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Closures of Bergman–Besov Spaces in the Weighted Bloch Spaces on the Unit Ball

Nihat Gökhan Göğüş[1] · **Faruk Yilmaz**[2]

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Abstract

In this paper, via invertible radial differential operators, we characterize the closures of the Bergman–Besov spaces in the weighted Bloch spaces on the unit ball. The results of this paper generalize some previous results of Wen Xu and Ruhan Zhao. We first show on the way that the Bergman–Besov space is contained in the weighted little Bloch space.

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1 Introduction

Let $\mathbb{B}_n = \{z : |z| \leq 1\}$ be the open unit ball in $\mathbb{C}^n$ and let $\mathbb{S}_n = \{z : |z| = 1\}$ be the unit sphere in $\mathbb{C}^n$. Let $H(\mathbb{B}_n)$ and H^∞ denote the spaces of all and bounded holomorphic functions on $\mathbb{B}_n$, respectively.

Let ν be the normalized Lebesgue measure on $\mathbb{B}_n$. When $n = 1$, $d\nu(z) = dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$ is the normalized area measure on the unit disc $\mathbb{B}_1 = \mathbb{D}$.

For $q \in \mathbb{R}$, we define the following measures on $\mathbb{B}_n$:

$$d\nu_q(z) = (1 - |z|^2)^q d\nu(z).$$

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✉ Nihat Gökhan Göğüş
nggogus@sabanciuniv.edu

Faruk Yilmaz
yilmaz@ahievran.edu.tr

1 Faculty of Engineering and Natural Sciences, Sabanci University, 34956 Tuzla, Istanbul, Turkey

2 Department of Mathematics, Faculty of Arts and Sciences, Kirşehir Ahi Evran University, Kirşehir, Turkey

For $0 < p < \infty$, the Lebesgue classes with respect to ν_q will be denoted by L^p_q. By Proposition 2.3 [7] we know that Lebesgue classes of essentially bounded functions on $\mathbb{B}_n$ with respect to ν_q are same for any $q \in \mathbb{R}$, and we will denote it by L^∞. Let m be a nonnegative integer such that $q + pm > -1$. Then the Bergman–Besov space B^p_q consists of all $f \in H(\mathbb{B}_n)$ for which

$$(1 - |z|^2)^m \frac{\partial^m f}{\partial z_1^{\gamma_1} ... \partial z_n^{\gamma_n}} \in L^p_q$$

for every multi-index $\gamma = (\gamma_1, ..., \gamma_n)$ with $|\gamma| = m$.

The spaces B^2_q are reproducing kernel Hilbert spces whose kernels play an important role in the study of all Bergman–Besov spaces B^p_q. Hence we follow Kaptanoğlu [7] and use invertible radial differential operators $D^t_s : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ of order $t \in \mathbb{R}$ for any $s \in \mathbb{R}$ which are compatible with the kernels. Also consider the linear transformation I^t_s defined by

$$I^t_s f(z) = (1 - |z|^2)^t D^t_s f(z),$$

where $f \in H(\mathbb{B}_n)$.

For $q \in \mathbb{R}$ and $0 < p < \infty$, the Bergman–Besov space B^p_q is

$$B^p_q = \{f \in H(\mathbb{B}_n) : I^t_s f(z) \in L^p_q \text{ for some } s, t \in \mathbb{R} \text{ satisfy } q + pt > -1\}.$$

Then $\|f\|_{B^p_q} := \left\|I^t_s f\right\|_{L^p_q}$ for such s, t defines a norm on B^p_q for $p \geq 1$, and quasinorm for $0 < p < 1$.

It is known that each B^p_q space contains all polynomials, see [9]. If $q > -1$, then one can take $t = 0$ to get that the Bergman–Besov spaces B^p_q are the weighted Bergman spaces $A^p_q = L^p_q \cap H(\mathbb{B}_n)$. Also B^2_{-1} is the Hardy space H^2, B^2_{-n} is the Drury–Arveson space and $B^2_{-(n+1)}$ is the Dirichlet space.

For $\alpha \in \mathbb{R}$, the weighted Bloch space $\mathcal{B}_\alpha$ is the class of all functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|_{\mathcal{B}_\alpha} := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\alpha |I^t_s f(z)| < \infty,$$

for some $s, t \in \mathbb{R}$ satisfying $\alpha + t > 0$. Then $\|.\|_{\mathcal{B}_\alpha}$ defines a norm for any such s, t.

