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MATRIX APPLICATION OF POWER INCREASING SEQUENCES TO INFINITE SERIES AND FOURIER SERIES

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We consider a generalization, under weaker conditions, of the main theorem on quasi-σ-power increasing sequences applied to $|A, \theta_n|_k$ summability factors of infinite series and Fourier series. We obtain some new and known results related to the basic summability methods.

1. Introduction

Definition 1.1. *A positive sequence* (b_n) *is said to be an almost increasing sequence if there exists a positive increasing sequence* (c_n) *and two positive constants* M *and* N *such that* $Mc_n \leq b_n \leq Nc_n$ *(see [1]).*

Definition 1.2. *A positive sequence* (X_n) *is said to be a quasi-*σ*-power increasing sequence if there exists a constant* $K = K(\sigma, X) \geq 1$ *such that*

$$Kn^\sigma X_n \geq m^\sigma X_m$$

for all $n \geq m \geq 1.$

Every almost increasing sequence is a quasi-σ-power increasing sequence for any nonnegative σ but the converse is not true for $\sigma > 0$ (see [13]). For any sequence (λ_n), we can write

$$\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1} \qquad \text{and} \qquad \Delta\lambda_n = \lambda_n - \lambda_{n+1}.$$

Definition 1.3. *A sequence* (λ_n) *is said to be sequence of bounded variation denoted by* $(\lambda_n) \in \mathcal{BV}$ *if*

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty.$$

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α with $\alpha > -1$ for the sequences (s_n) and (na_n), respectively, i.e., (see [8])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \qquad \text{and} \qquad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v,$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

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Definition 1.4. *A series* $\sum a_n$ *is said to be summable* $|C,\alpha|_k,\ k \geq 1,$ *if (see [10, 12])*

$$\sum_{n=1}^{\infty} n^{k-1}\left|u_n^\alpha - u_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty} \frac{1}{n}|t_n^\alpha|^k < \infty.$$

If we take $\alpha = 1,$ then the $|C,\alpha|_k$ summability reduces to the $|C,1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \qquad (P_{-i} = p_{-i} = 0,\ \ i \geq 1).$$

A sequence-to-sequence transformation

$$w_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines a sequence (w_n) of the Riesz mean or, simply, of the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [11]).

Definition 1.5. *A series* $\sum a_n$ *is said to be* $|\bar{N}, p_n|_k,\ k \geq 1,$ *summable if (see [2])*

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In a special case where $p_n = 1$ for all values of n (resp., $k = 1$), the $|\bar{N}, p_n|_k$ summability is the same as the $|C,1|_k$ $\big($resp., $|\bar{N}, p_n|\big)$ summability.

2. Known Results

The following theorem deals with the $|\bar{N}, p_n|_k$ summability factors of infinite series under weaker conditions.

Theorem 2.1 [7]. *Let* (X_n) *be a quasi-*σ*-power increasing sequence. If the sequences* $(X_n),\ (\lambda_n),$ *and* (p_n) *satisfy the conditions*

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m), \tag{2.3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}\frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.4}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.5}$$

then the series $\sum a_n\lambda_n$ *is* $|\bar{N}, p_n|_k,\ k \geq 1,$ *summable.*

3. Application of Absolute Matrix Summability to Infinite Series

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix with nonzero diagonal entries. Then A defines a sequence-to-sequence transformation, which maps a sequence $s = (s_n)$ into $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, \ldots .$$

Definition 3.1. *Let* (θ_n) *be any sequence of positive real numbers. A series* $\sum a_n$ *is called* $|A, \theta_n|_k,\ k \geq 1,$ *summable if (see [14, 15])*

