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Facilitating the development of Preservice teachers' proportional reasoning in geometric similarity problems using augmented reality activities

Muhammet Arican[1] · Bilal Özçakir[2]

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Abstract

The literature reports preservice teachers' overuse of proportionality when solving geometric similarity problems with nonproportional relationships. Changing this type of error is reported as difficult even after applying certain interventions. As a solution to this type of error, this study used augmented reality activities to facilitate the development of preservice mathematics teachers' proportional reasoning. The data of this qualitative study included 17 preservice teachers' written responses to a paper-and-pencil test with five problems, which had been applied before and after the implementation of the augment reality activities, and video recordings collected during the augment reality implementation process. A case study methodology was used in designing the study in which the collected data were analyzed using a content analysis method. The preservice teachers' first test responses showed that although they were good at solving problems with regular figures, they had difficulty solving the problem with irregular figures. In this specific problem, the preservice teachers expected a proportional relationship between the areas of the two irregular figures. Their difficulties appeared to be a result of not being able to calculate the areas of these two figures by tiling or multiplying length and width that they used for regular figures. After the implementation of the augmented reality activities, which provided a dynamic representation of similar figures, the preservice teachers' overuse of proportionality drastically decreased. This finding suggested the contribution of the augmented reality technology on the development of the preservice teachers' proportional reasoning.

Keywords Augmented reality · Geometric similarity · Preservice teachers · Proportional reasoning

✉ Muhammet Arican
muhammetarican@gmail.com

Extended author information available on the last page of the article

1 Introduction

One of the fundamental goals of instructional programs is to help students to recognize and use connections among mathematical concepts and ideas (National Council of Teachers of Mathematics 2000). Proportional reasoning and geometric similarity are two of these connected concepts both require students' understanding of multiplicative relationships between quantities involved (Lee and Yim 2014). The connection is more apparent when stretching or shrinking a geometric figure to create a similar figure since corresponding sides of this new figure stretch or shrink in proportion to the sides of the original figure. Students can be introduced with geometric similarity concept as early as in elementary school, and their proportional reasoning develop in conjunction with the notions of similarity (Lehrer et al. 2002). However, in schools, geometric similarity has been usually taught detached from proportional reasoning (e.g., Fuys et al. 1988).

Two types of multiplicative relationships can be detected between two similar figures: The multiplicative relationship between the lengths of corresponding sides of two similar figures (i.e., between ratio), and the multiplicative relationship between the lengths of sides within each figure (i.e., within ratio) (Common Core Standards Writing Team 2011; Cox and Lo 2014). In both cases, the ratios formed by the corresponding lengths of between figures and within figures equal to a constant number. Hence, a proportional relationship presents between the lengths of corresponding sides of two similar figures. In the literature, the between ratio is usually referred to as *enlargement factor* (or *scale factor*) that indicates the size of an enlargement or shrinking of a figure to a similar one. Therefore, in the geometric similarity problems, the enlargement factor can be considered as the constant of proportionality (i.e., $\frac{x}{y} = k$).

Although proportional reasoning is needed in solving geometric similarity problems, researchers (e.g., Arican 2019; Ayan and Isiksal-Bostan 2019; Ben-Chaim et al. 2007; Cox 2013; De Bock et al. 2002; De Bock et al. 1998; Ekawati et al. 2015; Hull 2000) reported students', preservice teachers' (PSTs), and in-service teachers' difficulties with solving similarity problems. One of the most reported difficulty is the overuse of proportionality when solving similarity problems that require comparison of areas and volumes of two similar figures. Despite nonproportional relationships exist in the area and volume problems, some students, PSTs, and even in-service teachers tend to apply proportional strategies to solve these problems. However, while the area of a two-dimensional similar figure increases by the square of the enlargement factor (i.e., k^2), the volume of a three-dimensional similar figure increases by the cube of the enlargement factor (i.e., k^3). These researchers also noted that this type of error is difficult to change even after instruction or applying other type of intervention methods. According to Seago et al. (2013), students' tendency to apply proportional strategies to solve area and volume problems is a result of "traditional non-transformations-based" (p. 75) definition of similarity taught in classrooms. Hence, they suggest using a dynamic transformations-based approach in teaching geometric similarity in classrooms.

As stated above, PSTs and in-service teachers also have similar difficulties as students about proportions. However, in comparison to students, there is a lack of information on PSTs' and in-service teachers' proportional reasoning and difficulties that they have with solving geometric similarity problems. Only a limited number of studies (e.g., Arican et al. 2018; Brown et al. 2020; Gerretson 2004; Lee and Yim 2014; Seago et al. 2013; Seago et al. 2014) investigated PSTs' and in-service teachers' proportional reasoning within the scope of geometric similarity. Therefore, in the current study, we used augment reality (AR) technology in designing dynamic transformations-based activities to develop the PSTs' proportional reasoning within the context of geometric similarity. These AR activities were designed with a purpose to assist PSTs in understanding the connection between proportionality and geometric similarity. To estimate the development of the PSTs' proportional reasoning, they were given a geometric similarity test before and after the implementation of the AR activities. The following research questions are investigated:

1. How did preservice teachers respond to the geometric similarity problems before and after the implementation of augmented reality activities?
2. How did augmented reality activities facilitate preservice teachers' proportional reasoning in the area of geometric similarity?

2 Background

In this section, we present a review of literature on proportional reasoning and AR technology and its applications in education.

2.1 Proportional reasoning

Proportional reasoning is described as an important concept in middle school (Lamon 2007; Lobato and Ellis 2010) that students need to succeed in science and higher mathematics (Kilpatrick et al. 2001). Proportional reasoning "consists of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities" (Lamon 2007, pp. 637–638). Therefore, it has been treated as a certain form of multiplicative reasoning (Lesh et al. 1988).

Although understanding the concept of proportional reasoning should assists students in solving similarity problems, this connection is not always explicitly stated in school textbooks (Denton 2017). On the other hand, the studies that investigated the connection between proportionality and similarity mostly focused on investigating students', PSTs', and in-service teachers' solution strategies, performances, and difficulties. Some of these difficulties were reported as follows: Not recognizing proportional relationships between corresponding sides of similar figures (e.g., De Bock et al. 1998; Lamon 1993; Lee and Yim 2014), failing to consider proportionality when solving similarity problems (e.g., Arican et al. 2018; Cox 2013), inappropriately applying proportional strategies for solving similarity problems involving comparison of areas and volumes (e.g., Ayan and Isiksal-Bostan 2019; De Bock et al. 2002; De Bock et al. 1998; Van Dooren et al. 2004), and having difficulty with discovering the impact of enlargement on irregular figures (e.g., De Bock et al. 1998).

