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Calderón–Zygmund operators in Morrey spaces

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Abstract This paper deals with mapping properties of classical Calderón–Zygmund operators in local and global Morrey spaces.

Keywords Calderón–Zygmund operators · Morrey spaces

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1 Introduction

Let T_0,

$$(T_0 f)(x) = \lim_{\varepsilon \downarrow 0} \int\limits_{y \in \mathbb{R}^n, |y| \ge \varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.1}$$

with

$$\Omega \in C^1(\mathbb{S}^{n-1}), \quad \int\limits_{\mathbb{S}^{n-1}} \Omega(\sigma)\, \mathrm{d}\sigma = 0, \tag{1.2}$$

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be the classical Calderón–Zygmund operators, where $f \in \operatorname{dom} T_0 = D(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$. Let $1 < p < \infty$. Then there is a constant $c = c_p > 0$ such that

$$\|T_0 f \,|L_p(\mathbb{R}^n)\| \le c\,\|f\,|L_p(\mathbb{R}^n)\|, \quad \text{for all } f \in \operatorname{dom} T_0 = D(\mathbb{R}^n). \tag{1.3}$$

This is a cornerstone of harmonic analysis in $\mathbb{R}^n$ since more than 60 years beginning with [5]. There are numerous papers and books dealing with various generalizations of (1.1)–(1.3). The related classical theory in $L_p(\mathbb{R}^n)$ may be found in [34, Chapter II], [35, Chapters VI, VII] and [37, Chapter XI]. Furthermore, a lot of attention has been paid to the question to which extent one can replace $L_p(\mathbb{R}^n)$, $1 < p < \infty$, in (1.3) by other spaces on $\mathbb{R}^n$ (more or less related to L_p-spaces). In particular one tries to reduce mapping properties for T_0 in more general spaces to (1.3) combined with

$$|(T_0 f)(x)| \le c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y, \quad f \in D(\mathbb{R}^n), \quad x \notin \operatorname{supp} f. \tag{1.4}$$

This transference method goes back to [33] and had been elaborated afterwards with respect to Morrey spaces by [10,13,19]. We refer in particular to the most recent paper [26] where arguments of this type have been applied to Morrey-type spaces based on so-called grand Lebesgue spaces. But the step from the Lebesgue spaces $L_p(\mathbb{R}^n)$, $1 < p < \infty$, to local Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$ and global Morrey spaces $L^r_p(\mathbb{R}^n)$ where $0 \le \frac{n}{p} + r < \frac{n}{p}$ (defined below) by transference arguments causes a serious problem which must be treated with greater care than in some related papers. Since $D(\mathbb{R}^n)$ is dense in $L_p(\mathbb{R}^n)$, $1 < p < \infty$, one can extend T_0 according to (1.1)–(1.3) by completion to $L_p(\mathbb{R}^n)$. By transference arguments we try to prove (1.3) with global Morrey spaces $L^r_p(\mathbb{R}^n)$ in place of $L_p(\mathbb{R}^n)$. But $D(\mathbb{R}^n)$ is not dense in $L^r_p(\mathbb{R}^n)$, $1 < p < \infty, 0 < \frac{n}{p} + r < \frac{n}{p}$, as shown at the beginning of Sect. 2.2. Rescue comes from the recent paper [1] which rises harmonic analysis in many respects, including the Calderón–Zygmund theory, from $L_p(\mathbb{R}^n)$, $1 < p < \infty$, to the global Morrey spaces $L^r_p(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le \frac{n}{p} + r < \frac{n}{p}$, based on the crucial observation that $D(\mathbb{R}^n)$ is dense in the predual of $L^r_p(\mathbb{R}^n)$ (which is described in [1] explicitly with the help of Hausdorff capacities and Muckenhoupt weights). We are not interested here in the predual of $L^r_p(\mathbb{R}^n)$, but in the predual of this predual. This is the completion of $D(\mathbb{R}^n)$ in $L^r_p(\mathbb{R}^n)$, denoted as $\mathring{L}^r_p(\mathbb{R}^n)$. Hence $L^r_p(\mathbb{R}^n)$ is the bidual of $\mathring{L}^r_p(\mathbb{R}^n)$,

$$\mathring{L}^r_p(\mathbb{R}^n)'' = (\mathring{L}^r_p(\mathbb{R}^n)')' = L^r_p(\mathbb{R}^n), \quad 1 < p < \infty, \quad 0 \le \frac{n}{p} + r < \frac{n}{p}.$$

This restores the above-described situation for T_0 in $L_p(\mathbb{R}^n)$. The rest is a matter of duality.

