac{\tau}{\kappa}$ is a constant [10,16]. It is an unit speed slant helix if and only if $\frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{(\tau^2 + \kappa^2)^{3/2}}$ is a constant [8].*

2.2. **Singular Spherical Curves.** In this subsection, we give the concept about spherical Legendre curves (or Legendre immersion) and its spherical frontal (or front) curve. For more detail and background about Legendre singularity theory and singular spherical curves (see [1, 2, 7, 17, 19]).

Definition 2.1. *Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a space curve. If $\dot{\boldsymbol{\alpha}}(t_0) = 0$ at $t_0 \in I$ then t_0 is called a singular point of the curve $\boldsymbol{\alpha}$. Also, if $\dot{\boldsymbol{\alpha}}(t_0) \times \ddot{\boldsymbol{\alpha}}(t_0) = 0$ (i.e. $\kappa(t_0) = 0$) at $t_0 \in I$ then t_0 is called an inflection point of the curve $\boldsymbol{\alpha}$.*

By Definition 2.1, we see that the Frenet Frame of any space curve is not well-defined at any singular or inflection point of the curve. Now, we give the following useful concepts for regular or singular spherical curves from perspective of Legendre curves. Let us take the set $\Delta = \left\{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0\right\}$ as a 3-dimensional manifold.

Definition 2.2. *$(\boldsymbol{\alpha}, \boldsymbol{\vartheta}) : I \to \Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ is called a spherical Legendre curve, if $\langle \dot{\boldsymbol{\alpha}}(t), \boldsymbol{\vartheta}(t) \rangle = 0$ for all $t \in I$. Then, the curve $\boldsymbol{\alpha} : I \to \mathbb{S}^2$ is called a spherical frontal and the curve $\boldsymbol{\vartheta} : I \to \mathbb{S}^2$ is called a dual of $\boldsymbol{\alpha}$.*

Definition 2.3. *Let the spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta}) : I \to \Delta$ be an immersion (i.e. that $(\dot{\boldsymbol{\alpha}}(t), \dot{\boldsymbol{\vartheta}}(t)) \neq 0$ for all $t \in I$). Then, $\boldsymbol{\alpha}$ is called a spherical front.*

Let $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ be a spherical Legendre curve. Unlike the Frenet frame of the curve $\boldsymbol{\alpha}$, if the spherical frontal $\boldsymbol{\alpha}$ have singular points then we can construct a well-defined moving frame along the spherical frontal $\boldsymbol{\alpha}$. Now, we can define $\boldsymbol{\mu}(t) = \boldsymbol{\alpha}(t) \times \boldsymbol{\vartheta}(t)$ is an unit vector on $\mathbb{S}^2$, which are orthogonal to both $\boldsymbol{\alpha}$ and $\boldsymbol{\vartheta}$. Thus, $\{\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{\mu}\}$ is an orthonormal frame of $\mathbb{R}^3$ and it is called the Legendre frame along the spherical frontal $\boldsymbol{\alpha}$ on $\mathbb{S}^2$. Then, we get the Legendre Frenet type formulas of the frontal $\boldsymbol{\alpha}$ are given by

$$\dot{\boldsymbol{\alpha}}(t) = \mathfrak{m}(t)\boldsymbol{\mu}(t), \quad \dot{\boldsymbol{\vartheta}}(t) = \mathfrak{n}(t)\boldsymbol{\mu}(t), \quad \dot{\boldsymbol{\mu}}(t) = -\mathfrak{m}(t)\boldsymbol{\alpha}(t) - \mathfrak{n}(t)\boldsymbol{\vartheta}(t), \tag{2.3}$$

where the pair of functions $(\mathfrak{m}, \mathfrak{n}) = \left(\left\langle \dot{\boldsymbol{\vartheta}}, \boldsymbol{\mu} \right\rangle, \langle \dot{\boldsymbol{\alpha}}, \boldsymbol{\mu} \rangle\right)$ is the Legendre curvature of the spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$.

Remark 2.2. *Let's note that: If t_0 is a singular point of the $\boldsymbol{\alpha}$ (respectively, the $\boldsymbol{\vartheta}$), then $\mathfrak{m}(t_0) = 0\,(respectively,\, \mathfrak{n}(t_0) = 0)$. If $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ is a spherical Legendre immersion, then $(\mathfrak{m}(t), \mathfrak{n}(t)) \neq (0, 0)$ for all $t \in I$. Especially, if $\mathfrak{n}(t_0) = 0$ then t_0 is called an inflection point of the front $\boldsymbol{\alpha}$.*

Moreover, for congruent Legendre curves and the existence-uniqueness theorem of Legendre curves, see [17]. Now, let us give the following definitions of the evolute of a spherical front ([17]) and the involute of a spherical front ([14]).

Definition 2.4. *Let the Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta}) : I \to \Delta$ be an immersion. Then, the evolute $\mathcal{E}v(\boldsymbol{\alpha}) : I \to \mathbb{S}^2$ of the spherical front $\boldsymbol{\alpha}$ is defined by*

$$\mathcal{E}v(\boldsymbol{\alpha})(t) = \pm \frac{\mathfrak{n}(t)}{\sqrt{\mathfrak{m}^2(t) + \mathfrak{n}^2(t)}} \boldsymbol{\alpha}(t) \mp \frac{\mathfrak{m}(t)}{\sqrt{\mathfrak{m}^2(t) + \mathfrak{n}^2(t)}} \boldsymbol{\vartheta}(t).$$

Definition 2.5. *Let the $(\boldsymbol{\alpha}, \boldsymbol{\vartheta}) : I \to \Delta$ be a Legendre curve (or Legendre immersion). Then, the involute $\mathcal{I}nv(\boldsymbol{\alpha}, t_0) : I \to \mathbb{S}^2$ of the spherical frontal (or front) $\boldsymbol{\alpha}$ at $t_0 \in I$ is defined by* $\mathcal{I}nv(\boldsymbol{\alpha}, t_0)(t) = \cos\left(\int\limits_{t_0}^{t} \mathfrak{m}(t)dt\right)\boldsymbol{\alpha}(t) - \sin\left(\int\limits_{t_0}^{t} \mathfrak{m}(t)dt\right)\boldsymbol{\mu}(t).$

2.3. **Regular Spherical Fronts.** In this subsection, when $\boldsymbol{\alpha}$ is a regular spherical curve as a special case, we give relationships between Legendre apparatus $\{\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{\mu}, \mathfrak{m}, \mathfrak{n}\}$ and Frenet apparatus $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}, \kappa, \tau\}$ (or Sabban apparatus $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{e}, \kappa_g\}$) of the curve $\boldsymbol{\alpha}$.

By using the formulas (2.1), (2.2) and (2.3), we get easily the following corollaries.