As stated above, one of the most persistent issues in solving similarity problems is the overuse of proportionality. This issue suggests problems with individuals' proportional reasoning because a competent proportional reasoner should be able to detect proportional and nonproportional relationships and distinguish these two relationships from each other. In order to remedy students' overuse of proportionality, Van Dooren et al. (2004) conducted a teaching experiment with eight grade students using experimental lessons. In the control group, who took regular classroom topics, students persisted on applying proportional strategies in solving nonproportional similarity problems in the pre-test and retention test. In the experimental group, many students' spontaneous use of proportional strategies drastically decreased in the post-test after the implementation of experimental lessons. However, in the experimental group, some students still applied proportional strategies to solve nonproportional problems.

Seago et al. (2013) suggested application of a geometric transformations-based (i.e., rotations, reflections, translations, and dilations) approach in teaching similarity concept for remedying difficulties that students, PSTs, and in-service teachers encounter in learning and teaching similarity. According to Seago et al. (2013), the traditional similarity definition that emphasizes "corresponding side lengths [of two figures] are in the same proportion" is a static perspective to define similarity (p. 77). On the other hand, transformations-based definition, "A figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations," provides a dynamic perspective to define similarity (Seago et al. 2013, p. 77). In order to compare the effectiveness of static and transformations-based approaches on teaching similarity, Seago et al. (2013) designed a study with several teachers. Some of these teachers used a static approach and others used a transformations-based approach. In general, students in static group attended to numerical relationships in two figures, students in transformations group conceptualized similarity as enlarging or shrinking figures proportionally. Furthermore, conducting professional development sessions with in-service teachers, Seago et al. (2014) provided in-service teachers with video cases that were specially designed from a transformations-based perspective for improving their understanding of similarity. The post-test results showed gains in the teachers' content knowledge of similarity. Testing students of these teachers, Seago et al. (2014) also observed gains in these students' content knowledge.

Although application of a transformations-based perspective can improve teachers' content knowledge of similarity, Cunningham and Rappa (2016) observed teachers' difficulties with solving a similarity problem that was designed from a transformations-based perspective. In this particular study, while all of the 15 teachers correctly solved a similarity problem requiring a traditional static perspective, seven of them failed to solve the similarity problem requiring a transformations perspective. This finding suggested teachers' unfamiliarity with the transformations perspective. In this study, Cunningham and Rappa (2016) presented both problems in classical paper-pencil forms. However, as reported by researchers (e.g., Brown et al. 2020; Denton 2017; Gerretson 2004), using a dynamic geometry software (DGS) in presenting the transformation-based problem could have contributed to seven failing teachers' understanding of the transformations perspective. Therefore, in the current study, we designed similarity activities using AR technology to provide a dynamic transformations-based perspective to enhance PSTs' understanding of the connection between similarity and proportionality.

2.2 Augmented reality

Technological tools can provide learners opportunity to investigate geometric similarity problems dynamically. These tools form a link between graphical representation of a geometrical figure and its symbolic or numeric representations (Hohenwarter and Jones 2007). Hence, one can investigate both graphical and symbolic/numeric relations among geometric figures from different orientations. In mathematics education, DGS provide this feature via constructing hot-links between multiple representations of a geometric figure (Hohenwarter and Jones 2007). Yet learning experiences on the DGS have still limited two dimensional projections of three dimensional objects on computer screens. Therefore, all dynamic manipulations on the DGS rely on indirect input mechanism like controlling with mouse and keyboard or touching on the screen. However, AR, which is a relatively new technology, provides some unique opportunities to eliminate this limitation of screen-based technologies (Özçakır 2017).

Regarding the AR technology, there is a lack of information on the potential of AR on the development of students' and PSTs' proportional reasoning in similarity problems. The AR technology can be regarded as a type of the virtual reality (VR) technology. AR allows learners to experience their real environment with superimposed virtual objects. In AR based learning settings, students and PSTs have unique opportunities of working directly with shared virtual objects collaboratively in their real environments rather than in virtual environments as in VR. Therefore, as described in Fig. 1, the AR technology can be defined as a bridge between virtual worlds and real world (Milgram et al. 1995).

In AR supported learning environments, students interact with virtual elements through using screens of mobile devices or head-mounted displays (Alcañiz et al. 2010). The AR technology serves as a learning and teaching medium. The most important feature of the AR technology is that both students and teachers have opportunities to see virtual objects as if they really exist in real environments, and they can interact with these objects in many ways, such as by touching, walking around, holding, etc. (Özçakır 2017) (Fig. 2). Moreover, the AR technology provides instant feedback to students and teachers about their physical actions on these virtual objects projected on real world.

Previous research indicated that the AR technology, as a medium of learning, reduces learners' mental effort on learning tasks, has a positive influence on learning process, enhances understanding of concepts, improves learners' comprehension on mathematical tasks, increases learners' motivations towards lessons, and develops learners' long-term memory retention of mathematical concepts (e.g., Chen 2006; Estapa and Nadolny 2015; Haniff and Baber 2003; Kaufmann 2004; Kaufmann and Dünser 2007; Lindgren and Moshell 2011; Özdemir and Özçakır 2019; Vincenzi et al. 2003; Wang and Dunston 2006). Moreover, AR experiences enhance students'

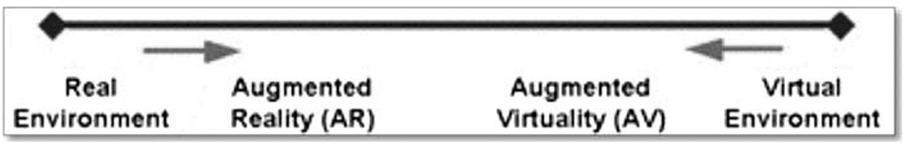

Fig. 1 Reality-virtuality continuum

Fig. 2 Augmented reality experiences via head-mounted displays

motivation in learning settings and increase their comprehension of mathematical concepts by means of higher involvement with the mathematical content to be learned (Coimbra et al. 2015).

In mathematics classes, it is important to use physical models and concrete materials by which teachers can provide multiple representations of mathematical concepts to increase students' interaction with learning tasks. However, these models and materials are static and usually not suitable for physical modifications. Hence, teachers cannot always modify these models and materials to use them in different learning settings. On the other hand, the AR technology allows teachers to modify existing models and materials and transfer them into other classroom settings. Furthermore, AR enhances teacher student interactions in learning settings by allowing students to actively participate in the learning process and providing unique opportunities for interactions (Jesionkowska et al. 2020). Therefore, the use of AR technology in mathematics classrooms may benefit both teachers and students.

The use of AR technology in education is an emerging area (Mitchell 2011). Hence, in recent years, researchers (e.g., Cerqueira and Kirner 2012; Coimbra et al. 2015; Estapa and Nadolny 2015; Özdemir and Özçakır 2019) have been paying attention to implement the AR technology in mathematics classrooms for enhancing students' achievements, understanding of some curricular units, and motivations towards learning mathematics. In terms of proportional reasoning, only a few studies (e.g., Mitchell 2011; Mitchell and DeBay 2012) investigated this concept designing AR activities. For instance, Mitchell (2011) investigated five in-service teachers' implementation of an AR activity and its paper version to support students' acquisition of multiple proportional reasoning strategies. The AR activity and paper version were applied to experimental group and control group, respectively. The findings showed that the AR activity was interesting to students and enhanced student engagement in comparison to its paper version. Similarly, implementing 17 AR sessions to motivate students to practice their proportional reasoning skills, Mitchell and DeBay (2012) determined that AR increased the students' academic engagement, created situated learning experiences, supported their thinking, and allowed transferring their proportional reasoning skills to standardized testing contexts. Thus, in this study, the AR technology used at enhancing the PSTs' proportional reasoning via providing dynamic and interactive virtual shapes for various geometric similarity tasks.