The weighted little Bloch space $\mathcal{B}_{\alpha 0}$ consists of functions $f \in \mathcal{B}_\alpha$ such that

$$\lim_{|z| \to 1^-} (1 - |z|^2)^\alpha |I^t_s f(z)| = 0,$$

where $s, t \in \mathbb{R}$ satisfies $\alpha + t > 0$.

It is known that $\mathcal{B}_{\alpha 0}$ is the closure of the set of polynomials in $\mathcal{B}_\alpha$, see [8]. If $\alpha = 0$, then the spaces $\mathcal{B}_0$ and $\mathcal{B}_{00}$ are the usual Bloch and little Bloch spaces. Also from definitions we have that $\mathcal{B}_\alpha \subset \mathcal{B}_{\beta 0} \subset \mathcal{B}_\beta$ for $\alpha < \beta$, see [8].

It is also known that the above definitions are independent of $s, t \in \mathbb{R}$ under the conditions $\alpha + t > 0$ and $q + pt > -1$ (see [10]). Furthermore these definitions are also independent of the particular type of the derivative. Namely one can use the holomorphic gradient and the usual radial derivative in place of D_s^t, see [10,18].

In [10], Kaptanoğlu and Üreyen gave the precise inclusion relations among Bergman–Besov spaces and weighted Bloch spaces on the unit ball of $\mathbb{C}^n$. They showed that if $\alpha \in \mathbb{R}$, then $B^p_{\alpha p-(1+n)} \subset \mathcal{B}_\alpha$ and $\mathcal{B}_\alpha$ cannot be replaced by a smaller weighted Bloch space. Hovewer, the inclusion relation between Bergman–Besov spaces and weighted little Bloch spaces was not clear from their result. In the case of unit disc, $n = 1$, it is well known that Besov spaces are contained in the little Bloch space for $1 \leq p < \infty$. In this direction our first result is an extension of the above well known fact. We would like to thank A. E. Üreyen for providing us with a proof of the following theorem.

Theorem 1.1 *Let $\alpha \in \mathbb{R}$ and $0 < p < \infty$. Then the Bergman–Besov space $B^p_{\alpha p-(n+1)}$ is strictly contained in the weighted little Bloch space $\mathcal{B}_{\alpha 0}$.*

In [2], Anderson, Clunie and Pommerenke asked the closure of the space of all bounded analytic functions H^∞ in the Bloch norm. This problem is still an open problem. The motivation for this type of work is a result of Peter Jones that gives a description of the closure of BMOA in the Bloch space $\mathcal{B}$ (see Theorem 9 of [1] and [6] for proof). There has been many results on this topic, namely to determine the closure of various subspaces of the Bloch space in the Bloch norm, see [3–5,11,14–16].

If X is a subspace of the weighted Bloch space $\mathcal{B}_\alpha$, then $C_{\mathcal{B}_\alpha}(X)$ will denote the closure of X in the $\mathcal{B}_\alpha$-norm, and the distance from $f \in \mathcal{B}_\alpha$ to the subspace X in the $\mathcal{B}_\alpha$-norm will be denoted by $dist_{\mathcal{B}_\alpha}(f, X)$.

Let $\alpha \in \mathbb{R}$ and $\epsilon > 0$. If $f \in H(\mathbb{B}_n)$, we define the level set $\Omega_\epsilon^{\alpha,t}(f)$ by

$$\Omega_\epsilon^{\alpha,t}(f) = \{z \in \mathbb{B}_n : (1 - |z|^2)^\alpha |I_s^t f(z)| \geq \epsilon\},$$

where $s, t \in \mathbb{R}$ with $\alpha + t > 0$.

For the closure problem, our result is the following theorem.