Corollary 2.1. *Let $\boldsymbol{\alpha}$ be a regular spherical front with the Frenet apparatus $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}, \kappa, \tau\}$ and the Legendre apparatus $\{\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{\mu}, \mathfrak{m}, \mathfrak{n}\}$, then the relationship formulas are obtained by*

$$\begin{cases} \boldsymbol{t} = \varepsilon \boldsymbol{\mu}, \\ \boldsymbol{n} = -\varepsilon \left(\frac{\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}} \boldsymbol{\alpha} + \frac{\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}} \boldsymbol{\vartheta} \right), \\ \boldsymbol{b} = \frac{\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}} \boldsymbol{\alpha} - \frac{\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}} \boldsymbol{\vartheta}, \end{cases}$$

where $\varepsilon = \det(\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{t})$ and

$$\begin{cases} \kappa = \dfrac{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}{|\mathfrak{m}|} = \sqrt{1 + \left(\dfrac{\mathfrak{n}}{\mathfrak{m}}\right)^2}, \\ \tau = \dfrac{\mathfrak{m}\dot{\mathfrak{n}} - \dot{\mathfrak{m}}\mathfrak{n}}{\mathfrak{m}\left(\mathfrak{m}^2+\mathfrak{n}^2\right)} = \dfrac{\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)^{\cdot}}{\mathfrak{m}\left(1 + \left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)^2\right)}. \end{cases}$$

Corollary 2.2. *Let $\boldsymbol{\alpha}$ be a regular spherical front with the Sabban apparatus $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{e}, \kappa_g\}$ and the Legendre apparatus $\{\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{\mu}, \mathfrak{m}, \mathfrak{n}\}$, then the relationship formulas are obtained by*

$$\begin{cases} \boldsymbol{\alpha} = \boldsymbol{\alpha}, \\ \boldsymbol{t} = \varepsilon \boldsymbol{\mu}, \\ \boldsymbol{e} = -\varepsilon \boldsymbol{\vartheta}, \end{cases}$$

where $\varepsilon = \det(\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{t})$ and

$$\kappa_g = \frac{\mathfrak{n}}{|\mathfrak{m}|}.$$

Corollary 2.3. *By Remark 2.1, Corollary 2.1 and Corollary 2.2, we have the following statements:*

- $\mathfrak{m} = 0$ *iff $\boldsymbol{\alpha}$ is a point on $\mathbb{S}^2$.*
- $\mathfrak{n} = 0$ *iff $\boldsymbol{\alpha}$ is a great circle on $\mathbb{S}^2$ (i.e. $\kappa_g = 0$).*
- $\frac{\mathfrak{n}}{\mathfrak{m}} = \lambda$ *is a constant iff $\boldsymbol{\alpha}$ is a circle with radius $\frac{1}{\sqrt{1+\lambda^2}}$ on $\mathbb{S}^2$.*
- $\frac{\mathfrak{m}^2 \left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)^{\cdot}}{(\mathfrak{m}^2+\mathfrak{n}^2)^{3/2}}$ *is a non-zero constant iff $\boldsymbol{\alpha}$ is a spherical helix on $\mathbb{S}^2$.*

3. Orthogonal-Type Legendre Mates

In this section, we introduce three kinds of orthogonal type Legendre mates of spherical Legendre curve by using Legendre frame $\{\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{\mu}\}$ along its spherical frontal or front on $\mathbb{S}^2$. Moreover, we obtain some characterizations between Legendre curvatures of orthogonal-type Legendre mates.

Definition 3.1. *We say that spherical Legendre curves $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$ are first orthogonal-type Legendre mates, if $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_\star$ are orthogonal at the correspondence points of the curves.*

Definition 3.2. *We say that spherical Legendre curves $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)$ are second orthogonal-type Legendre mates, if $\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}_\star$ are orthogonal at the correspondence points of the curves.*

Definition 3.3. *We say that spherical Legendre curves $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)$ are third orthogonal-type Legendre mates, if $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_\star$ are orthogonal at the correspondence points of the curves.*

Moreover, we call that the front (or frontal) $\boldsymbol{\alpha}_\star$ is orthogonal-type Legendre mate of the front (or frontal) $\boldsymbol{\alpha}$, if $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)$ are orthogonal-type Legendre mates.

Theorem 3.1. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_\star$ be spherical front of spherical Legendre immersion $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)$, respectively. Then, the front $\boldsymbol{\alpha}_\star$ is second orthogonal-type Legendre mate of the front $\boldsymbol{\alpha}$ if and only if $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)=(\pm\boldsymbol{\vartheta},-\boldsymbol{\alpha})$ is Legendre immersion whose the curvature*

$$\left(\mathfrak{m}_\star\frac{dt_\star}{dt},\mathfrak{n}_\star\frac{dt_\star}{dt}\right)=(\mathfrak{n},\mp\mathfrak{m})\,,$$

or if the front $\boldsymbol{\alpha}$ has no inflection point, then $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)=\left(\frac{\pm\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha}+\frac{\mp\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}\,,\ \boldsymbol{\mu}\right)$ is Legendre immersion whose the curvature

$$\left(\mathfrak{m}_\star\frac{dt_\star}{dt},\mathfrak{n}_\star\frac{dt_\star}{dt}\right)=\left(\frac{\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)^{\cdot}}{1+\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)^2},\pm\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}\right).$$

Proof. Assume that the spherical front $\boldsymbol{\alpha}_\star$ be second orthogonal-type Legendre mate of the spherical front $\boldsymbol{\alpha}$. Then, $\boldsymbol{\vartheta}_\star$ and $\boldsymbol{\vartheta}$ are orthogonal at the correspondence points of the curves. So, there exists a differentiable function $\theta : I\to\mathbb{R}$ such that

$$\boldsymbol{\alpha}_\star(t_\star)=\cos\theta(t)\,\boldsymbol{\alpha}(t)+\sin\theta(t)\,\boldsymbol{\vartheta}(t)\,,\tag{3.1}$$

for all $t,t_\star\in I$. By taking the derivative of (3.1) with respect to parameter t and using Legendre Frenet type formulas (2.3), we get

$$\mathfrak{m}_\star\frac{dt_\star}{dt}\boldsymbol{\mu}_\star=-\left(\dot{\theta}\sin\theta\right)\boldsymbol{\alpha}+\left(\dot{\theta}\cos\theta\right)\boldsymbol{\vartheta}+(\mathfrak{m}\cos\theta+\mathfrak{n}\sin\theta)\,\boldsymbol{\mu}.\tag{3.2}$$

After taking the vector product of both sides of (3.2) by $\boldsymbol{\alpha}_\star$ and then using (3.1), we have

$$\mathfrak{m}_\star\frac{dt_\star}{dt}\boldsymbol{\vartheta}_\star=-\sin\theta\,(\mathfrak{m}\cos\theta+\mathfrak{n}\sin\theta)\,\boldsymbol{\alpha}+\cos\theta\,(\mathfrak{m}\cos\theta+\mathfrak{n}\sin\theta)\,\boldsymbol{\vartheta}-\dot{\theta}\boldsymbol{\mu}.\tag{3.3}$$