3 Methods

3.1 Research design

This study investigated potential contributions of the AR technology on the development of the PSTs' proportional reasoning when solving geometric similarity problems. An exploratory multiple-case study methodology (e.g., Yin 2009) was followed when designing this study because our main purpose of conducting this study was to explore the development of PSTs' proportional reasoning. Therefore, each individual participant constituted a case, and a multiple-case study methodology best suited the scope of this study.

3.2 Participants and recruitment procedure

The participants of this study included 17 second-year PSTs (15 female and 2 male) who were enrolled in the middle school mathematics program of a Turkish university. The PSTs were contacted in an instructional material design course on mathematics during the spring semester of 2019. The course was taught by the first author and focused on designing instructional materials for teaching middle school mathematics topics. This group of PSTs was included in the study because the course content also included the use of educational technologies in learning settings. Hence, the PSTs had a chance to explore the contributions of the AR technology in a learning setting. During the course, the PSTs did not have any university level instruction on proportional relationships and geometric similarity.

3.3 The data collection and analysis

The data included the PSTs' written responses to a paper-and-pencil test with five problems (see Appendix) that had been administrated before and after the implementation of the AR activities. Furthermore, video recordings were collected during the implementation of the AR activities. In the second week of the semester, which was 14 weeks long, the PSTs were given the paper-and-pencil test that they were required to complete in 50 min. After collecting their responses, the second author designed the AR activities (see Appendix) with four tasks in which the design process took several weeks. After designing the AR activities, we provided the PSTs with these activities in the twelfth week. The PSTs were randomly divided in seven groups in which three groups included three PSTs and four groups included two PSTs. Each PSTs was provided with a paper-and-pencil form of activity problems, and each group was given an Android tablet that included the AR software to work on those problems. These tablets were also used as a data collection tool via video recording the PSTs' works in these groups. The implementation of the AR activities took two class hours, 100 min, in which both authors presented in the classroom.

Before the AR implementation process, the PSTs stated that they were familiar with AR. They mentioned that the second author of this study provided them with some information about AR and its applications in education during an educational technology course that they took a year ago. To refresh the PSTs' knowledge on AR, the second author explained one more time how to use tablets when working on the AR activities. During the implementation process, the PSTs worked together on the activities and shared their understanding of problems with their group mates and with the authors. The authors responded to the PSTs' questions and asked questions to the PSTs about the activities by circulating between groups. One week after completing the implementation of the AR activities, the PSTs were given the paper-and-pencil test again to understand changes in their reasoning.

We used a content analysis method (e.g., Hsieh and Shannon 2005) to analyze the PSTs' responses to the paper-and-pencil test. To conduct this content analysis, we recorded the PSTs' responses in an Excel file. In the Excel file, the PSTs' first test responses and second test responses were recorded side-by-side in front of their names. For instance, after entering their first test responses to the five problems, we entered their test second responses for the same problems besides of them. After recording the PSTs' responses, we coded these responses as correct, partially correct, wrong, and no answer and generated tables using these codes. In these tables, we presented findings using descriptive statistics (i.e., frequencies and percentages). Furthermore, the collected video recordings were watched by the two authors and transcribed verbatim for research purposes.

3.4 The geometric similarity test

The geometric similarity test included five paper-and-pencil problems (see Appendix). Three of these five problems included comparison of two-dimensional regular figures, one included comparison of two-dimensional irregular figures, and another problem included comparison of three-dimensional regular figures. In the first problem, which is adapted from Beckmann (2013), the PSTs were given a 6 by 5 rectangle, A, that is formed by xs. The PSTs were told that this rectangle was $\frac{5}{2}$ times of another rectangle, B, and so they were asked to represent this rectangle. The PSTs could recognize that rectangle B was two-fifths of rectangle A. Hence taking two-fifths of A could yield a 6 by 2 rectangle, which represents rectangle B. By providing this problem, we aimed at investigating if the PSTs were able identify the multiplicative relationship between the areas that two rectangles covered. However, we expected some PSTs to focus on the multiplicative relationship between the edges rather than the areas.

In the second problem, which was adapted from Arican et al. (2018), a trapezoid was enlarged by a factor of $\frac{3}{2}$ and converted to a similar trapezoid. The PSTs were asked to calculate the lengths of edges in this enlarged trapezoid. There was a proportional relationship between the lengths of their corresponding edges. Hence, we aimed at investigating if the PSTs were able to recognize that the factor of enlargement was preserved between the lengths of corresponding edges of these two trapezoids.

The third problem was adapted from De Bock et al. (2002). Similar to the second problem, this problem also involved enlargement by some factor, which was three. However, the enlargement was applied to an irregular two-dimensional figure. By providing this problem, we aimed at examining the PSTs' understanding of the nonproportional relationship between the sizes of two figures and amount of paint required to paint them. The ratio formed by the areas of two horse figures was equal to the square of the enlargement factor (i.e., $\frac{Area\ of\ the\ small\ figure}{Area\ of\ the\ large\ figure} = k^2$ in which k was equal to three). It was difficult for the PSTs to apply traditional methods to calculate their areas because the two figures had irregular shapes. Therefore, we provided the amount of paint as an indirect measure of areas.

In the fourth problem, similar to the third problem, we investigated the same nonproportional relationship using two regular two-dimensional figures (i.e., rectangles). There was a nonproportional relationship between the amounts of seeds required to plant two similar shaped gardens. The ratio formed by the areas of two gardens was equal to the square of the enlargement factor (i.e., $\frac{Area\ of\ the\ small\ garden}{Area\ of\ the\ large\ garden} = k^2$ in which k was equal to five). Therefore, the amount of seeds needed for planting the large garden was 25 times (i.e., 5^2) the amount of seeds required for the small garden.

In the last problem, we investigated a nonproportional relationship using three-dimensional regular figures (i.e., cubes). In this problem, the ratio formed by the volumes of two cubes were equal to the cube of the enlargement factor (i.e., $\frac{Volume\ of\ the\ small\ cube}{Volume\ of\ the\ large\ cube} = k^3$ in which k was equal to two). The PSTs were expected to recognize that enlarging the size of each box by a factor of two would have increased the volume of the box by eight (i.e., 2^3). Therefore, the PSTs should calculate that the enlarged box can take 64 cube-shaped olive oil cans.