Theorem 1.2 *Let $1 \leq p < \infty$, $\alpha \in \mathbb{R}$, $q \leq \alpha p - (n + 1)$ and choose $t \in \mathbb{R}$ such that $0 < \alpha + t$. If $f \in \mathcal{B}_\alpha$, then the following conditions are equivalent:*

(i) $f \in \mathcal{B}_{\alpha 0}$.
(ii) $f \in C_{\mathcal{B}_\alpha}(B^p_q)$.
(iii) *For every $\epsilon > 0$,*

$$\int_{\Omega_\epsilon^{\alpha,t}(f)} (1 - |z|^2)^{-(n+1)} d\nu(z) < \infty.$$

The holomorphic Besov spaces $B_p(\mathbb{B}_n)$ defined by Zhu in [18] are our Bergman–Besov spaces $B^p_{-(n+1)}$ for $p \geq 1$ (see [7] p.∼392). In [15] Xu gave distance estimates for Besov spaces B_p. In our case, if $\alpha = 0$, then we have the usual Bloch space $\mathcal{B}_0$, and Besov spaces $B^p_{-(n+1)}$ for $p \geq 1$. Hence Theorem 1.2 includes Xu results (see Theorem 3 of [15]). Further, if we are in the unit disc, i.e. $n = 1$, then B^p_{-2} are Besov spaces on the unit disc. These spaces are denoted by B_p by Zhao in [16]. Theorem 1.2 generalizes Theorem 8 of [16] to higher dimensions.

If $q > \alpha p - (n+1)$, then there is no inclusion relation between B_q^p and $\mathcal{B}_\alpha$, hence, we consider the closure of $B_q^p \cap \mathcal{B}_\alpha$ in the weighted Bloch norm. By Kaptanoğlu and Üreyen's result, if $q > \alpha p - 1$, then the closure of $B_q^p \cap \mathcal{B}_\alpha$ in the weighted Bloch norm is trivial. It remains to consider the case $\alpha p - (n+1) < q \leq \alpha p - 1$. In this case, we have the following result.

Theorem 1.3 *Let* $1 \leq p < \infty$, $\alpha \in \mathbb{R}$, $\alpha p - (n+1) < q \leq \alpha p - 1$ *and choose* $t \in \mathbb{R}$ *such that* $\alpha + t > 0$. *If* $f \in \mathcal{B}_\alpha$, *then the following conditions are equivalent:*

(i) $f \in C_{\mathcal{B}_\alpha}(B_q^p \cap \mathcal{B}_\alpha)$.

(ii) There exists $t_0 \geq t$ *with* $\alpha + t_0 > n$ *and* $q + pt_0 > -1$ *such that for every* $\epsilon > 0$,

$$\int_{\Omega_\epsilon^{\alpha,t_0}(f)} (1 - |z|^2)^{q-\alpha p} d\nu(z) < \infty.$$

The paper is organized as follows. In Sect. 2, we give some background on Bergman–Besov spaces. In Sect. 3 we will give some required lemmas and in Sect. 4, we prove Theorem 1.1 as Theorem 4.1. In the final section we prove Theorem 1.2 as Theorem 5.1.

Throughout this paper, we will write $f \lesssim g$ if there exists a constant C such that $f \leq Cg$. Also, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$.

2 Background on Bergman–Besov Spaces

For an n-tuple of nonnegative integers $\gamma = (\gamma_1, ..., \gamma_n)$, we willl write $\gamma! = \prod_{i=1}^n \gamma_i!$ and $|\gamma| = \sum_{i=1}^n \gamma_i$. If we are given a point $z = (z_1, ..., z_n) \in \mathbb{C}^n$, then we will write $z^\gamma = \prod_{i=1}^n z_i^{\gamma_i}$. With this notation, for every $f \in H(\mathbb{B}_n)$, we can find coefficients $\hat{f}(\gamma) \in \mathbb{C}$ such that

$$f(z) = \sum_\gamma \hat{f}(\gamma) z^\gamma,$$

where the sum is taken over all n-tuple of nonnegative integers. Furhermore, for $k \in \mathbb{N}$, we can write

$$f_k(z) = \sum_{|\gamma|=k} \hat{f}(\gamma) z^\gamma$$

and hence we get

$$f(z) = \sum_{k=0}^\infty f_k(z).$$

This is called homogeneous expansion of f.

The spaces B^2_q are reproducing kernel Hilbert spaces whose kernels play a big role in the study of all B^p_q.

Definition 2.1 For $q \in \mathbb{R}$ and $z, w \in \mathbb{B}_n$, the Bergman–Besov kernels are

$$K_q(z,w) := \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{1+n+q}} = \sum_{k=0}^{\infty} \frac{(1+n+q)_k}{k!} \langle z,w\rangle^k & q > -(1+n) \\ {}_2F_1(1,1;1-(n+q);\langle z,w\rangle) = \sum_{k=0}^{\infty} \frac{k!\langle z,w\rangle^k}{(1-(n+q))_k}, & q \le -(1+n) \end{cases}$$

where ${}_2F_1$ is the usual hypergeometric function and $(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$ is the Pochhammer symbol.