Now, by applying hypothesis (i.e. $\boldsymbol{\vartheta}_\star\perp\boldsymbol{\vartheta}$) after taking the inner product of both sides of (3.3) by the vector $\boldsymbol{\vartheta}$, we obtain

$$\cos\theta\,(\mathfrak{m}\cos\theta+\mathfrak{n}\sin\theta)=0.\tag{3.4}$$

Then, we consider the following two cases:

Case 1. If $\cos\theta=0$ in (3.4) then $\sin\theta=\pm1$. So, by using (3.1), (3.2) and (3.3), we get

$$\begin{aligned}\boldsymbol{\alpha}_\star &= \pm\boldsymbol{\vartheta}, &(3.5)\\ \mathfrak{m}_\star\frac{dt_\star}{dt}\boldsymbol{\mu}_\star &= \mp\mathfrak{n}\boldsymbol{\mu}, &(3.6)\\ \mathfrak{m}_\star\frac{dt_\star}{dt}\boldsymbol{\vartheta}_\star &= -\mathfrak{n}\boldsymbol{\alpha}. &(3.7)\end{aligned}$$

Without lost of generality, after taking the inner product of both sides of (3.6) and (3.7), we have

$$\mathfrak{m}_\star \frac{dt_\star}{dt} = \mathfrak{n},$$

and so,

$$\begin{aligned} \boldsymbol{\vartheta}_\star &= -\boldsymbol{\alpha}, & (3.8)\\ \boldsymbol{\mu}_\star &= \pm\boldsymbol{\mu}. & (3.9)\end{aligned}$$

Moreover, after basic calculations, we obtain also

$$\mathfrak{n}_\star \frac{dt_\star}{dt} = \mp\mathfrak{m}.$$

Finally, by using (3.5), (3.8) and (3.9) and taking into account Legendre Frenet type formulas,

$$\begin{aligned} \langle \boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star \rangle &= \langle \pm\boldsymbol{\vartheta}, -\boldsymbol{\alpha}\rangle = 0,\\ \left\langle \frac{d}{dt_\star}(\boldsymbol{\alpha}_\star), \boldsymbol{\vartheta}_\star \right\rangle &= \frac{dt}{dt_\star}\langle \pm\mathfrak{n}\boldsymbol{\mu}, -\boldsymbol{\alpha}\rangle = 0,\end{aligned}$$

and since $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ is a spherical Legendre immersion,

$$\left(\mathfrak{m}_\star \frac{dt_\star}{dt}, \mathfrak{n}_\star \frac{dt_\star}{dt}\right) = (\mathfrak{n}, \mp\mathfrak{m}) \neq (0,0).$$

Thus, $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\pm\boldsymbol{\vartheta}, -\boldsymbol{\alpha})$ is also a spherical Legendre immersion.

Case 2. If $\mathfrak{m}\cos\theta + \mathfrak{n}\sin\theta = 0$ in (3.4) then we can choose as

$$\tan\theta = -\frac{\mathfrak{m}}{\mathfrak{n}}, \quad \cos\theta = \frac{\pm\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}, \quad \sin\theta = \frac{\mp\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}} \tag{3.10}$$

under the assumption $\mathfrak{n} \neq 0$ (i.e the front $\boldsymbol{\alpha}$ has no inflection point). Hence, by using (3.1), (3.2) and (3.3), we get

$$\begin{aligned} \boldsymbol{\alpha}_\star &= \frac{\pm\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha} + \frac{\mp\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}, & (3.11)\\ \mathfrak{m}_\star \frac{dt_\star}{dt}\boldsymbol{\mu}_\star &= -\dot{\theta}\left(\sin\theta\boldsymbol{\alpha} - \cos\theta\boldsymbol{\vartheta}\right), & (3.12)\\ \mathfrak{m}_\star \frac{dt_\star}{dt}\boldsymbol{\vartheta}_\star &= -\dot{\theta}\boldsymbol{\mu}. & (3.13)\end{aligned}$$

Without lost of generality, after taking the inner product of both sides of (3.12) and (3.13), by using (3.10), we have

$$\mathfrak{m}_\star \frac{dt_\star}{dt} = -\dot{\theta} = \frac{\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)^{\cdot}}{1+\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)^2},$$

and so, by using (3.10), (3.12) and (3.13), we get

$$\begin{aligned} \boldsymbol{\vartheta}_\star &= \boldsymbol{\mu}, & (3.14)\\ \boldsymbol{\mu}_\star &= \frac{\mp\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha} + \frac{\mp\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}. & (3.15)\end{aligned}$$

Moreover, after basic calculations, we obtain also

$$\mathfrak{n}_\star \frac{dt_\star}{dt} = \pm\sqrt{\mathfrak{m}^2 + \mathfrak{n}^2}.$$

Finally, by using (3.11), (3.14) and (3.15) and taking into account Legendre Frenet type formulas,

$$\begin{aligned} \langle \boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star \rangle &= \left\langle \frac{\pm\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha} + \frac{\mp\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}, \boldsymbol{\mu} \right\rangle = 0, \\ \left\langle \frac{d}{dt_\star}(\boldsymbol{\alpha}_\star), \boldsymbol{\vartheta}_\star \right\rangle &= \frac{dt}{dt_\star} \frac{\mathfrak{n}^2 \left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)^{\cdot}}{\left(\mathfrak{m}^2+\mathfrak{n}^2\right)^{3/2}} \langle \mp\mathfrak{m}\boldsymbol{\alpha} \mp \mathfrak{n}\boldsymbol{\vartheta}, \boldsymbol{\mu} \rangle = 0, \end{aligned}$$

and since $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ is a spherical Legendre immersion and $\mathfrak{n} \neq 0$,

$$\left(\mathfrak{m}_\star \frac{dt_\star}{dt}, \mathfrak{n}_\star \frac{dt_\star}{dt}\right) = \left(\frac{\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)^{\cdot}}{1+\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)^2}, \pm\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}\right) \neq (0,0).$$

Thus, $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = \left(\frac{\pm\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha} + \frac{\mp\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}, \boldsymbol{\mu}\right)$ is also a spherical Legendre immersion under the assumption $\mathfrak{n} \neq 0$.

Conversely, we suppose that the spherical Legendre immersion $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$ is given by (3.5),(3.8) or (3.11),(3.14). Then, it is clear that $\boldsymbol{\vartheta}_\star$ and $\boldsymbol{\vartheta}$ are orthogonal at the correspondence points of the curves. Consequently, the proof is complete. □

Thus, if $\boldsymbol{\alpha}$ is a spherical frontal then we can give the following theorem for second orthogonal-type Legendre mate of the frontal $\boldsymbol{\alpha}$.