A linear bounded operator T acting in $L^r_p(\mathbb{R}^n)$, hence $T : L^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$, is called an extension of T_0 to $L^r_p(\mathbb{R}^n)$ if it coincides on $D(\mathbb{R}^n)$ with (1.1). Similarly for $\mathring{L}^r_p(\mathbb{R}^n)$ and $\mathcal{L}^r_p(\mathbb{R}^n)$. It is the main aim of this paper to prove the following assertion.

Theorem 1.1 *Let T_0 be given by* (1.1), (1.2) *with dom $T_0 = D(\mathbb{R}^n)$. Let $1 < p < \infty$, $0 \le \frac{n}{p} + r < \frac{n}{p}$.*

(i) *Let Ω be not identically zero. Then there is no linear and bounded extension of T_0 to the local Morrey space $\mathcal{L}^r_p(\mathbb{R}^n)$.*

(ii) *There is a linear and bounded extension T of T_0 to $\mathring{L}^r_p(\mathbb{R}^n)$,*

$$T : \quad \mathring{L}^r_p(\mathbb{R}^n) \hookrightarrow \mathring{L}^r_p(\mathbb{R}^n).$$

(iii) *There are linear and bounded extensions T of T_0 to the global Morrey spaces $L^r_p(\mathbb{R}^n)$,*

$$T : \quad L^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n).$$

We collect in Sect. 2 some definitions and preliminaries. The proof of the Theorem will be given in Sect. 3. In Sect. 4 we add some comments and further references.

2 Definitions and preliminaries

2.1 Definitions

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$. Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differential functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Let $D(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ be the collection of all functions $f \in S(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$. As usual $D'(\mathbb{R}^n)$ stands for the space of all distributions in $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $1 \le p < \infty$, is the standard complex Banach space with respect to the Lebesgue measure, normed by

$$\|f \,|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \right)^{1/p}.$$

Similarly $L_p(M)$ where M is a measurable subset of $\mathbb{R}^n$. As usual $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$ denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be the usual cubes in $\mathbb{R}^n$, $n \in \mathbb{N}$, with sides of length 2^{-j} parallel to the axes of coordinates and $2^{-j}m$ as the lower left corner. As usual, $L_p^{\mathrm{loc}}(\mathbb{R}^n)$ collects all locally p-integrable functions, hence $f \in L_p(M)$ for any bounded measurable set M in $\mathbb{R}^n$.

Definition 2.1 Let $1 \le p < \infty$ and $0 \le \frac{n}{p} + r < \frac{n}{p}$.

(i) Then $\mathcal{L}^r_p(\mathbb{R}^n)$ collects all $f \in L^{\rm loc}_p(\mathbb{R}^n)$ such that

$$\|f\,|\mathcal{L}^r_p(\mathbb{R}^n)\| = \sup_{J\in\mathbb{N}_0, M\in\mathbb{Z}^n} 2^{J(\frac{n}{p}+r)}\|f\,|L_p(Q_{J,M})\|$$

is finite.

(ii) Then $L^r_p(\mathbb{R}^n)$ collects all $f \in L^{\rm loc}_p(\mathbb{R}^n)$ such that

$$\|f\,|L^r_p(\mathbb{R}^n)\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} 2^{J(\frac{n}{p}+r)}\|f\,|L_p(Q_{J,M})\| \tag{2.1}$$

is finite.

Remark 2.2 These are the well-known local Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$ and global Morrey spaces $L^r_p(\mathbb{R}^n)$. They are Banach spaces. Of course, $L^{-n/p}_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. Let $w_\gamma(x) = (1+|x|^2)^{\gamma/2}$, $\gamma \in \mathbb{R}$, and let $L_p(\mathbb{R}^n, w_\gamma)$, $1 \le p < \infty$, be the corresponding L_p-space, normed by

$$\|f\,|L_p(\mathbb{R}^n, w_\gamma)\| = \|w_\gamma f\,|L_p(\mathbb{R}^n)\|, \quad 1 \le p < \infty, \quad \gamma \in \mathbb{R}.$$