3.5 The augmented reality activities

The AR activities were designed from scratch on using the Unity 3D engine with Vuforia AR SDK and C# programming language. The AR activities were designed as marker-based AR, so that the AR activity sheets included qr-codes as markers to initiate AR experiences. These qr-codes were detected by the cameras on tablets. Moreover, the AR activities had some interactivity layers to provide a game based interactive learning environment for the purpose of enhancing active learning. The main purpose of using AR in education is to provide a virtual multi-dimensional object in a real working environment without substituted fabric of reality (Özçakır 2017). Hence, in this study, the AR activities included tasks with not only two-dimensional objects but also three-dimensional figures to improve the PSTs' understanding of geometric similarity.

The AR activities included four AR tasks on geometric similarity. First two tasks designed to investigate the PSTs' understanding of geometric similarity in two-dimensional regular geometric figures. The first activity involved comparison of squares, and the second activity involved comparison of rectangles. In general, the PSTs were asked to find out how many a given square or a given rectangle there were in a larger square or rectangle. Since these two tasks included touch for assemble interactivity layer, the PSTs were supposed to form the geometric figure in question, which was either a square or a rectangle, by touching on virtual objects that were projected on a qr-code marker (Fig. 3). Therefore, the PSTs were able to assemble the required figure by touching on the screens of their tablets.

The third task was about filling a cube shaped box with some cube shaped bricks. Hence, this task was designed to investigate the PSTs' understanding of volumetric relationships in two cubes. As in the first two tasks, this activity had touch for assemble interactivity. The PSTs were asked to fill a container by bricks by touching on their screens and assembling pieces. Therefore, they had to think similarity concept from a three-dimensional perspective. The virtual bricks and containers were projected on a qr-code marker via the AR software (Fig. 4). Learners were able to interact with these bricks and containers in natural ways of interactions because of the basic feature of the AR. Hence, AR allowed the PSTs to easily count the number of bricks needed for the container by only inspecting with their eyes and circling around these containers to analyze their dimensions.

The last task was slightly different form the remaining three tasks in terms of the geometric figures used. While the first three tasks included regular figures, the last task included irregular geometric figures. For this task, the AR software provided some virtual irregular figures (i.e., varying sized dinosaur figures) on two walls of a school that the PSTs were asked to paint by throwing paint bombs. Therefore, the AR software involved touch for bomb interactivity for this task. In the story of the task, the PST were asked to paint these irregular figures by throwing paint bombs as they needed to paint the whole area via touching on virtual figures (Fig. 5). Thus, they were needed to think about the areas that dinosaur figures covered on the walls to find out the required number of paint bombs to complete painting.

The AR activities were designed to provide an interactive learning environment that aimed at facilitating the PSTs' active participation in the learning process and to reveal their mathematical thinking. At the beginning of the study, the PSTs were informed about how they could use the AR software uploaded in android tablets by providing some example AR tasks. Since the AR activities involved only basic direct ways of interactions such as walking around, turning over or touching inputs, the use of the AR software was easy for these PSTs. Therefore, no technical support was necessary throughout administration of the tasks.

4 Results

The PSTs' responses to the paper-and-pencil test and findings obtained from video recordings are presented in the following pages.

4.1 The Preservice teachers' responses to the paper-and-pencil test

The PSTs' first and second responses to the five problems are presented in Table 1. In Table 1, the PSTs' responses are categorized as correct, partially correct, wrong, and no answer. Table 1 highlights the changes in the PSTs' responses before and after the implementation of the AR activities.

In the Rectangle problem, 10 PSTs (59%) were able to represent rectangle B both in their first time and second time taking the test. Hence, there was not an increment in the number of correct responses. Among 10 PSTs, eight of them were able to represent rectangle B in both tests. Two and five PSTs who provided incorrect representations in the first and second tests, respectively, appeared to focus on the multiplicative relationship between the edges of rectangles rather than their areas. As represented in Fig. 6, taking two-fifths of the lengths of edges in rectangle A, these PSTs expected rectangle B to have edges with 2.4 and 2 xs. Hence, all five incorrect responses in the second test being on this issue suggested the PSTs' incorrect interpretations of the problem context. In the first test, one PST incorrectly expanded the edge with six xs by $\frac{5}{2}$ and obtained a 15 by 5 rectangle.

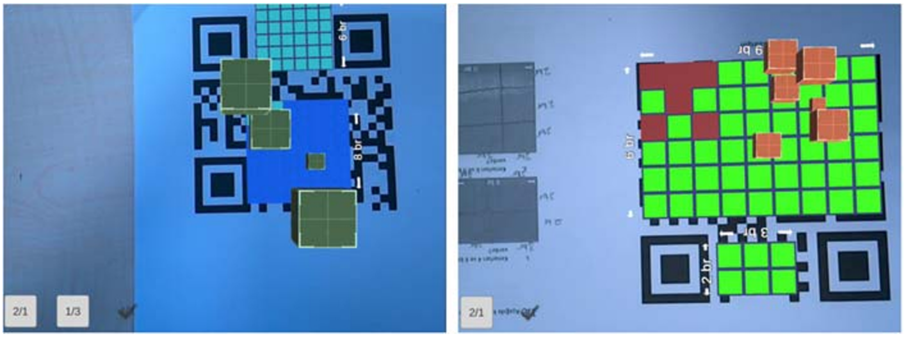

Fig. 3 The AR activities about forming a square and a rectangle using unit squares

The same PST did not provide a response in the second test. A PST who made a calculation error and obtained a 6 by 3 rectangle in the first test was able to calculate the correct answer in the second test. Moreover, a PST expanded both edges by $\frac{5}{2}$ and obtained a 15 by 12.5 rectangle in the first test. The same PST represented rectangle B with 2.4 and 2 xs in the second test. In addition, two PSTs who provided partially correct responses in the first test obtained wrong answers in the second test. Finally, one of the two PSTs who provided correct responses in the first test provided wrong answer in the second test, and the other PST did not provide a response in the second test.

In the Trapezoid problem, 10 (59%) and 14 PSTs (82%) were able to calculate the lengths of edges in the first test and second test, respectively. All the PSTs who provided incorrect answers in the first and second tests tended to add the original length of edges over their enlarged lengths. These PSTs obtained the lengths as 7.5 cm and 12.5 cm (e.g., $3\ cm + 3\ cm \times \frac{3}{2} = 7.5\ cm$ and $5\ cm + 5\ cm \times \frac{3}{2} = 12.5\ cm$) (Fig. 7). These PSTs appeared to not understand that the enlargement on the original trapezoid preserved the factor of enlargement that is formed between the lengths of corresponding edges of these two trapezoids (e.g., $\frac{\frac{9}{2}}{3} = \frac{\frac{15}{2}}{5} = \frac{3}{2}$). Similarly, the ratio formed by the lengths of short and long edges within each trapezoid is also preserved in two trapezoids (e.g., $\frac{3}{5} = \frac{\frac{9}{2}}{\frac{15}{2}}$). However, none of the PSTs recognized the preservation of this within ratio. Overall, the increment in the number of PSTs who provided correct answers in the second test and transition from wrong to correct answers suggested an improvement in the PSTs' understanding of the proportional relationship between the lengths of their corresponding edges.