Theorem 3.2. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_\star$ be spherical frontal of spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$, respectively. Then, the frontal $\boldsymbol{\alpha}_\star$ is second orthogonal-type Legendre mate of the frontal $\boldsymbol{\alpha}$ if and only if $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\pm\boldsymbol{\vartheta}, -\boldsymbol{\alpha})$ is Legendre curve whose the curvature*

$$\left(\mathfrak{m}_\star \frac{dt_\star}{dt}, \mathfrak{n}_\star \frac{dt_\star}{dt}\right) = (\mathfrak{n}, \mp\mathfrak{m}),$$

or $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\cos\theta\, \boldsymbol{\alpha} + \sin\theta \boldsymbol{\vartheta}, \boldsymbol{\mu})$ is Legendre curve whose the curvature

$$\left(\mathfrak{m}_\star \frac{dt_\star}{dt}, \mathfrak{n}_\star \frac{dt_\star}{dt}\right) = \left(-\dot{\theta}, \mathfrak{m}\cos\theta - \mathfrak{n}\sin\theta\right),$$

where $\theta : I \to \mathbb{R}$ is a differentiable function.

Proof. Assume that $\boldsymbol{\alpha}$ is a spherical frontal. Then, all remaining possible cases are valid except for the condition $(\mathfrak{m}, \mathfrak{n}) \neq 0$ in Theorem 3.1. Hence, the proof is clear. □

Theorem 3.3. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_\star$ be spherical front of spherical Legendre immersion $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$, respectively. Then, the front $\boldsymbol{\alpha}_\star$ is third orthogonal-type Legendre mate of the front $\boldsymbol{\alpha}$ if and only if $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = \left(\pm\boldsymbol{\mu}, \frac{-\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha} + \frac{\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}\right)$ is Legendre immersion whose the curvature*

$$\left(\mathfrak{m}_\star \frac{dt_\star}{dt}, \mathfrak{n}_\star \frac{dt_\star}{dt}\right) = \left(\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}, \pm\frac{\mathfrak{m}\dot{\mathfrak{n}} - \mathfrak{n}\dot{\mathfrak{m}}}{\mathfrak{m}^2+\mathfrak{n}^2}\right),$$

or if the front $\boldsymbol{\alpha}$ has no inflection point, then

$$(\boldsymbol{\alpha}_{\star}, \boldsymbol{\vartheta}_{\star}) = \left(\cos\left(\int \mathfrak{m}dt\right)\boldsymbol{\alpha} - \sin\left(\int \mathfrak{m}dt\right)\boldsymbol{\mu}\,\textbf{,}\ \mp\sin\left(\int \mathfrak{m}dt\right)\boldsymbol{\alpha} \mp \cos\left(\int \mathfrak{m}dt\right)\boldsymbol{\mu}\right)$$

is Legendre immersion whose the curvature

$$\left(\mathfrak{m}_{\star}\frac{dt_{\star}}{dt}, \mathfrak{n}_{\star}\frac{dt_{\star}}{dt}\right) = \left(\pm\mathfrak{n}\sin\left(\int \mathfrak{m}dt\right)\ , \ \mathfrak{n}\cos\left(\int \mathfrak{m}dt\right)\right).$$

Proof. We omit the proof since it is analogous to the proof of Theorem 3.1. □

Thus, if $\boldsymbol{\alpha}$ is a spherical frontal then we can give the following theorem for third orthogonal-type Legendre mate of the frontal $\boldsymbol{\alpha}$.

Theorem 3.4. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_{\star}$ be spherical frontal of spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_{\star}, \boldsymbol{\vartheta}_{\star})$, respectively. Then, the frontal $\boldsymbol{\alpha}_{\star}$ is third orthogonal-type Legendre mate of the frontal $\boldsymbol{\alpha}$ if and only if*

$$(\boldsymbol{\alpha}_{\star}, \boldsymbol{\vartheta}_{\star}) = \left(\cos\left(\int \mathfrak{m}dt\right)\boldsymbol{\alpha} - \sin\left(\int \mathfrak{m}dt\right)\boldsymbol{\mu}\,\textbf{,}\ \mp\sin\left(\int \mathfrak{m}dt\right)\boldsymbol{\alpha} \mp \cos\left(\int \mathfrak{m}dt\right)\boldsymbol{\mu}\right)$$

is Legendre curve whose the curvature

$$\left(\mathfrak{m}_{\star}\frac{dt_{\star}}{dt}, \mathfrak{n}_{\star}\frac{dt_{\star}}{dt}\right) = \left(\pm\mathfrak{n}\sin\left(\int \mathfrak{m}dt\right)\ , \ \mathfrak{n}\cos\left(\int \mathfrak{m}dt\right)\right).$$

Proof. Assume that $\boldsymbol{\alpha}$ is a spherical frontal. Then, all remaining possible cases are valid except for the condition $(\mathfrak{m}, \mathfrak{n}) \neq 0$ in Theorem 3.3. Hence, the proof is clear. □

Theorem 3.5. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_{\star}$ be spherical front of spherical Legendre immersion $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_{\star}, \boldsymbol{\vartheta}_{\star})$, respectively. Then, the front $\boldsymbol{\alpha}_{\star}$ is first orthogonal-type Legendre mate of the front $\boldsymbol{\alpha}$ if and only if $(\boldsymbol{\alpha}_{\star}, \boldsymbol{\vartheta}_{\star}) = (\pm\boldsymbol{\vartheta}, \mp\boldsymbol{\alpha})$ is Legendre immersion whose the curvature*

$$\left(\mathfrak{m}_{\star}\frac{dt_{\star}}{dt}, \mathfrak{n}_{\star}\frac{dt_{\star}}{dt}\right) = (\mp\mathfrak{n}, \mp\mathfrak{m})$$

or, $(\boldsymbol{\alpha}_{\star}, \boldsymbol{\vartheta}_{\star}) = \left(\pm\boldsymbol{\mu}, -\frac{\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha} + \frac{\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}\right)$ is Legendre immersion whose the curvature

$$\left(\mathfrak{m}_{\star}\frac{dt_{\star}}{dt}, \mathfrak{n}_{\star}\frac{dt_{\star}}{dt}\right) = \left(\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}, \pm\frac{\mathfrak{m}\dot{\mathfrak{n}} - \mathfrak{n}\dot{\mathfrak{m}}}{\mathfrak{m}^2+\mathfrak{n}^2}\right)$$

or, if $\boldsymbol{\alpha}$ is a great circle (i.e. that $\mathfrak{n} = 0$) on $\mathbb{S}^2$ and $\theta : I \to \mathbb{R} - \{2k\pi \mid k \in \mathbb{Z}\}$ is a differentiable function then $(\boldsymbol{\gamma}_{\star}, \boldsymbol{\vartheta}_{\star}) = \left(\cos\theta\boldsymbol{\vartheta} + \sin\theta\boldsymbol{\mu}\,, \ \frac{-\dot{\theta}\boldsymbol{\gamma}+\mathfrak{m}\sin^2\theta\boldsymbol{\vartheta}-\mathfrak{m}\sin\theta\cos\theta\boldsymbol{\mu}}{\sqrt{\mathfrak{m}^2\sin^2\theta+\left(\dot{\theta}\right)^2}}\right)$ is Legendre immersion whose the curvature