Then it follows from

$$L^r_p(\mathbb{R}^n) \hookrightarrow \mathcal{L}^r_p(\mathbb{R}^n) \hookrightarrow \mathcal{L}_p(\mathbb{R}^n) = \mathcal{L}^{-n/p}_p(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n, w_\gamma) \hookrightarrow S'(\mathbb{R}^n)$$

if $\gamma < -n/p$ that all spaces can be considered in the framework of $S'(\mathbb{R}^n)$. We used arguments of this type in [39, p. 22] to reduce periodic function spaces to weighted function spaces. The above spaces are usually attributed to Morrey [18]. But Morrey himself (and also high-ranking mathematicians including John Nash, Jürgen Moser and Olga Ladyshenskaya following this path) were only interested in related integral inequalities in connection with smoothness properties (Hölder-continuity) of solutions of nonlinear elliptic and parabolic equations. The reformulation in terms of function spaces goes back to Campanato, Brudnyi and Peetre in the 1960s, [2,3,6,7,22,23] using quite similar notation as in the above Definition 2.1. Nowadays also some other notation are in common use, in particular

$$\mathcal{L}^r_p(\mathbb{R}^n) = \mathcal{M}_{u,p}(\mathbb{R}^n) \quad \text{and} \quad L^r_p(\mathbb{R}^n) = M_{u,p}(\mathbb{R}^n) \quad \text{with } r = -n/u,$$

$1 < p \le u < \infty$. Further references and properties may be found in [40, Chapter 3], in particular characterizations of $\mathcal{L}^r_p(\mathbb{R}^n)$ in terms of wavelets. This will not be needed here.

Remark 2.3 We mention two simple well-known properties which will be of some use for us later on. Let $\lambda > 0$ and $f \in L^r_p(\mathbb{R}^n)$. Then

$$\|f(\lambda\cdot)\,|L^r_p(\mathbb{R}^n)\| = \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{J\left(\frac{n}{p}+r\right)} \left(\int\limits_{0\le y_l-2^{-J}M_l\le 2^{-J}} |f(\lambda y)|^p \,\mathrm{d}y \right)^{1/p}$$

$$= \lambda^r \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} (\lambda^{-1}2^J)^{\left(\frac{n}{p}+r\right)} \left(\int\limits_{0\le y_l-\lambda 2^{-J}M_l\le \lambda 2^{-J}} |f(y)|^p \,\mathrm{d}y \right)^{1/p}$$

$$\sim \lambda^r \,\|f\,|L^r_p(\mathbb{R}^n)\|.$$

Hence, $L^r_p(\mathbb{R}^n)$ is (essentially) homogeneous of degree r. Let T_0 be given by (1.1) and $f \in D(\mathbb{R}^n)$. With $\lambda > 0$ one has

$$(T_0 f(\lambda\cdot))(x) = \lim_{\varepsilon\downarrow 0} \int\limits_{|y|\ge\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(\lambda x - \lambda y)\,\mathrm{d}y = (T_0 f)(\lambda x).$$

Hence, T_0 is homogeneous of degree zero.

2.2 Preliminaries

Let $1 \le p < \infty$ and $0 < \frac{n}{p} + r < \frac{n}{p}$. Let $f(x) = |x|^r$ if $|x| \le 1$ and $f(x) = 0$ if $|x| > 1$. Then $f \in L^r_p(\mathbb{R}^n)$ and one has for some $c > 0$ and all $g \in L_\infty(\mathbb{R}^n)$,

$$2^{J(\frac{n}{p}+r)} \left(\int\limits_{Q_{J,0}} |f(x) - g(x)|^p \,\mathrm{d}x \right)^{1/p} \ge c > 0, \quad J \ge J_g \in \mathbb{N}.$$

In particular, $\{g \in L_\infty(\mathbb{R}^n),\ \operatorname{supp} g \text{ compact}\}$ is a subset of $L^r_p(\mathbb{R}^n)$, but not dense.

Definition 2.4 Let $1 < p < \infty$ and $0 \le \frac{n}{p} + r < \frac{n}{p}$. Then $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ is the completion of $D(\mathbb{R}^n)$ in $L^r_p(\mathbb{R}^n)$.