Table 1 shows that the most problematic problem in the first test was the Advertisement problem. In the first test, only one PST was able to provide the correct answer, 36 ml paint. Two PSTs provided partially correct answers in which one PST made a calculation error and obtained 31 ml as her answer, and one PST wrote that "The figure is not regular, so we cannot say it is 12 ml" but did not calculate the answer. The remaining 14 PSTs expected a directly proportional relationship between the sizes of figures and amount of paint. Hence, many of these PSTs incorrectly applied the cross-multiplication strategy and obtained 12 ml as their answers (Fig. 8). On the other hand, in the second test, 10 PSTs (59%) were able to obtain the correct answer by recognizing the amount of paint was proportional to the square of the enlargement factor. Moreover, two PSTs who provided incorrect answers in the first test provided partially correct

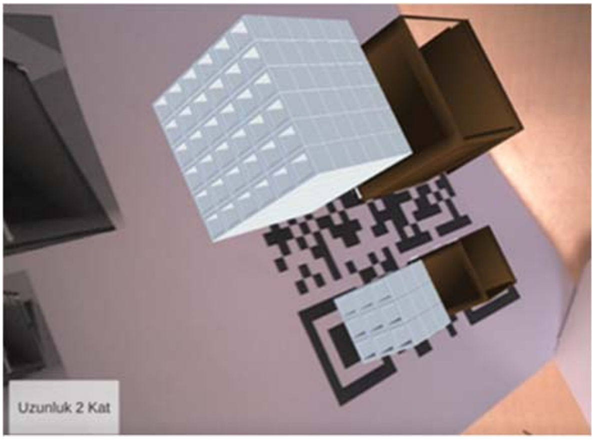

Fig. 4 The AR activity about volumetric relationships

answers in the second test, and five PSTs did not change their incorrect answers. The PSTs' second test responses suggested that in addition to the increment in the height of the figure, they also attended to the increment in the width of the enlarged horse figure. These PSTs usually wrote that "Besides the height, the width also increases by 3. So, the amount of paint increases by 9." Therefore, the increment in the number of PSTs who provided correct answers and transition from wrong to correct answers suggested an improvement in their understanding of the nonproportional relationship (i.e., the amount of paint required for the enlarged horse figure is 9 times the amount of paint used for the small figure) between the sizes of two figures and amount of paint used.

On the contrary to the Advertisement problem, the PSTs were mostly successful at obtaining the correct answer in the Farm problem. Sixteen and 17 PSTs obtained the correct answer in the first test and second test, respectively. The PSTs who provided correct answers, first, calculated the areas of two rectangles. Next, recognizing the directly proportional relationship between the area of a rectangle and amount of seeds needed, they applied the cross-multiplication strategy (Fig. 9).

In the Olive Oil problem, while all PSTs obtained the correct answer in the first test, 15 PSTs obtained the correct answer in the second test. One of these two PSTs made a mistake in calculating the volume of the box and obtained an incorrect answer, and the other PST expected the number of olive oil cans to be placed in the box being equal to 8^3. In both tests, after taking an arbitrary measure (e.g., 1 unit, 1 cm, *x* cm, etc.) for the length of the olive oil can, the PSTs calculated the volumes of the olive oil can and box. Next, they applied the cross-multiplication strategy to calculate the answer (Fig. 10). The PSTs usually calculated the correct answer by forming a proportional relationship between the volume of the box and number of olive oil cans. However, some PSTs calculated the correct answer by recognizing the multiplicative factor between volumes being eight. Hence, these PSTs multiplied eight olive oil cans by 8 and obtained 64 cans.

4.2 The findings obtained from video recordings

This section summarizes some key findings obtained from the video recordings collected during the implementation of the AR activities. Although the PSTs had some

Fig. 5 The AR activity about irregular figures

previous experience with AR technology, they puzzled about using tablets in the beginning of the AR session. The video recordings showed the PSTs' initial tendencies to solve AR activities by either tiling a large figure with the small figure and then counting the number of small figures inside the large figure or calculating areas (or volumes) of two similar figures and then figuring out how many small figures can be placed in the large figure. The analysis of the PSTs' written responses to the first two AR activities showed that none of them recognized the square of the enlargement factor was yielding the number of small figures tiled in the large figure. Similarly, in the third

Table 1 The Preservice Teachers' Responses to the Paper-and-Pencil Test

			Second Time Taking the Test			
			C	**PC**	**W**	**NA**
Rectangle	First Time Taking the Test	**C**	8	–	1	1
		PC	–	–	2	–
		W	2	–	2	1
			C	**PC**	**W**	**NA**
Trapezoid		**C**	10	–	–	–
		W	4	–	3	–
			C	**PC**	**W**	**NA**
Advertisement		**C**	1	–	–	–
		PC	2	–	–	–
		W	7	2	5	–
			C	**PC**	**W**	**NA**
Farm		**C**	16	–	–	–
		W	1	–	–	–
			C	**PC**	**W**	**NA**
Olive Oil		**C**	15	–	2	–

Note. C: Correct; PC: Partially Correct; W: Wrong: and NA: No Answer.

activity, none of the PSTs recognized that the cube of the enlargement factor was yielding the number of cube-shaped bricks placed into the cube-shaped boxes. Therefore, being able to calculate the area or the volume of a given figure appeared to direct the PSTs to apply conventional solution methods.

As we stated in the Methods section, during the implementation of the AR activities, we circulated between groups and asked questions to the PSTs about their understandings of the relationships presented in those activities. For instance, the following conversations are reported from a group of PSTs while they were working on the third AR activity:

> PST1: Wait a little, let me count [counting number of small cubes in the large cube]. One, two, three, four, five, and six. Six times six is 36 and 36 times 6 is…
> PST2: 216.
> Author1: How have you been doing? Did you understand the problem? It is asking how many times more bricks you can put in the box.
> PST2: Yes, we understand.
> Author1: How many times?
> PST1: The same result, eight times.
> Author1: If you had increased the length of the box by three times, then what would be your answer?
> PST1: I would need to take the cube of that again…what do you mean by three times?
> Author1: I mean if the length of box was increased from 3 units to 9 units. What would be the easiest way to solve this problem?
> PST1: Three of these [pointing at 1 unit cubes] will be one…279 [she miscalculated, the correct calculation should be 81 times 9 is equal to 729]?
> PST2: Is not it going to be 27 times?
> PST1: Uhhh [surprised], I understand, it is the cube of three. I got it.