$$\mathfrak{m}_{\star}\frac{dt_{\star}}{dt}, \mathfrak{n}_{\star}\frac{dt_{\star}}{dt} \quad = \left(\sqrt{\mathfrak{m}^2\sin^2\theta + \quad \dot{\theta} \quad ^2}, \frac{\mathfrak{m}\sin\theta \quad \ddot{\theta} \quad - \dot{\mathfrak{m}}\sin\theta \quad \dot{\theta} \quad - 2\mathfrak{m}\cos\theta \quad \dot{\theta} \quad ^2 - \mathfrak{m}^3\sin^2\theta\cos\theta}{\mathfrak{m}^2\sin^2\theta + \quad \dot{\theta} \quad ^2}\right).$$

Proof. We omit the proof since it is analogous to the proof of Theorem 3.1. □

Thus, if $\boldsymbol{\alpha}$ is a spherical frontal then we can give the following theorem for first orthogonal-type Legendre mate of the frontal $\boldsymbol{\alpha}$.

Theorem 3.6. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_\star$ be spherical frontal of spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$, respectively. Then, the frontal $\boldsymbol{\alpha}_\star$ is first orthogonal-type Legendre mate of the frontal $\boldsymbol{\alpha}$ if and only if $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\pm\boldsymbol{\vartheta}, \mp\boldsymbol{\alpha})$ is Legendre curve whose the curvature*

$$\left(\mathfrak{m}_\star \frac{dt_\star}{dt}, \mathfrak{n}_\star \frac{dt_\star}{dt}\right) = (\mp\mathfrak{n}, \mp\mathfrak{m}),$$

or, if the frontal $\boldsymbol{\alpha}$ is a great circle (i.e. $\mathfrak{n} = 0$) on $\mathbb{S}^2$ and $\theta : I \to \mathbb{R} - \{2k\pi \mid k \in \mathbb{Z}\}$ is a differentiable function then $(\boldsymbol{\gamma}_\star, \boldsymbol{\vartheta}_\star) = \left(\cos\theta\boldsymbol{\vartheta} + \sin\theta\boldsymbol{\mu}, \frac{-\dot{\theta}\boldsymbol{\gamma} + \mathfrak{m}\sin^2\theta\boldsymbol{\vartheta} - \mathfrak{m}\sin\theta\cos\theta\boldsymbol{\mu}}{\sqrt{\mathfrak{m}^2\sin^2\theta + (\dot{\theta})^2}}\right)$ is a Legendre curve whose the curvature

$$\mathfrak{m}_\star \frac{dt_\star}{dt}, \mathfrak{n}_\star \frac{dt_\star}{dt} = \left(\sqrt{\mathfrak{m}^2\sin^2\theta + \ \dot{\theta}\ ^2}, \frac{\mathfrak{m}\sin\theta\ \ \ddot{\theta}\ \ - \dot{\mathfrak{m}}\sin\theta\ \ \dot{\theta}\ \ - 2\mathfrak{m}\cos\theta\ \ \dot{\theta}\ ^2 - \mathfrak{m}^3\sin^2\theta\cos\theta}{\mathfrak{m}^2\sin^2\theta + \ \dot{\theta}\ ^2}\right).$$

Proof. Assume that $\boldsymbol{\alpha}$ is a spherical frontal. Then, all remaining possible cases are valid except for the condition $(\mathfrak{m}, \mathfrak{n}) \neq 0$ in Theorem 3.5. Hence, the proof is clear. □

3.1. Results and Some Examples for Orthogonal-Type Legendre Mates.

Corollary 3.1. *Let $\boldsymbol{\alpha}$ be the spherical front of a spherical Legendre immersion $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$. Then, the evolute of $\boldsymbol{\alpha}$ is a second orthogonal-type Legendre mate of the spherical front $\boldsymbol{\alpha}$.*

Corollary 3.2. *Let $\boldsymbol{\alpha}$ be the spherical frontal (or front) of a spherical Legendre curve (or immersion) $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$. Then, the involute of $\boldsymbol{\alpha}$ is a third orthogonal-type Legendre mate of the spherical frontal (or front) $\boldsymbol{\alpha}$.*

Corollary 3.3. *Let $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\boldsymbol{\vartheta}, \boldsymbol{\alpha})$ be spherical Legendre curves (or immersions) on Δ. Then, the frontal (or front) $\boldsymbol{\alpha}_\star$ is both first and second orthogonal-type Legendre mate of the frontal (or front) $\boldsymbol{\alpha}$.*

Corollary 3.4. *Let $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = \left(\boldsymbol{\mu}, \frac{-\mathfrak{n}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\alpha} + \frac{\mathfrak{m}}{\sqrt{\mathfrak{m}^2+\mathfrak{n}^2}}\boldsymbol{\vartheta}\right)$ be the spherical Legendre immersions on Δ. Then, the spherical front $\boldsymbol{\alpha}_\star$ is both first and third orthogonal-type Legendre mate of the spherical front $\boldsymbol{\alpha}$.*

Corollary 3.5. *Let $\boldsymbol{\alpha}$ be a regular spherical front with the Frenet apparatus $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}, \kappa, \tau\}$ and the Legendre apparatus $\{\boldsymbol{\alpha}, \boldsymbol{\vartheta}, \boldsymbol{\mu}, \mathfrak{m}, \mathfrak{n}\}$. Then, the following statements are satisfied:*

- *If $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\pm\boldsymbol{t}, -\boldsymbol{b})$, then the spherical front $\boldsymbol{\alpha}_\star$ is both first and third orthogonal-type Legendre mate of the spherical front $\boldsymbol{\alpha}$ such that $\frac{\mathfrak{n}_\star}{\mathfrak{m}_\star} = \pm\frac{\tau}{\kappa}$.*
- *If $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\boldsymbol{b}, \pm\boldsymbol{t})$, then the spherical front $\boldsymbol{\alpha}_\star$ is second orthogonal-type Legendre mate of the spherical front $\boldsymbol{\alpha}$ such that $\frac{\mathfrak{m}_\star}{\mathfrak{n}_\star} = \mp\frac{\tau}{\kappa}$.*

Now, we give some examples of orthogonal-type Legendre mates of a spherical front or frontal on $\mathbb{S}^2$.