Remark 2.5 By the above observation one has $\overset{\circ}{L}{}^r_p(\mathbb{R}^n) \ne L^r_p(\mathbb{R}^n)$ if $0 < \frac{n}{p} + r < \frac{n}{p}$. This is well known and goes back to [24] and has also been mentioned in [44] (essentially with the same counter-example as above) and in [1]. It makes sense to interpret the dual spaces $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)'$ of $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ in the context of the dual pairings $(D(\mathbb{R}^n), D'(\mathbb{R}^n))$ or, likewise, $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. This dual space has been determined in [1] explicitly with the help of Muckenhoupt weights and Hausdorff capacities. A detailed description of this dual space will not be needed here. But as mentioned in [1] the set of all continuous functions with compact support is a dense subset in $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)'$. From the explicit norm given there it follows that $D(\mathbb{R}^n)$ is also dense in $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)'$. Then it makes sense to deal again with the dual of $\mathring{L}^r_p(\mathbb{R}^n)'$ in the context of the above-mentioned distributional pairings with the following outcome.

Proposition 2.6 *Let* $1 < p < \infty$ *and* $0 < \frac{n}{p} + r < \frac{n}{p}$. *Then*

$$\mathring{L}^r_p(\mathbb{R}^n)'' = (\mathring{L}^r_p(\mathbb{R}^n)')' = L^r_p(\mathbb{R}^n). \tag{2.2}$$

Remark 2.7 It extends $L^{-n/p}_p(\mathbb{R}^n)'' = L_p(\mathbb{R}^n)'' = L_p(\mathbb{R}^n)$, $1 < p < \infty$, to the Morrey spaces $L^r_p(\mathbb{R}^n)$. This remarkable assertion is due to [1, Section 3.2]. There one finds the necessary norm equivalences combined with references to [29,44]. The observation (2.2) is the counterpart of

$$cmo(\mathbb{R}^n)' = h_1(\mathbb{R}^n) \quad \text{and} \quad h_1'(\mathbb{R}^n) = bmo(\mathbb{R}^n). \tag{2.3}$$

Here $h_1(\mathbb{R}^n) = F^0_{1,2}(\mathbb{R}^n)$ is the inhomogeneous version of the Hardy space $H_1(\mathbb{R}^n)$. Similarly $bmo(\mathbb{R}^n) = F^0_{\infty,2}(\mathbb{R}^n)$ is the inhomogeneous version of $BMO(\mathbb{R}^n)$. Details and historical comments may be found in [38, pp. 37,38]. The space $cmo(\mathbb{R}^n)$ is the completion of $D(\mathbb{R}^n)$ in $bmo(\mathbb{R}^n)$. Both duality assertions in (2.3) are covered by [9, Theorems 5, 6, 9] and the literature mentioned there. We refer in this context also to [43, Section 7.3]. This book deals with some Morrey versions $A^{s,\tau}_{p,q}(\mathbb{R}^n)$ of the nowadays well-known spaces $A^s_{p,q}(\mathbb{R}^n)$ with $A = B$ and $A = F$. The preduals of some spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$ are characterized in [43, Chapter 7] again with the help of some Hausdorff capacities similarly as in [1]. Whether related counterparts of Proposition 2.6 combined with the arguments in this paper are of some use remains to be seen.

3 Proof of the Theorem

Step 1 We prove part (i). We may assume that $\Omega(\sigma_0) > 0$ for some $\sigma_0 \in \mathbb{S}^{n-1}$. Let $\Omega(\sigma) > 0$ for all $\sigma \in \mathbb{S}^{n-1}$ with $|\sigma - \sigma_0| \le |\sigma_1 - \sigma_0|$ and some $\sigma_1 \in \mathbb{S}^{n-1}$, $\sigma_1 \neq \sigma_0$. Let $\varphi_K \in D(\mathbb{R}^n)$ with compact support in the sectoral domain

$$\{y \in \mathbb{R}^n : \; y = |y|\sigma, \; |\sigma - \sigma_0| \le |\sigma_1 - \sigma_0|, \; |y| \ge K_1\},$$