The kernel K_q is the reproducing kernel of the Hilbert space B^2_q.

Definition 2.2 For any $s, t \in \mathbb{R}$ the radial differential operator D^t_s defined on $H(\mathbb{B}_n)$ by

$$D^t_s f(z) := \sum_{k=0}^{\infty} d_k(s,t) f_k(z) := \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k(z),$$

where $c_k(s)$ is the coefficient of $\langle z,w\rangle^k$ in $K_q(z,w)$ (see [10]).

With this definition, it is known that

$$D^0_s = I, \qquad D^u_{s+t} D^t_s = D^{t+u}_s, \qquad (D^t_s)^{-1} = D^{-t}_{s+t}$$

for any s, t, u.

A nice property for the radial differential operator D^t_s is the identity

$$D^t_q K_q(z,w) = K_{q+t}(z,w)$$

for any q, t, where differentiation is performed on the variable z (see [10]).

In [10], Kaptanoğlu and Üreyen gave the precise inclusion relations among Bergman–Besov spaces and Bloch-type spaces on the unit ball of $\mathbb{C}^n$. Namely, they proved the following theorems.

Theorem 2.3 (Theorem 1.5 of [10]) *Given B^p_q, we have the inclusions*

$$\mathcal{B}_{<\frac{1+q}{p}} \subset B^p_q \subset \mathcal{B}_{\frac{1+n+q}{p}},$$

where the symbol $\mathcal{B}_{<\frac{1+q}{p}}$ denotes any one of the spaces $\mathcal{B}_b$ with $b < \frac{1+q}{p}$.

An equivalent statement of Theorem 2.3 is the following: Given $\mathcal{B}_\alpha$, the inclusions $B^p_{\alpha p-(1+n)} \subset \mathcal{B}_\alpha \subset B^p_{>\alpha p-1}$ hold.

Theorem 2.4 (Theorem 1.6 of [10]) *Let B^p_q be given.*

(i) If $p \leq P$*, then* $B^p_q \subset B^P_Q$ *if and only if*

$$\frac{1+n+q}{p} \leq \frac{1+n+Q}{P}.$$

(ii) If $P < p$*, then* $B^p_q \subset B^P_Q$ *if and only if*

$$\frac{1+q}{p} < \frac{1+Q}{P}.$$

3 Some Lemmas

In this section, we will give some known lemmas that we need in order to prove Theorem 1.2. First, we begin with the following integral estimate of Rudin.

Lemma 3.1 (Proposition 1.4.10 of [13]) *Suppose* $c > 0$ *and* $t > -1$*. Then*

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^t d\nu(w)}{|1-\langle z, w\rangle|^{n+1+t+c}} \approx \frac{1}{(1-|z|^2)^c}.$$

We will also use the following Minkowski integral inequality which exchanges the order of integration.

Lemma 3.2 (Theorem 3.3.5 of [12]) *Let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \lambda)$ *be* σ*-finite measure spaces and let* $f(x, y)$ *be an* $\mathcal{A} \times \mathcal{B}$ *measurable function. If* $1 \leq p < \infty$*, then*

$$\left(\int_Y \left(\int_X |f(x,y)| d\mu(x)\right)^p d\lambda(y)\right)^{1/p} \leq \int_X \left(\int_Y |f(x,y)|^p d\lambda(y)\right)^{1/p} d\mu(x).$$

4 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. We first give an example of a function that is in the weighted little Bloch space $\mathcal{B}_{\alpha 0}$ but not in the Bergman–Besov space $B^p_{\alpha p-(n+1)}$ for $\alpha \in \mathbb{R}$. Our candidate is a function constructed by Kaptanoğlu and Üreyen(see Example 3.4. of [10]).

Recall that a sequence $\{n_k\}$ of positive integers has Hadamard gaps if there exists $c > 1$ such that $n_{k+1} \geq c n_k$ for all $k \geq 1$.