Example 3.1. *Let $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ be a spherical Legendre immersion on Δ whose the curvature*

$$(\mathfrak{m}(t), \mathfrak{n}(t)) = \left(-2\sqrt{6}\cos t, 2\sqrt{6}\sin t\right),$$

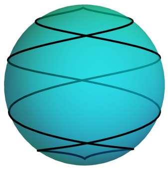
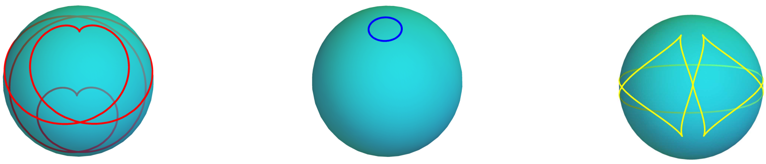


Figure 1: The spherical front $\boldsymbol{\alpha}$ which is a spherical helix.



(a) (b) (c)

Figure 2: The first, second and third orthogonal-type Legendre mates of $\boldsymbol{\alpha}$, respectively.

where the spherical front $\boldsymbol{\alpha}$ is a spherical helix (see Figure 1) which is given by

$$\boldsymbol{\alpha}(t) = \left(\frac{1}{5}(-3\cos 4t - 2\cos 6t), \frac{1}{5}(-3\sin 4t - 2\sin 6t), \frac{2}{5}\sqrt{6}\sin t\right)$$

and its dual $\boldsymbol{\vartheta}$ is also a spherical helix which is given by

$$\boldsymbol{\vartheta}(t) = \left(\frac{1}{5}(2\sin 6t - 3\sin 4t), \frac{1}{5}(3\cos 4t - 2\cos 6t), \frac{2}{5}\sqrt{6}\cos t\right).$$

Then,

- *the spherical Legendre immersion $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = (\boldsymbol{\vartheta}, \boldsymbol{\alpha})$ is the first orthogonal-type Legendre mate of $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ whose the curvature*

$$(\mathfrak{m}_\star(t), \mathfrak{n}_\star(t)) = \left(-2\sqrt{6}\sin t, 2\sqrt{6}\cos t\right).$$

It is also second orthogonal-type Legendre mate of $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ by Corollary 3.3. and the spherical front $\boldsymbol{\alpha}_\star$ is also a spherical helix (see Figure 2A).

- *the spherical Legendre immersion*

$$(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = \left(\frac{\mathfrak{n}}{\sqrt{\mathfrak{m}^2 + \mathfrak{n}^2}} \boldsymbol{\alpha} + \frac{-\mathfrak{m}}{\sqrt{\mathfrak{m}^2 + \mathfrak{n}^2}} \boldsymbol{\vartheta} \,, \ \boldsymbol{\mu} \right)$$

is the second orthogonal-type Legendre mate of $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ whose the curvature

$$(\mathfrak{m}_\star(t), \mathfrak{n}_\star(t)) = \left(1, 2\sqrt{6}\right).$$

The spherical front $\boldsymbol{\alpha}_\star$ is also evolute of the front $\boldsymbol{\alpha}$ by Corollary 3.1. and it is a circle (see Figure 2B).

- *the spherical Legendre curve*

$$(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star) = \quad \cos \quad \int \mathfrak{m} dt \quad \boldsymbol{\alpha} - \sin \quad \int \mathfrak{m} dt \quad \boldsymbol{\mu} \boldsymbol{,} \sin \quad \int \mathfrak{m} dt \quad \boldsymbol{\alpha} + \cos \quad \int \mathfrak{m} dt \quad \boldsymbol{\mu}$$

is the third orthogonal-type Legendre mate of $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ whose the curvature

$$(\mathfrak{m}_\star(t), \mathfrak{n}_\star(t)) = \left(2\sqrt{6} \sin\left(2\sqrt{6} \sin t\right) \sin t \ , \ 2\sqrt{6} \cos\left(2\sqrt{6} \sin t\right) \sin t\right)$$

The spherical frontal $\boldsymbol{\alpha}_\star$ is also involute of the frontal $\boldsymbol{\alpha}$ by Corollary 3.2. and it is a spherical slant helix (see Figure 2C).

Example 3.2. *Let $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ be a spherical Legendre curve on Δ whose the curvature*

$$(\mathfrak{m}(t), \mathfrak{n}(t)) = (-\cos(\sin t) \sin t, -\sin(\sin t) \sin t) \,,$$

where the spherical frontal $\boldsymbol{\alpha}$ is a spherical slant helix which is given by

$$\boldsymbol{\alpha}(t) = \begin{pmatrix} \frac{1}{2} \quad -2\sin \quad \sqrt{2}t \quad \sin(\sin t)\cos t + \sqrt{2}\cos \quad \sqrt{2}t \quad \sin(\sin t)\sin t - \sqrt{2}\cos \quad \sqrt{2}t \quad \cos(\sin t) \\ \frac{1}{2} \quad \sqrt{2}\sin \quad \sqrt{2}t \quad \sin(\sin t)\sin t + 2\cos \quad \sqrt{2}t \quad \sin(\sin t)\cos t - \sqrt{2}\sin \quad \sqrt{2}t \quad \cos(\sin t) \\ -\frac{1}{\sqrt{2}}\left(\sin(\sin t)\sin t + \cos(\sin t)\right) \end{pmatrix}$$

(see Figure 3)

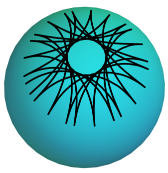


Figure 3: The spherical frontal $\boldsymbol{\alpha}$ which is a spherical slant helix

and its dual $\boldsymbol{\vartheta}$ is also spherical slant helix which is given by

$$\boldsymbol{\vartheta}(t) = \begin{pmatrix} \frac{1}{2} \quad 2\sin \quad \sqrt{2}t \quad \cos(\sin t)\cos t - \sqrt{2}\cos \quad \sqrt{2}t \quad \sin(\sin t) - \sqrt{2}\cos \quad \sqrt{2}t \quad \cos(\sin t)\sin t \\ -\frac{1}{2}\cos(\sin t) \quad \sqrt{2}\sin \quad \sqrt{2}t \quad \sin t + 2\cos \quad \sqrt{2}t \quad \cos t + \sqrt{2}\sin \quad \sqrt{2}t \quad \tan(\sin t) \\ \frac{1}{\sqrt{2}}\left(\cos(\sin t)\sin t - \sin(\sin t)\right) \end{pmatrix}.$$

(a) (b)

Figure 4: The first and second orthogonal-type Legendre mate of $\boldsymbol{\alpha}$, respectively.