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty,$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take

$$\theta_n = \frac{P_n}{p_n},$$

then we obtain the $|A, p_n|_k$-summability (see [16]). At the same time, if we take

$$\theta_n = n,$$

then we get the $|A|_k$-summability (see [18]). Moreover, if we take

$$\theta_n = \frac{P_n}{p_n} \quad \text{and} \quad a_{nv} = \frac{p_v}{P_n},$$

then we have the $|\bar{N}, p_n|_k$-summability. Furthermore, if we take

$$\theta_n = n, \quad a_{nv} = \frac{p_v}{P_n}, \quad \text{and} \quad p_n = 1$$

for all values of n, then the $|A, \theta_n|_k$-summability reduces to the $|C, 1|_k$-summability (see [10]). Finally, if we take

$$\theta_n = n \quad \text{and} \quad a_{nv} = \frac{p_v}{P_n},$$

then we obtain the $|R, p_n|_k$-summability (see [3]).

4. Main Results

The Fourier series play an important role in various areas of applied mathematics and mechanics. Recently, some papers devoted to the absolute matrix summability of infinite series and Fourier series have been published (see [5, 6, 19–21]). The aim of the present paper is to generalize Theorem 2.1 for the $|A, \theta_n|_k$-summability method for these series.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \dots,$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \qquad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots .$$

It is worth noting that $\bar{A}$ and $\hat{A}$ are, respectively, the well-known matrices of series-to-sequence and series-to-series transformations. Thus, we get

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{4.1}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{4.2}$$

By using this notation, we arrive at the following theorem:

Theorem 4.1. *Let $k \geq 1$ and let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \dots, \tag{4.3}$$

$$a_{n-1,v} \geq a_{nv} \quad \textit{for} \quad n \geq v + 1, \tag{4.4}$$

$$\sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} = O(a_{nn}). \tag{4.5}$$

Suppose that (X_n) is a quasi-σ-power increasing sequence and that $(\theta_n a_{nn})$ is a nonincreasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions (2.1)–(2.3) of Theorem 2.1 and

$$\sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad \textit{as} \quad m \to \infty, \tag{4.6}$$

$$\sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} \frac{|t_n|^k}{n X_n^{k-1}} = O(X_m) \quad \textit{as} \quad m \to \infty, \tag{4.7}$$

then the series $\sum a_n \lambda_n$ is $|A, \theta_n|_k$, $k \geq 1$, summable.

Note that if we take

$$A=(\bar{N},p_n) \quad \text{and} \quad \theta_n=\frac{P_n}{p_n},$$

then conditions (4.6), (4.7) are reduced to (2.4), (2.5). In addition, condition (4.5) is satisfied by condition (2.3). Therefore, we arrive at Theorem 2.1.

We need the following lemmas to prove our theorem:

Lemma 4.1 [17]. *It follows from the conditions (4.3) and (4.4) of Theorem 4.1 that*

$$\sum_{v=0}^{n-1}|\bar{\Delta}a_{nv}|\leq a_{nn},$$

$$\hat{a}_{n,v+1}\geq 0,$$

$$\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}=O(1).$$

Lemma 4.2 [4]. *Under the conditions of Theorem 2.1, the following relations are true:*

$$nX_n|\Delta\lambda_n|=O(1) \quad \textit{as} \quad n\to\infty,$$

$$\sum_{n=1}^{\infty}X_n|\Delta\lambda_n|<\infty.$$

Proof of Theorem 4.1. Let (I_n) denote the A-transform of the series $\sum_{n=1}^{\infty}a_n\lambda_n$. Thus, by (4.1) and (4.2), we find

$$\bar{\Delta}I_n=\sum_{v=1}^{n}\hat{a}_{nv}a_v\lambda_v.$$

Applying Abel's transformation to this sum, we obtain

$$\begin{aligned}\bar{\Delta}I_n&=\sum_{v=1}^{n}\hat{a}_{nv}a_v\lambda_v\frac{v}{v}=\sum_{v=1}^{n-1}\Delta\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)\sum_{r=1}^{v}ra_r+\frac{\hat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n}ra_r\\&=\sum_{v=1}^{n-1}\Delta\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v+\hat{a}_{nn}\lambda_n\frac{n+1}{n}t_n\\&=\sum_{v=1}^{n-1}\bar{\Delta}a_{nv}\lambda_v t_v\frac{v+1}{v}+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v t_v\frac{v+1}{v}+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v}+a_{nn}\lambda_n t_n\frac{n+1}{n}\\&=I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.\end{aligned}$$