The exchanges above showed the PSTs' initial tendencies to calculate results by counting objects and calculating the volumes of brick and box. The PST1 responded that "I would need to take the cube of that again." However, she implied taking the cube of 9 units to calculate the volume of the enlarged box, which she mistakenly calculated as 279. Hence, as in this example, some PSTs did not initially recognize that

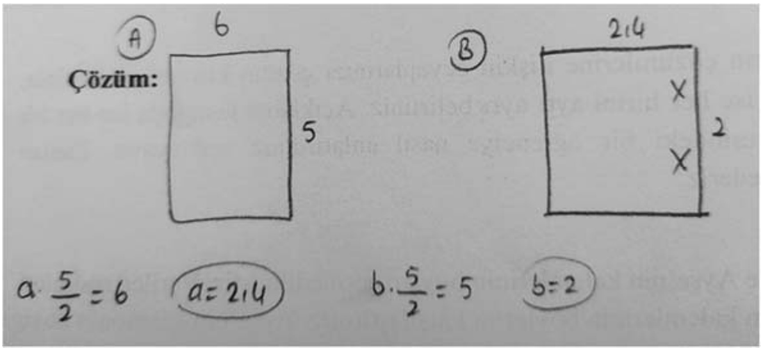

Fig. 6 A PST's incorrect representation of a similar rectangle

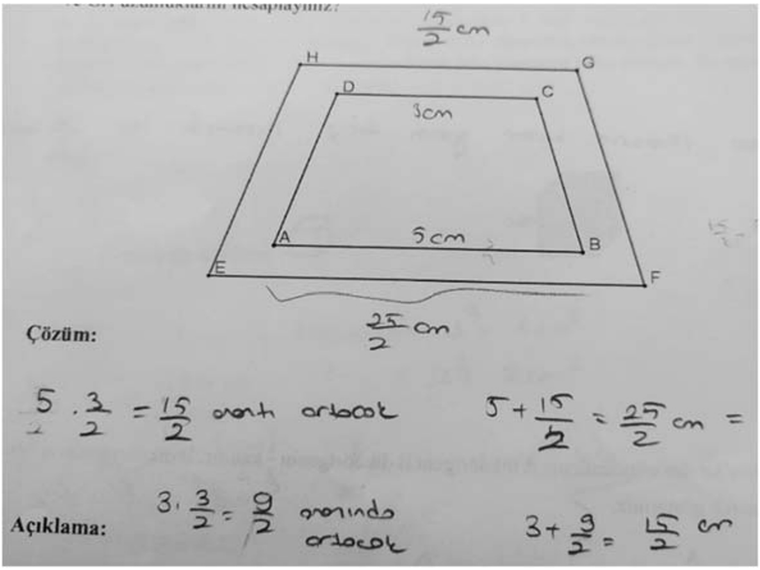

Fig. 7 A PST's incorrect solution for the trapezoid problem

the cube of the enlargement factor (i.e., $2^3 = 8$) was yielding the relationship between how many times more bricks could be placed in the enlarged box. In the exchanges, PST1 recognized this relationship after receiving questions from the first author.

In the first test, most of the PSTs had difficulty answering the Advertisement problem. Hence, the Kindergarten activity (see Appendix) was designed to facilitate the PSTs' understanding of the nonproportional relationship (i.e., $\frac{Area\ of\ the\ small\ figure}{Area\ of\ the\ large\ figure} = k^2$) presented between the areas of two irregular figures. The following conversations occurred between the members of the same group when working on the first problem of the Kindergarten activity. In the problem, the PSTs were asked to calculate the number of cans required to paint a dinosaur figure with 40 cm length.

PST1: How many paint bombs did we throw?
PST2: Eight.
PST1: How many for this [pointed at the figure with 20 cm length]?
PST2: Two. It was two and now eight.

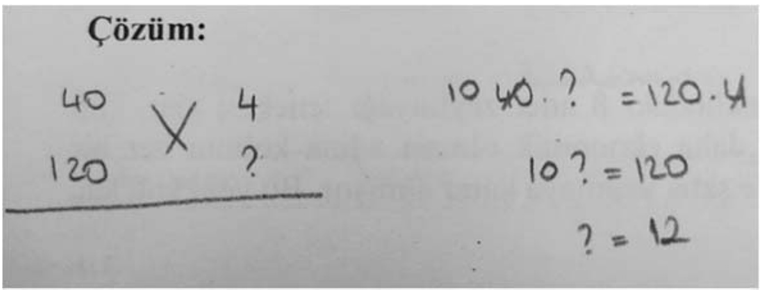

Fig. 8 A PST's incorrect cross-multiplication strategy in the advertisement problem

PST1: It is taking the cube again. It was calculated based on the volume.
PST2: For this one [pointed at the figure with 20 cm length], we used two cans of paint and for this one [pointed at the figure with 40 cm length] eight.
PST1: It is taking the cube again…calculated based on the volume. But why it is calculated by the cube…?
PST2: It should be the area.
PST1: There are two cans of paint for 20 cm, then for 40 cm it is eight…
PST2: I do not know.
PST1: There are two [cans of paint] for 20 and eight [cans of paint] for 40.
PST2: There are two [cans of paint] for 20 and eight [cans of paint] for 40. How many times did it increase?
PST1: It increases by 3 [she meant to say the cube of two]. What did we found for the figure with 60 cm length? Was it 18?
PST2: 18.
PST1: There were 18 for 60.
PST2: It is three times of 20…nine times…uhh it is the square of….when the length increases by 2, the number of cans increases by 4. When it increases by 3, then paint increases by 9. It is the area.

The exchanges above demonstrated two PSTs' exploration of the relationship between the sizes of two irregular dinosaur figures and number of cans required to paint these figures. The PSTs initially expected the number of cans required to paint the figure with 40 cm length to be the cube of the enlargement factor. Next, they realized that taking the cube of this enlargement factor was related with calculating the volume. Hence, this created a disequilibrium in their understanding since they knew that the painting was related with the areas of two figures. Later, PST2 recognized that they should have taken the square of the enlargement factor instead of taking the cube of it. In our examination of the videos obtained from the remaining six groups, we observed similar conversations. Therefore, this AR activity facilitated these two PSTs' recognition of the multiplicative relationship (i.e., the amount of paint was proportional to the square of the enlargement factor) between the sizes of two dinosaur figures and amount of paint needed.

5 Conclusions and discussion

This study aimed at developing the PSTs' proportional reasoning in the context of geometric similarity by means of implementing the AR activities. The potential