Thus, $\boldsymbol{\mu}$ is a spherical helix which is given by

$$\boldsymbol{\mu}(t)=\begin{pmatrix} \frac{1}{2}\left(-\sqrt{2}\cos\left(\sqrt{2}t\right)\cos t-2\sin\left(\sqrt{2}t\right)\sin t\right) \\ \frac{1}{2}\left(2\cos\left(\sqrt{2}t\right)\sin t-\sqrt{2}\sin\left(\sqrt{2}t\right)\cos t\right) \\ \frac{\cos t}{\sqrt{2}} \end{pmatrix}.$$

Then,

- *the spherical Legendre curve $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)=(\boldsymbol{\vartheta},\boldsymbol{\alpha})$ is both first and second orthogonal-type Legendre mate of $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ whose the curvature*

$$(\mathfrak{m}_\star(t),\mathfrak{n}_\star(t))=(\sin(\sin t)\sin t,\cos(\sin t)\sin t)$$

  by Corollary 3.3 (see Figure 4A).
- *for $\theta(t)=\cos t$, the spherical Legendre curve $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)=(\cos\theta\,\boldsymbol{\alpha}+\sin\theta\,\boldsymbol{\vartheta}\,,\,\boldsymbol{\mu})$ is second orthogonal-type Legendre mate of $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ whose the curvature*

$$(\mathfrak{m}_\star(t),\mathfrak{n}_\star(t))=(-\sin t,-\cos(\sin t+\cos t)\sin t)\,,$$

  by Theorem 3.2 (see Figure 4B).

Example 3.3. *Let $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ be a spherical Legendre curve on Δ whose the curvature*

$$(\mathfrak{m}(t),\mathfrak{n}(t))=\left(-\frac{t^3\sqrt{t^{10}+25t^2+16}}{t^{10}+t^8+1},\frac{20\sqrt{t^{10}+t^8+1}}{t^{10}+25t^2+16}\right),$$

where the spherical frontal $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha}(t)=\frac{1}{\sqrt{t^{10}+t^8+1}}\left(1,t^4,t^5\right),$$

(see Figure 5) and its dual $\boldsymbol{\vartheta}$ is given by

$$\boldsymbol{\vartheta}(t)=\frac{1}{\sqrt{t^{10}+25t^2+16}}\left(t^5,-5t,4\right).$$

Thus, $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu}(t)=\frac{1}{\sqrt{\left(t^{10}+25t^2+16\right)\left(t^{10}+t^8+1\right)}}\left(t^4\left(5t^2+4\right),-4+t^{10},-t\left(t^8+5\right)\right).$$

Then,

- *the spherical Legendre curve $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)=(\boldsymbol{\vartheta},\boldsymbol{\alpha})$ is both first and second orthogonal-type Legendre mate of $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ whose the curvature*

$$(\mathfrak{m}_\star(t),\mathfrak{n}_\star(t))=\left(-\frac{20\sqrt{t^{10}+t^8+1}}{t^{10}+25t^2+16},\frac{t^3\sqrt{t^{10}+25t^2+16}}{t^{10}+t^8+1}\right),$$

by Corollary 3.3 (see Figure 6a).

- *for $\theta(t)=\arctan t$, the spherical Legendre curve $(\boldsymbol{\alpha}_\star,\boldsymbol{\vartheta}_\star)=(\cos\theta\,\boldsymbol{\alpha}+\sin\theta\,\boldsymbol{\vartheta}\,,\,\boldsymbol{\mu})$ is second orthogonal-type Legendre mate of $(\boldsymbol{\alpha},\boldsymbol{\vartheta})$ whose the curvature*

$$(\mathfrak{m}_\star(t),\mathfrak{n}_\star(t))=\left(\frac{1}{t^2+1},\frac{-t}{\sqrt{t^2+1}}\left(\frac{\sqrt{t^{10}+25t^2+16}t^2}{t^{10}+t^8+1}+\frac{20\sqrt{t^{10}+t^8+1}}{t^{10}+25t^2+16}\right)\right),$$

by Theorem 3.2 (see Figure 6b).

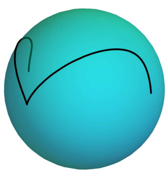
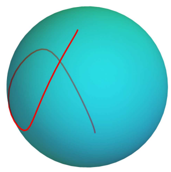


Figure 5: The spherical frontal $\boldsymbol{\alpha}$



(a) (b)

Figure 6: The first and second orthogonal-type Legendre mate of $\boldsymbol{\alpha}$, respectively.

4. Parallel-Type Legendre Mates

In this section, we introduce three kinds of parallel type Legendre mates of spherical Legendre curve by using Legendre frame along its spherical frontal on $\mathbb{S}^2$. But, we show that there is no parallel-type spherical Legendre mates.

Definition 4.1. *We say that spherical Legendre curves $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$ are first parallel-type Legendre mates, if $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_\star$ are parallel at the correspondence points of the curves.*

Moreover, we call that the front (or frontal) $\boldsymbol{\alpha}_\star$ is parallel-type Legendre mate of the front (or frontal) $\boldsymbol{\alpha}$, if $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$ are parallel-type Legendre mates.

Theorem 4.1. *Let $\boldsymbol{\alpha}$ be the spherical frontal of a spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$. Then, there exist no first parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$.*

Proof. Assume that the spherical frontal $\boldsymbol{\alpha}_\star$ be first parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$. Then, $\boldsymbol{\alpha}_\star$ and $\boldsymbol{\alpha}$ are parallel at the correspondence points of the curves. So, there exists a non-zero constant λ such that

$$\boldsymbol{\alpha}_\star(t_\star) = \lambda\boldsymbol{\alpha}(t), \tag{4.1}$$

for all $t, t_\star \in I$. By taking the inner product of both sides of (4.1), we get

$$\lambda = \pm 1,$$

since $\boldsymbol{\alpha}_\star$ and $\boldsymbol{\alpha}$ are spherical. Thus, $\boldsymbol{\alpha}_\star = \pm\boldsymbol{\alpha}$. Consequently, we proof that there exist no first parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$. □

Definition 4.2. *We say that spherical Legendre curves $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$ are second parallel-type Legendre mates, if $\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}_\star$ are parallel at the correspondence points of the curves.*

Theorem 4.2. *Let $\boldsymbol{\alpha}$ be the spherical frontal of a spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$. Then, there exist no second parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$.*

Proof. Assume that the spherical frontal $\boldsymbol{\alpha}_\star$ be second parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$. Then, $\boldsymbol{\vartheta}_\star$ and $\boldsymbol{\vartheta}$ are parallel at the correspondence points of the curves. So, there exists a differentiable function $\theta : I \to \mathbb{R}$ such that

$$\boldsymbol{\alpha}_\star(t_\star) = \cos\theta(t)\,\boldsymbol{\alpha}(t) + \sin\theta(t)\,\boldsymbol{\vartheta}(t), \tag{4.2}$$

for all $t, t_\star \in I$. By taking the derivative of (4.2) with respect to parameter t and using Legendre Frenet type formulas (2.3), we get

$$\mathfrak{m}_\star \frac{dt_\star}{dt}\boldsymbol{\mu}_\star = -\left(\dot{\theta}\sin\theta\right)\boldsymbol{\alpha} + \left(\dot{\theta}\cos\theta\right)\boldsymbol{\vartheta} + (m\cos\theta + n\sin\theta)\,\boldsymbol{\mu} \tag{4.3}$$