Let $0 < p < \infty$ and set $q := \alpha p - (n+1)$. Define

$$G_{qp}(z) := \sum_k 2^{k(1+q)/p} W_{2^k}(z), z \in \mathbb{B}_n,$$

where W_m are the Ryll–Wojtaszczyk polynomials with the properties

$$\|W_m\|_{L^\infty(\sigma)} = 1 \text{ and } \|W_m\|_{L^p(\sigma)} \gtrsim 1, \quad 0 < p < \infty.$$

Here σ is the normalized Lebesgue measure on $\mathbb{S}_n$.

Then

$$\sum_k (2^k)^{-(1+q)} \left\| 2^{k(1+q)/p} W_{2^k}(z) \right\|^p_{L^p(\sigma)} \gtrsim \sum_k (2^k)^{-(1+q)} 2^{k(1+q)} = \infty$$

Thus by Theorem 3.3 of [10], $G_{qp} \notin B^p_q$. On the other hand

$$\begin{aligned} \sup_k (2^k)^{-\alpha} \left\| 2^{k(1+q)/p} W_{2^k}(z) \right\|_{L^\infty(\sigma)} &= \sup_k (2^k)^{-\alpha} 2^{k(1+q)/p} \left\| W_{2^k}(z) \right\|_{L^\infty(\sigma)} \\ &= \sup_k 2^{k((1+q)/p-\alpha)} < \infty, \end{aligned}$$

since $\frac{1+q}{p} < \alpha$. Hence by Theorem 3.3 of [10], $G_{qp} \in \mathcal{B}_\alpha$. Furthermore

$$\lim_{k\to\infty} (2^k)^{-\alpha} \left\| 2^{k(1+q)/p} W_{2^k}(z) \right\|_{L^\infty(\sigma)} = \lim_{k\to\infty} 2^{k(\frac{1+q}{p}-\alpha)} = 0,$$

since $\frac{1+q}{p} - \alpha = -\frac{n}{p}$. Hence by Proposition 63 of [17], $G_{qp} \in \mathcal{B}_{\alpha 0}$.

Theorem 4.1 *Let $\alpha \in \mathbb{R}$ and $0 < p < \infty$. Then the Bergman–Besov space $B^p_{\alpha p-(n+1)}$ is strictly contained in the weighted little Bloch space $\mathcal{B}_{\alpha 0}$.*

Proof By Theorem 2.3 of [10], it is enough to show that $B^p_{\alpha p-(n+1)} \subset \mathcal{B}_{\alpha 0}$ for $\alpha = 0$.

Let $f \in B^p_{-(n+1)}$. Then for any real number s, $D^1_s f(z) \in B^p_{p-(n+1)}$. It follows from Corollary 6.5 of [9] that

$$\lim_{|z|\to 1^-} (1-|z|^2)|D^1_s f(z)| = 0.$$

Thus $f \in \mathcal{B}_{00}$ and $B^p_{-(n+1)} \subseteq \mathcal{B}_{00}$. The above function $G_{qp}(z) \in \mathcal{B}_{\alpha 0} \setminus B^p_q$, where $q = \alpha p - (n+1)$. That finishes the proof. □

5 Proof of Theorem 1.2

Recall that, given $\alpha \in \mathbb{R}$, we know from Theorem 2.3 that $B^p_{\alpha p-(n+1)} \subset \mathcal{B}_\alpha$. Thus we can study $C_{\mathcal{B}_\alpha}(B^p_{\alpha p-(n+1)})$, the closure of $B^p_{\alpha p-(n+1)}$ in the weighted Bloch space $\mathcal{B}_\alpha$. Further let $f \in \mathcal{B}_\alpha$ and $\epsilon > 0$. Recall that the level set $\Omega^{\alpha,t}_\epsilon(f)$ for f is

$$\Omega^{\alpha,t}_\epsilon(f) = \{z \in \mathbb{B}_n : (1-|z|^2)^\alpha |I^t_s f(z)| \geq \epsilon\},$$

where $s, t \in \mathbb{R}$ with $\alpha + t > 0$. Let $\chi_{\Omega^{\alpha,t}_\epsilon(f)}$ be the characteristic function of the set $\Omega^{\alpha,t}_\epsilon(f)$. With this notation, Theorem 1.2 repeated is the following.