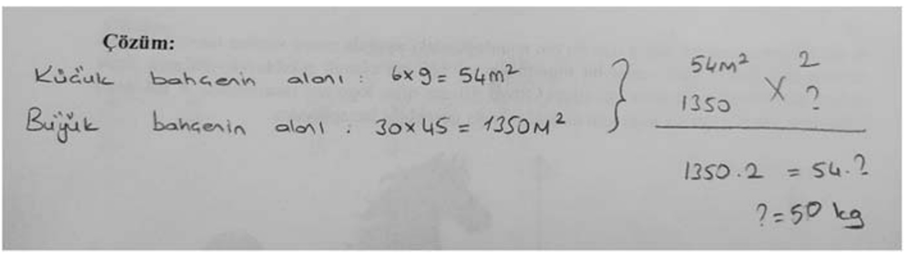

Fig. 9 A PST's correct cross-multiplication strategy in the garden problem

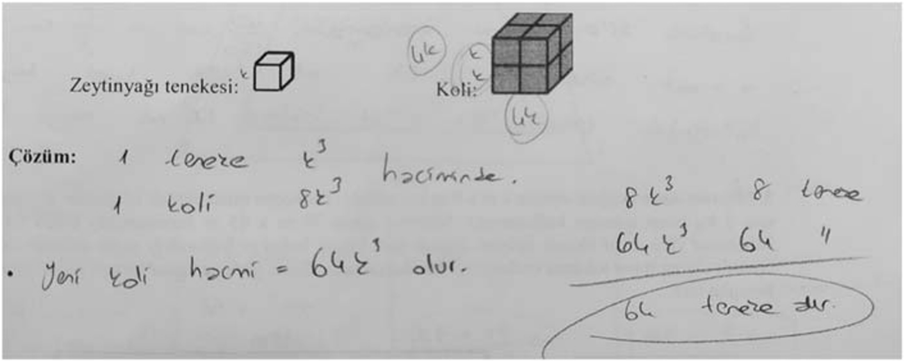

Fig. 10 A PST's correct cross-multiplication strategy in the olive oil problem

contributions of the AR activities on the development of the PSTs' proportional reasoning were investigated via applying a geometric similarity test before and after the implementation of the AR activities.

In the first research question, we investigated the PSTs' responses to the geometric similarity problems before and after the implementation of AR activities. The PSTs' first test responses suggested their abilities to solve similarity problems with two-dimensional and three-dimensional regular figures. However, in the first test, only one PST, who was the only participant provided correct answers to all five problems in the first and second tests, was able to solve the Advertisement problem that involved enlargement in the area of an irregular two-dimensional figure. This finding agreed with De Bock et al. (1998) study who reported students' difficulties with discovering the impact of enlargement on irregular figures. Hence, similar to students, the PSTs also had difficulties with solving similarity problems with irregular figures. Regarding the explanation of this difficulty, the PSTs' written solutions and video recordings showed that in the Advertisement problem, the PSTs could not tile the large horse figure with the small horse figure, so they could not count how many small horse figures there were in the large horse figure. Moreover, they could not apply traditional methods to calculate the areas of these two horse figures. Therefore, the PSTs had difficulties with solving this problem in the first test. After the implementation of the AR activities, the PSTs' incorrect responses to the Advertisement problem drastically decreased.

Beside the effect of figures being regular or irregular on the PSTs' performances in solving the similarity problems, we observed that familiarity with the problem context was another factor that appeared to affect their performances. The PSTs were more successful in the Farm and Olive Oil problems both in the first test and second test in comparison to the remaining three problems. These two problems included contexts that the PSTs used to solve in middle and high school. However, Rectangle, Trapezoid, and Advertisement problems included nonroutine contexts that they did not have experience with. While the number of correct responses to the Trapezoid and Advertisement problems increased in the second test, it stayed the same for the Rectangle problem. On the contrary to the familiarity with problem contexts, the number of dimensions of similar figures did not appear to affect the PSTs' performances. The Farm and Olive Oil problem, which included comparison of two-dimensional and three-dimensional regular figures, respectively, were almost answered by all the PSTs in the first test and second test.

Regarding the second research question, after the implementation of the AR activities, the number of correct responses was increased for the Trapezoid, Advertisement, and Farm problems. On the other hand, the number of correct responses stayed the same for the Rectangle problem and slightly decreased for the Olive Oil problem. It is important to state that many PSTs avoided applying proportional strategies in the second test for solving the Advertisement problem that suggested development in their understanding of the nonproportional relationship. Hence, similar to Gerretson (2004) who reported positive effects of DGS on PSTs' performances on geometric similarity problems, the AR activities used in this current study enhanced the PSTs' performances on similarity problems. This finding was also consistent with previous studies (e.g., Kaufmann 2004; Kaufmann and Dünser 2007; Özdemir and Özçakır 2019) that reported positive effects of AR on students' understanding of mathematical concepts. However, as reported by Van Dooren et al. (2004), some PSTs still applied proportional strategies to solve the Advertisement problem. This finding confirmed that changing PSTs' overuse of proportionality in nonproportional situations is an extremely challenging task.

In this study, the AR activities provided opportunities for the PSTs to work with dynamic and interchangeable virtual objects both interactively and collaboratively. Moreover, the PSTs were provided with instant feedbacks about their actions on these virtual objects through the AR technology. These features of AR assisted the PSTs to focus more on the learning tasks rather than focusing on the technical and procedural operations. Hence, as reported by Haniff and Baber (2003) and Wang and Dunston (2006), the AR technology reduced the PSTs' mental efforts on understanding mathematical concepts. In addition, AR activities enabled the PSTs to investigate proportionality in similarity problems from a dynamic-transformations perspective.

6 Limitations, implications, and suggestions for future research

In this study, we could not investigate the effect of the AR activities on the PSTs' performances on the similarity problems with three-dimensional irregular figures because designing an AR activity with three-dimensional irregular figures is a very complex process. However, similar to the Advertisement problem, we expect PSTs to have difficulties in solving similarity problems with three-dimensional irregular figures. Hence, future studies can design AR activities with three-dimensional irregular figures and investigate the effect of AR activities on students' or PSTs' performances. Moreover, this study was conducted with a relatively small group of PSTs. Therefore, we recommend future studies to be conducted with students and in-service teachers and to include larger sample sizes. The findings obtained from this study highlights that the AR technology can be used in developing PSTs' proportional reasoning and facilitating their ability to understand the connection between proportionality and geometric similarity. Thus, we encourage the implementation of AR-based activities in schools and teacher education programs to develop students' proportional reasoning to ease teachers' jobs in making connections between proportionality and geometric similarity.

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Availability of data and materials The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Authors' contributions MA taught the course and designed geometric similarity test items. BO designed augmented reality activities. Both authors reviewed the test items and augmented reality activities, analyzed preservice teachers' responses, and contributed to writing of the necessary sections of the manuscript. The authors read and approved the final manuscript.

Compliance with ethical standards

Competing interests The authors declare no potential competing interests with respect to the research, authorship, and publication of this article.