Moreover, we have

$$\boldsymbol{\vartheta}_\star = \pm\boldsymbol{\vartheta} \tag{4.4}$$

since $\boldsymbol{\vartheta}_\star$ and $\boldsymbol{\vartheta}$ are spherical and parallel from hypothesis. Then, by taking the inner product of both sides of (4.3) by the vector $\boldsymbol{\vartheta}$, we obtain

$$\dot{\theta}\cos\theta = 0.$$

Also, by using (4.2), (4.4) and (4.5), we get

$$\begin{aligned} \langle \boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star \rangle &= \sin\theta, \\ \left\langle \frac{d}{dt_\star}(\boldsymbol{\alpha}_\star), \boldsymbol{\vartheta}_\star \right\rangle &= \frac{dt}{dt_\star}\left(\dot{\theta}\cos\theta\right) = 0. \end{aligned}$$

However, since $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$ is a Legendre curve, it must be $\sin\theta = 0$ and so $\cos\theta = \pm 1$. Thus, we obtain that $\boldsymbol{\alpha}_\star = \pm\boldsymbol{\alpha}$ by using (4.2). As a result, we proof that there exist no second parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$. $\square$

Definition 4.3. *We say that spherical Legendre curves $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$ and $(\boldsymbol{\alpha}_\star, \boldsymbol{\vartheta}_\star)$ are third parallel-type Legendre mates, if $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_\star$ are parallel at the correspondence points of the curves.*

Theorem 4.3. *Let $\boldsymbol{\alpha}$ be the spherical frontal of a spherical Legendre curve $(\boldsymbol{\alpha}, \boldsymbol{\vartheta})$. Then, there exist no third parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$.*

Proof. Assume that the spherical frontal $\boldsymbol{\alpha}_\star$ be third parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$. Then, $\boldsymbol{\mu}_\star$ and $\boldsymbol{\mu}$ are parallel at the correspondence points of the curves. So, there exists a differentiable function $\theta : I \to \mathbb{R}$ such that

$$\boldsymbol{\alpha}_\star(t_\star) = \cos\theta(t)\,\boldsymbol{\alpha}(t) + \sin\theta(t)\,\boldsymbol{\mu}(t), \tag{4.5}$$

for all $t, t_\star \in I$. By taking the derivative of (4.5) with respect to parameter t and using Legendre Frenet type formulas (2.3), we get

$$\mathfrak{m}_\star \frac{dt_\star}{dt}\boldsymbol{\mu}_\star = -\sin\theta\left(\mathfrak{m} + \dot{\theta}\right)\boldsymbol{\alpha} - \mathfrak{n}\sin\theta\,\boldsymbol{\vartheta} + \cos\theta\left(\mathfrak{m} + \dot{\theta}\right)\boldsymbol{\mu}. \tag{4.6}$$

After taking the vector product of both sides of (4.6) by $\boldsymbol{\alpha}_\star$ and then using (4.5), we have

$$\mathfrak{m}_\star \frac{dt_\star}{dt}\boldsymbol{\vartheta}_\star = \left(-\mathfrak{n}\sin^2\theta\right)\boldsymbol{\alpha} + \left(\mathfrak{m} + \dot{\theta}\right)\boldsymbol{\vartheta} + (\mathfrak{n}\sin\theta\cos\theta)\,\boldsymbol{\mu}. \tag{4.7}$$

Moreover, we have

$$\boldsymbol{\mu}_\star = \pm\boldsymbol{\mu} \tag{4.8}$$

since $\boldsymbol{\mu}_\star$ and $\boldsymbol{\mu}$ are spherical and parallel from hypothesis. Then, by taking the inner product of both sides of (4.7) by the vector $\boldsymbol{\mu}$, we obtain

$$\mathfrak{n}\sin\theta\cos\theta = 0. \tag{4.9}$$

Then, we consider the following three cases:

Case 1. If $\mathfrak{n} = 0$ in (4.9), then we have

$$\mathfrak{m}_\star \frac{dt_\star}{dt} = \mathfrak{m} + \dot{\theta}$$

from (4.6) and (4.7) without lost of generality. So,

$$\begin{aligned} \boldsymbol{\mu}_\star &= -\sin\theta\boldsymbol{\alpha} + \cos\theta\boldsymbol{\mu}, && (4.10)\\ \boldsymbol{\vartheta}_\star &= \boldsymbol{\vartheta}. && (4.11) \end{aligned}$$

Hence, by using (4.8) and (4.10), it must be $\cos\theta = \pm 1$ and so $\sin\theta = 0$. Thus, we obtain that $\boldsymbol{\alpha}_\star = \pm\boldsymbol{\alpha}$ by using (4.5). Consequently, this is a contradiction with assumption.

Case 2. If $\sin\theta = 0$ in (4.9) then $\cos\theta = \pm 1$. Thus, we obtain that $\boldsymbol{\alpha}_\star = \pm\boldsymbol{\alpha}$ by using (4.5). Consequently, this is a contradiction with assumption.

Case 3. If $\cos\theta = 0$ in (4.9) then $\sin\theta = \pm 1$. So, by using (4.5), we get

$$\boldsymbol{\alpha}_\star = \pm\boldsymbol{\mu}, \tag{4.12}$$

However, by taking the inner product of both sides of (4.12) with the vector $\boldsymbol{\mu}_\star$, we obtain that

$$\langle \boldsymbol{\mu}_\star, \boldsymbol{\mu} \rangle = 0.$$

Thus, this is a contradiction with the assumption. As a result, we proof that there exist no third parallel-type Legendre mate of the spherical frontal $\boldsymbol{\alpha}$. □

5. Conclusion

At the singular points of curves in three-dimensional Euclidean space, the curvature and torsion functions are undefined and the Frenet frame is not well defined. This naturally applies to spherical curves with singular points on the unit sphere. That is, the geodesic curvature function at the singular point of a spherical curve is undefined. However, it is possible to obtain a well-defined moving frame at singular points along the spherical curve. This frame is the Legendre frame defined along the spherical frontal (or spherical front) of a spherical Legendre curve (or Legendre immersion). In this sense, as regular or singular spherical curves, to define and examine to spherical frontal or spherical fronts is the first problem that comes to mind.

In this paper, inspired by the curve pairs of regular curves (involute-evolute, Bertrand, Mannheim, etc.) in classical differential geometry, we introduce that orthogonal-type and parallel-type Legendre mates (partner curves) of a spherical Legendre curve according to the Legendre frame along the front or frontal in the unit sphere as singular spherical curves. In particular, characterizations of orthogonal-type Lengendre mates according to Legendre curvature are obtained. It is proved that parallel-type Lengendre curve pairs do not exist.

By using similar technique in this paper, it can be easily applied to singular curves on two-dimensional hyperbolic space or de-Sitter space. Moreover, to define generalized orthogonal-type Legendre curve pairs in high-dimensional spaces and to examine their geometric properties will be a separate research topic.

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