To complete the proof of Theorem 4.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\theta_n^{k-1}|I_{n,r}|^k<\infty \quad \text{for} \quad r=1,2,3,4.$$

First, by applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,1}|^k\le\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\left|\frac{v+1}{v}\right|\left|\bar{\Delta}a_{nv}\right||\lambda_v||t_v|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\sum_{v=1}^{n-1}\left|\bar{\Delta}a_{nv}\right||\lambda_v|^k|t_v|^k\left\{\sum_{v=1}^{n-1}\left|\bar{\Delta}a_{nv}\right|\right\}^{k-1}.$$

By using

$$\Delta\hat{a}_{nv}=\hat{a}_{nv}-\hat{a}_{n,v+1}=\bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}=a_{nv}-a_{n-1,v}$$

and relations (4.3) and (4.4), we get

$$\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|=\sum_{v=1}^{n-1}|a_{nv}-a_{n-1,v}|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})$$

$$=\sum_{v=0}^{n-1}a_{n-1,v}-a_{n-1,0}-\sum_{v=0}^{n}a_{nv}+a_{n0}+a_{nn}$$

$$=1-a_{n-1,0}-1+a_{n0}+a_{nn}\le a_{nn}.$$

Further, since

$$\sum_{n=v+1}^{m+1}|\bar{\Delta}a_{nv}|\le a_{vv},$$

we obtain

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,1}|^k=O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}||\lambda_v|^k|t_v|^k\right\}$$

$$=O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\bar{\Delta}a_{nv}|$$

$$=O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}|\bar{\Delta}a_{nv}|$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||t_v|^k a_{vv}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\theta_r^{k-1}a_{rr}^k\frac{|t_r|^k}{X_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k\frac{|t_v|^k}{X_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 4.1 and Lemmas 4.1 and 4.2. Moreover, we find

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,2}|^k &\le \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\frac{v+1}{v}||\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v||t_v|\frac{X_v}{X_v}\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v|X_v\frac{1}{X_v^k}|t_v|^k\right\}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v|X_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v|X_v\frac{1}{X_v^k}|t_v|^k\right\}\left\{\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{1}{X_v^{k-1}}\frac{1}{v}|t_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}\hat{a}_{n,v+1}\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}v|\Delta\lambda_v|\frac{1}{X_v^{k-1}}\frac{1}{v}|t_v|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}\\
&= O(1)\sum_{v=1}^{m}v(\theta_v a_{vv})^{k-1}|\Delta\lambda_v|\frac{1}{vX_v^{k-1}}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}(\theta_r a_{rr})^{k-1}\frac{|t_r|^k}{rX_r^{k-1}}\\
&\qquad + O(1)m|\Delta\lambda_m|\sum_{r=1}^{m}(\theta_r a_{rr})^{k-1}\frac{|t_r|^k}{rX_r^{k-1}}
\end{aligned}
$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v + O(1)m|\Delta\lambda_m|X_m$$

$$= O(1)\sum_{v=1}^{m-1} vX_v|\Delta^2\lambda_v| + O(1)\sum_{v=1}^{m-1} X_v|\Delta\lambda_v| + O(1)m|\Delta\lambda_m|X_m$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 4.1 and Lemmas 4.1 and 4.2.