Appendix

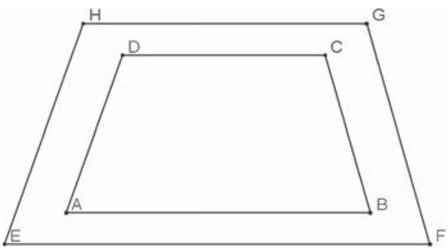
The Geometric Similarity Test

1. The rectangle A formed by *x*s is $\frac{5}{2}$ times of the rectangle B. Please draw rectangle B next to A.

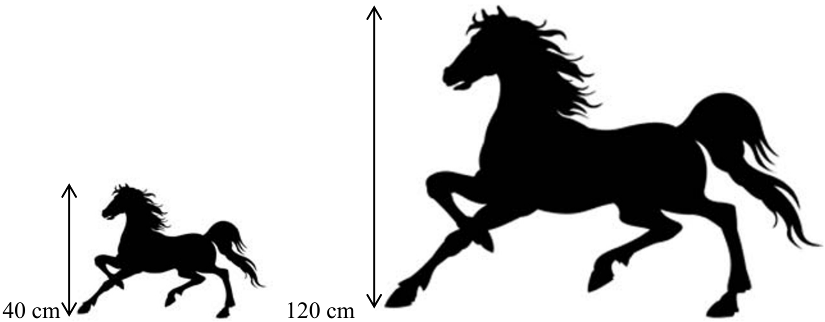
A

x x x x x x
x x x x x x
x x x x x x
x x x x x x
x x x x x x

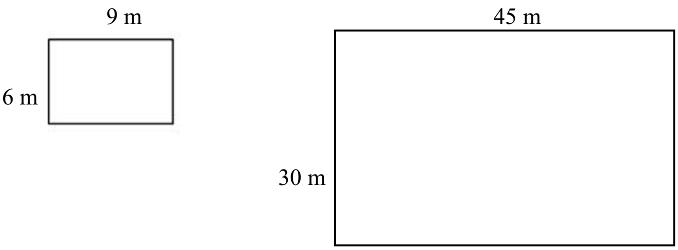
2. In the figure below, the edges of ABCD isosceles trapezoid are enlarged by $\frac{3}{2}$ and converted to EFGH isosceles trapezoid. The lengths of AB and CD edges are measured as 5 cm and 3 cm, respectively. Please calculat*e the lengths of EF and GH*?



3. An advertising agency has designed a 40 cm-long logo for a company. The company then requested an enlarged logo with 120 cm in length to be posted on billboards. If 4 ml paint is used in the design of the logo with 40 cm in length, calculate how much paint is required for the logo with 120 cm in length.



4. A farmer planted corn in his garden with 6 m x 9 m lengths. For doing this he used 2 kg of corn seed. He wants to plant corn in another garden with 30 m x 45 m lengths. He plans to use corn seeds in proportion to the amount of corn he used in his small garden. Please calculate how much seed he will need.



5. An olive oil company sells olive oils in cube-shaped boxes in which each can take 8 cube-shaped olive oil cans. In order to be more economical, the company decides to double each side of the boxes. Please calculate how many olive oil cans this new boxes can take.

Olive oil can: 

Box: 

The Augmented Reality Activities

The Augmented Reality Activities

1. You are given a square with a size of 2 units.

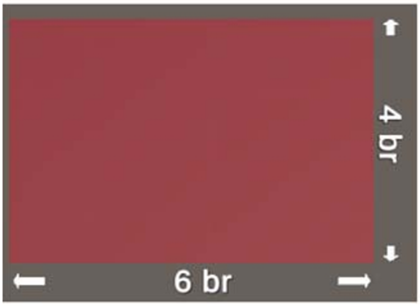
i. How many 2 unit squares are there in a 6 unit-square?

ii. How many 2 unit squares are there in a 8 unit-square?

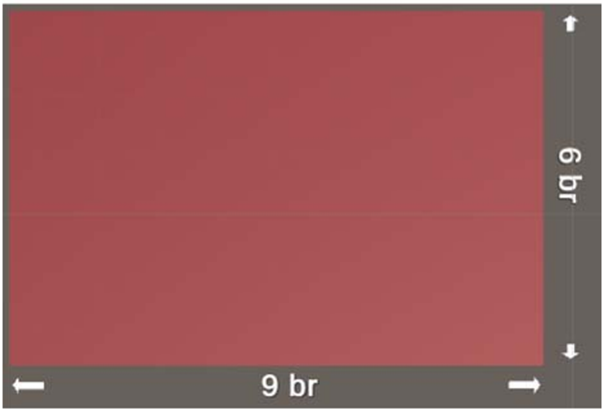
iii. How many 2 unit squares are there in a 12 unit-square?

2. You are given a rectangle with edges 3 units and 2 units in length.

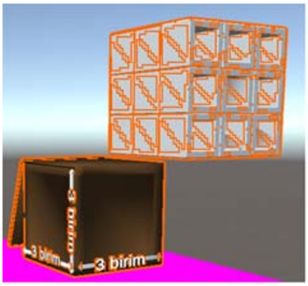
i. How many 3x2 rectangles are there in a 6x4 rectangle?

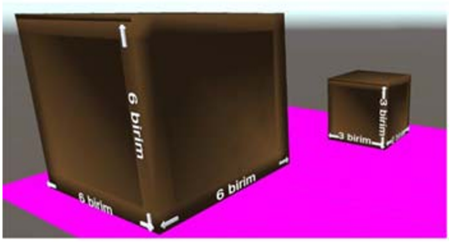


ii. How many 3x2 rectangles are there in a 9x6 rectangle?



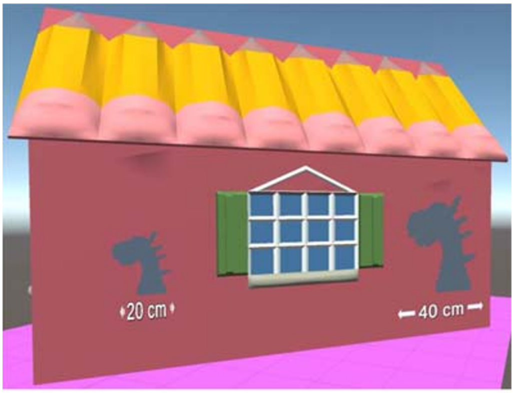
3. A company sells cube-shaped bricks with a side of 1 unit by packing them into cube-shaped boxes with a length of 3 units. If this company doubles the lengths of edges of these boxes, how many times more bricks can they put in each box?



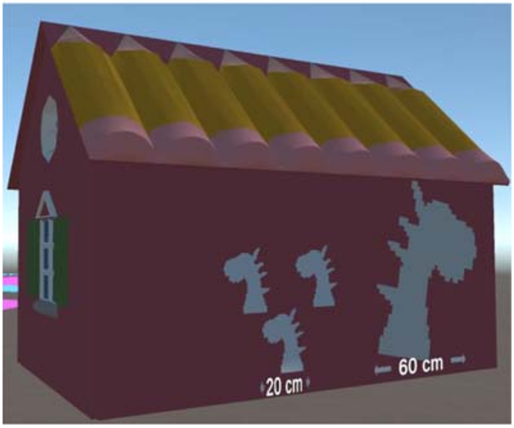


4. A teacher wants to paint dinosaur pictures on the walls of a kindergarten. The teacher was able to paint half of a 20 cm wide dinosaur picture with a can of paint. So, she asks help from a mathematics teacher to calculate how many cans of paint she needed in total.

i. How many cans of paint does she need for the dinosaurs with 20 cm and 40 cm widths that are shown in below on the right wall of the school?



ii. How many cans of paint does she need for the dinosaurs with 20 cm and 60 cm widths that are shown in below on the left wall of the school?



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Affiliations

Muhammet Arican[1] **· Bilal Özçakir**[2]

[1] Department of Mathematics and Science Education, Kirsehir Ahi Evran University, Kırşehir, Turkey

[2] Department of Mathematics and Science Education, Alanya Alaaddin Keykubat University, Alanya, Turkey