$0 \le \varphi_K \le 1$, and

$$\varphi_K(y) = 1 \quad \text{if} \quad \{y \in \mathbb{R}^n : \; y = |y|\sigma, \; |\sigma - \sigma_0| \le |\sigma_2 - \sigma_0|, \; K_2 \le |y| \le K\}$$

for some $\sigma_2 \in \mathbb{S}^{n-1}$ with $|\sigma_2 - \sigma_0| < |\sigma_1 - \sigma_0|$, where K_1, K_2 and K are natural numbers, $K_1 < K_2 \le K$. Then $\varphi_K \in \operatorname{dom} T_0$. If K_1 is chosen sufficiently large then one has by (1.1) for all $x \in \mathbb{R}^n$ with $|x| \le 1$ and some $c > 0$,

$$(T_0\varphi_K)(x) \ge c \log K.$$

Hence $(T_0\varphi_K)(x) \to \infty$ if $K \to \infty$, whereas $\|\varphi_K \,|\mathcal{L}^r_p(\mathbb{R}^n)\|$ is uniformly bounded as one checks easily. This proves part (i) of the Theorem.

Step 2 We prove part (ii). By the homogeneity properties mentioned in Remark 2.3 one may assume

$$f \in D(\mathbb{R}^n), \quad \operatorname{supp} f \subset \{y \in \mathbb{R}^n : |y| < 1\}. \tag{3.1}$$

Furthermore, to prove the requested estimate for $T_0 f$ in the counterpart of (2.1) it is sufficient to deal with the cubes $Q_{J,0}$ or the model cubes Q_J, centered at the origin with sides of length 2^{-J} parallel to the axes of coordinates, $J \in \mathbb{Z}$. The possibility to reduce the consideration to this model case follows from the translation-invariance of $L^r_p(\mathbb{R}^n)$ and of T_0. Let $J \in \mathbb{Z}$ and let $\{\varphi_j\}_{j=-\infty}^{J-J_0} \subset D(\mathbb{R}^n)$ be a canonical dyadic resolution of unity in $\mathbb{R}^n$ with

$$\varphi_{J-J_0}(x) = 1 \quad \text{if} \quad x \in Q_{J-J_0}, \quad \operatorname{supp} \varphi_{J-J_0} \subset Q_{J-J_0-1},$$

and

$$\operatorname{supp} \varphi_j \subset Q_{j-1} \setminus Q_{j+1}, \quad j \in \mathbb{Z}, \quad j < J - J_0, \tag{3.2}$$

where $J_0 \in \mathbb{N}$ will be chosen later on, independently of J. In particular,

$$\sum_{j=-\infty}^{J-J_0} \varphi_j(x) = 1, \quad x \in \mathbb{R}^n,$$

and

$$f = \sum_{j=-\infty}^{J-J_0} \varphi_j\, f = f_{J-J_0} + \sum_{j=-\infty}^{J-J_0-1} f_j.$$

Obviously

$$\|T_0 f \,|L_p(Q_J)\| \le \|T_0 f_{J-J_0} \,|L_p(Q_J)\| + \sum_{j=-\infty}^{J-J_0-1} \|T_0 f_j \,|L_p(Q_J)\|. \tag{3.3}$$

Recall that the extension of T_0 maps $L_p(\mathbb{R}^n)$ into itself. Then one has for the first term on the right-hand side of (3.3),

$$\begin{aligned} 2^{J\left(\frac{n}{p}+r\right)} \|T_0 f_{J-J_0} \,|L_p(Q_J)\| &\le c\, 2^{J\left(\frac{n}{p}+r\right)} \|f_{J-J_0} \,|L_p(\mathbb{R}^n)\| \\ &\le c\, 2^{J\left(\frac{n}{p}+r\right)} \|f \,|L_p(Q_{J-J_0-1})\| \\ &\le c'\, 2^{(J_0+1)\left(\frac{n}{p}+r\right)} \|f \,|L^r_p(\mathbb{R}^n)\|. \end{aligned} \tag{3.4}$$

We use (1.1) with f_j, $j < J - J_0$, in place of f, and $x \in Q_J$. We choose now $J_0 \in \mathbb{N}$ (independently of J) such that $x \in Q_J$ and $x - y \in Q_{j-1} \setminus Q_{j+1}$, $j < J - J_0$ ensures $|y| \sim 2^{-j}$. Then one has for $x \in Q_J$ by (1.1), (3.2) and Hölder's inequality

$$|T_0 f_j(x)| \le c\, 2^{nj} \|\varphi_j\, f\, | L_1(\mathbb{R}^n)\| \le c'\, 2^{\frac{jn}{p}}\, \|f\, | L_p(Q_{j-1})\| \le c'' 2^{-jr} \|f\, | L^r_p(\mathbb{R}^n)\|$$

and

$$2^{J(\frac{n}{p}+r)} \|T_0 f_j\, | L_p(Q_J)\| \le c\, 2^{(J-j)r} \|f\, | L^r_p(\mathbb{R}^n)\|. \tag{3.5}$$