Furthermore, as in $I_{n,1}$, we get

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,3}|^k \leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{|t_v|^k}{v}\right\}\left\{\sum_{v=1}^{n-1}\frac{1}{v}\hat{a}_{n,v+1}\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\lambda_{v+1}||\lambda_{v+1}|^{k-1}\frac{|t_v|^k}{v}\hat{a}_{n,v+1}$$

$$= O(1)\sum_{v=1}^{m}\frac{|t_v|^k}{v}\frac{1}{X_v^{k-1}}|\lambda_{v+1}|\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}\hat{a}_{n,v+1}$$

$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{|t_v|^k}{v}\frac{1}{X_v^{k-1}}|\lambda_{v+1}|\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}$$

$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_{v+1}|\frac{|t_v|^k}{v}$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 4.1 and Lemmas 4.1 and 4.2.

Finally, as in $I_{n,1}$, we get

$$\sum_{n=1}^{m}\theta_n^{k-1}|I_{n,4}|^k = O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k|\lambda_n|^k|t_n|^k$$

$$= O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k|\lambda_n|^{k-1}|\lambda_n||t_n|^k$$

$$= O(1) \sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \frac{1}{X_n^{k-1}} |\lambda_n||t_n|^k$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of hypotheses of the Theorem 4.1 and Lemmas 4.1 and 4.2.

Theorem 4.1 is proved.

5. Application of the Absolute Matrix Summability to Fourier Series

Let f be a periodic function with period 2π integrable (L) over $(-\pi, \pi)$. Without loss of generality, the constant term in the Fourier series of f can be set equal to zero and, hence,

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} C_n(x),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

We can write

$$\phi(t) = \frac{1}{2} \{ f(x+t) + f(x-t) \},$$

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \quad \alpha > 0.$$

It is well known that if $\phi_1(t) \in \mathcal{BV}(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C, 1)$-mean of the sequence $(nC_n(x))$ (see [9]).

By using this fact, Bor established the following main result for the trigonometric Fourier series:

Theorem 5.1 [7]. *Let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0, \pi)$ and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 2.1, then the series $\sum C_n(x)\lambda_n$ is $|\bar{N}, p_n|_k$, $k \geq 1$, summable.*

By using Theorem 5, we arrive at the following result for the $|A, \theta_n|_k$-summability.

Theorem 5.2. *Let A be a positive normal matrix satisfying the conditions of Theorem 4.1. Also let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0, \pi)$ and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 4.1, then the series $\sum C_n(x)\lambda_n$ is $|A, \theta_n|_k$, $k \geq 1$, summable.*

6. Applications

We can apply Theorems 4.1 and 5.2 to the weighted mean in which $A = (a_{nv})$ is defined as follows:

$$a_{nv} = \frac{p_v}{P_n} \quad \text{with} \quad 0 \le v \le n, \quad \text{where} \quad P_n = p_0 + p_1 + \ldots + p_n.$$

Thus, we get

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \qquad \text{and} \qquad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}.$$

Hence, the results presented in what follows can be easily verified.

7. Conclusions

1. If we take

$$\theta_n = \frac{P_n}{p_n}$$

in Theorems 4.1 and 5.2, then we have a result dealing with the $|A, p_n|_k$-summability.

2. If we take

$$\theta_n = n$$

in Theorems 4.1 and 5.2, then we obtain a result dealing with the $|A|_k$-summability.

3. If we take

$$\theta_n = \frac{P_n}{p_n} \quad \text{and} \quad a_{nv} = \frac{p_v}{P_n}$$

in Theorems 4.1 and 5.2, then we get Theorems 2.1 and 5.1, respectively.

4. If we take

$$\theta_n = n, \quad a_{nv} = \frac{p_v}{P_n}, \quad \text{and} \quad p_n = 1$$

for all values of n in Theorems 4.1 and 5.2, then we arrive at a new result for the $|C, 1|_k$-summability.

5. If we take

$$\theta_n = n \quad \text{and} \quad a_{nv} = \frac{p_v}{P_n}$$

in Theorems 4.1 and 5.2, then we deal with the $|R, p_n|_k$-summability.

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