Since $r < 0$ one obtains by (3.3), (3.4) and (3.5)

$$\|T_0 f\, | L^r_p(\mathbb{R}^n)\| \le c\, \|f\, | L^r_p(\mathbb{R}^n)\|.$$

Hence $T_0 f \in L^r_p(\mathbb{R}^n)$. It remains to show that $T_0 f \in \mathring{L}^r_p(\mathbb{R}^n)$. Let f be as above and $\varphi \in S(\mathbb{R}^n)$. Then one has by the usual dual pairing and the Fubini-Lebesgue theorem

$$\begin{aligned}(-1)^{|\alpha|}(T_0 f, D^\alpha \varphi) &= \lim_{\varepsilon\downarrow 0} \int\limits_{|y|\ge\varepsilon} \int\limits_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n}\, f(x-y)(-1)^{|\alpha|} D^\alpha \varphi(x)\, \mathrm{d}x\, \mathrm{d}y \\ &= (T_0 D^\alpha f, \varphi).\end{aligned}$$

Hence $T_0 f \in W^k_p(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. By embedding $T_0 f \in L^r_p(\mathbb{R}^n)$ is a C^∞ function. Furthermore one has by (1.1), (1.4) and (3.1)

$$|(T_0 f)(x)| \le c\, |x|^{-n} \quad \text{if} \quad |x| \ge 2.$$

Let $R \ge 2$. Then one has for cubes $Q_{J,M}$ with $Q_{J,M} \subset \{x \in \mathbb{R}^n : |x| > R\}$,

$$2^{J(\frac{n}{p}+r)} \|T_0 f\, | L_p(Q_{J,M})\| \le c\, 2^{Jr}\, R^{-n} \quad \text{if} \quad J \in \mathbb{N}_0. \tag{3.6}$$

Using $1 < p < \infty$ and again $|T_0 f(x)| \le c/|x|^n$ one obtains

$$2^{J(\frac{n}{p}+r)} \|T_0 f\, | L_p(Q_{J,M})\| \le c\, 2^{J(\frac{n}{p}+r)}\, R^{-n(1-\frac{1}{p})} \quad \text{if} \quad -J \in \mathbb{N}. \tag{3.7}$$

Let ψ_R be a smooth cut-off function with $\psi_R(x) = 1$ if $|x| \le R$. Then $\psi_R\, T_0 f \in D(\mathbb{R}^n)$ and it follows from (3.6), (3.7) with $1 < p < \infty$ and $0 \le \frac{n}{p} + r < \frac{n}{p}$,

$$\lim_{R\to\infty} \|T_0 f - \psi_R T_0 f\, | L^r_p(\mathbb{R}^n)\| = 0.$$

Hence $T_0 f \in \mathring{L}^r_p(\mathbb{R}^n)$. This shows that T_0 given by (1.1) with $\operatorname{dom} T_0 = D(\mathbb{R}^n)$ can be extended (uniquely) to $\mathring{L}^r_p(\mathbb{R}^n)$.

Step 3 Let T with $\operatorname{dom} T = \mathring{L}^r_p(\mathbb{R}^n)$ be the extension of T_0 according to part (ii) of the Theorem and Step 2. Then it follows from Proposition 2.6 and Banach space theory that the bidual $T'' = (T')'$ with $\operatorname{dom} T'' = L^r_p(\mathbb{R}^n)$ is an extension of T. As for the abstract background one may consult [42, pp. 112/113] and [25, pp. 35/36]. Furthermore in Remark 4.2 below we add a comment about the so-called extension property of Banach spaces.

4 Comments

Remark 4.1 Our arguments rely on two observations. First there is a possibility to transfer mapping properties of operators of type (1.1) in L_p-spaces, $1 < p < \infty$, with the help of decay assertions of type (1.4) to more general spaces, in our case global Morrey spaces $L^r_p(\mathbb{R}^n)$. This has been done in Step 2 of the above proof. It may work also in other similar situations. Near to us is [26], but decomposition techniques have been used in many other papers dealing with a wide range of operators in Morrey spaces. We refer in particular to [8,10,11,13–15,17,19,20,28,30–32,36]. Also the original Morrey and Campanato spaces have been modified in the above context in many respects. One may consult the recent papers [4,16,21] and the references within. The second basic ingredient in our approach is the possibility to use the duality (2.2) in the context of the dual pairings $(D(\mathbb{R}^n), D'(\mathbb{R}^n))$ or $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. Here we rely on [1]. Then one can circumvent that $D(\mathbb{R}^n)$ is not dense in $L^r_p(\mathbb{R}^n)$ as mentioned at the beginning of Sect. 2.2 and Remark 2.5. This observation seems to quite recent and not in common use so far. It is the main aim of the present paper to describe the interplay of these two ingredients as simple as possible.

Remark 4.2 According to [25, p. 133] a Banach space Y is said to have the extension property if every linear bounded operator $T_0 : X_0 \hookrightarrow Y$ defined on a closed subspace X_0 of an arbitrary Banach space X admits a linear and bounded extension $T : X \hookrightarrow Y$ such that its restriction to X_0 coincides with T_0, $T|X_0 = T_0$. We refer also to [41, p. 127]. Applied to the above situation one may think about $X = Y = L^r_p(\mathbb{R}^n)$ and $X_0 = \mathring{L}^r_p(\mathbb{R}^n)$. But there is no abstract assertion ensuring that $T_0 : \mathring{L}^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$ can be extended to $T : L^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$. According to [25, p. 134] with a reference to [12, p. 169] an infinite-dimensional Banach space with the extension property is necessarily non-separable. This applies to $L^r_p(\mathbb{R}^n)$ as a non-separable target space, but not to $\mathring{L}^r_p(\mathbb{R}^n)$ as an alternative separable target space. But sufficient conditions ensuring the extension property (for $L^r_p(\mathbb{R}^n)$) are apparently not known.

Remark 4.3 In [40] we dealt with the n-dimensional Navier-Stokes equations in the context of some (inhomogeneous) global spaces $A^s_{p,q}(\mathbb{R}^n)$ where $A \in \{B, F\}$ and some related local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ where $0 \le \frac{n}{p} + r < \frac{n}{p}$ is adapted to the above considerations. In particular one has

$$\mathcal{L}^r_p(\mathbb{R}^n) = \mathcal{L}^r L_p(\mathbb{R}^n) = \mathcal{L}^r F^0_{p,2}(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty.$$

Then one needs the so-called Leray projector which can be reduced to the Riesz transforms R_k,

$$(R_k f)(x) = c_n \lim_{\varepsilon \downarrow 0} \int\limits_{y \in \mathbb{R}^n, |y| \ge \varepsilon} \frac{y_k}{|y|^{n+1}} f(x-y)\, \mathrm{d}y, \quad k = 1, \dots, n.$$

This is a special case of (1.1). Assertions of type

$$R_k : \quad A^s_{p,q}(\mathbb{R}^n) \hookrightarrow A^s_{p,q}(\mathbb{R}^n), \quad 1 < p, q < \infty, \quad s \in \mathbb{R}, \tag{4.1}$$

are crucial for the approach to the Navier-Stokes equations as developed in [40]. There is no direct counterpart for the related local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. This negative outcome is apparently confirmed by part (i) of the Theorem. But according to the parts (ii) and (iii) the situation is more favorable for the related global spaces $L^r A^s_{p,q}(\mathbb{R}^n)$. In addition to (4.1) we needed for the spaces $A^s_{p,q}(\mathbb{R}^n)$ some lifting, sharp Michlin-Hörmander Fourier multiplier theorems, and sufficient conditions (or characterizations) ensuring that some spaces $A^s_{p,q}(\mathbb{R}^n)$ are multiplication algebras. Many of the needed assertions can be proved nowadays by using wavelet characterizations of $A^s_{p,q}(\mathbb{R}^n)$. The parts (ii) and (iii) of the Theorem suggest that it might well be possible to extend the theory as developed in [40] from $A^s_{p,q}(\mathbb{R}^n)$ to some spaces $L^r A^s_{p,q}(\mathbb{R}^n)$. For this purpose one could try to combine relevant arguments from [40] with corresponding wavelet expansions for the spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ as obtained recently in [27].

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