Theorem 5.1 *Let $1 \leq p < \infty$, $\alpha \in \mathbb{R}$, $q \leq \alpha p - (n+1)$ and choose $t \in \mathbb{R}$ such that $0 < \alpha + t$. If $f \in \mathcal{B}_\alpha$, then the following conditions are equivalent:*

(i) $f \in \mathcal{B}_{\alpha 0}$.
(ii) $f \in C_{\mathcal{B}_\alpha}(B_q^p)$.
(iii) *For every* $\epsilon > 0$,

$$\int_{\Omega_\epsilon^{\alpha,t}(f)} (1-|z|^2)^{-(n+1)} d\nu(z) < \infty. \tag{1}$$

Proof Theorem 2.4 implies that $B_q^p \subset B_{\alpha p-(n+1)}^p$ whenever $q \le \alpha p-(n+1)$. Hence it is enough to prove theorem only for $q := \alpha p-(n+1)$. Since B_q^p contains all polynomials, and it is known that the closure of the set of polynomials in the weighted Bloch space $\mathcal{B}_\alpha$ is just the weighted little Bloch space $\mathcal{B}_{\alpha 0}$, we obtain that the closure of B_q^p in $\mathcal{B}_\alpha$ contains $\mathcal{B}_{\alpha 0}$. On the other hand, by Theorem 1.1, $B_q^p \subset \mathcal{B}_{\alpha 0}$. It is clear that the closure of B_q^p in $\mathcal{B}_\alpha$ is contained in $\mathcal{B}_{\alpha 0}$. Thus $\mathcal{B}_{\alpha 0}$ equals to the closure of B_q^p in $\mathcal{B}_\alpha$, and so statement (i) is equivalent to statement (ii).

(ii) $\rightarrow$ (iii): Let $f \in C_{\mathcal{B}_\alpha}(B_q^p)$ and $\epsilon > 0$. Then there exists a function $g \in B_q^p$ such that $\|f-g\|_{\mathcal{B}_\alpha} \le \frac{\epsilon}{2}$ for some $s, t_1 \in \mathbb{R}$ satisfying $\alpha + t_1 > 0$. By our assumption $\alpha + t > 0$, we can take $t_1 = t$. Given such $t \in \mathbb{R}$, let $p_0 > p$ be such that $p_0 > \frac{n}{\alpha+t}$. Set $q_0 = \alpha p_0-(n+1)$. Then $q_0+p_0 t > -1$ and by Theorem 2.4, we have $B_q^p \subset B_{q_0}^{p_0}$. Also for such $s, t \in \mathbb{R}$ with $\alpha + t > 0$, let $z \in \mathbb{B}_n$. Then since

$$(1-|z|^2)^\alpha |I_s^t f(z)| \le (1-|z|^2)^\alpha |I_s^t (f(z)-g(z))| + (1-|z|^z)^\alpha |I_s^t g(z)|,$$

we have that

$$\Omega_\epsilon^{\alpha,t}(f) \subseteq \Omega_{\frac{\epsilon}{2}}^{\alpha,t}(g).$$

Hence, since $g \in B_q^p \subset B_{q_0}^{p_0}$, we have

$$\begin{aligned}
\infty &> \int_{\mathbb{B}_n} |I_s^t g(z)|^{p_0} (1-|z|^2)^{q_0} d\nu(z) \\
&\ge \int_{\Omega_{\frac{\epsilon}{2}}^{\alpha,t}(g)} [(1-|z|^2)^\alpha |I_s^t g(z)|]^{p_0} (1-|z|^2)^{q_0-\alpha p_0} d\nu(z) \\
&\ge \left(\frac{\epsilon}{2}\right)^{p_0} \int_{\Omega_{\frac{\epsilon}{2}}^{\alpha,t}(g)} (1-|z|^2)^{q_0-\alpha p_0} d\nu(z) \\
&\ge \left(\frac{\epsilon}{2}\right)^{p_0} \int_{\Omega_\epsilon^{\alpha,t}(f)} (1-|z|^2)^{q_0-\alpha p_0} d\nu(z).
\end{aligned}$$

Thus, $\int_{\Omega_\epsilon^{\alpha,t}(f)} (1-|z|^2)^{-(n+1)} d\nu(z) < \infty$.

(iii) $\rightarrow$ (ii): Fix $\epsilon > 0$ and let $f \in \mathcal{B}_\alpha$ satisfy (1). Since $\alpha + t > 0$, choose $s \in \mathbb{R}$ such that $\alpha < s+1$. Then by (4) in [8] for $f \in \mathcal{B}_\alpha$, we have the integral representation

$$f(z) = \frac{(1+s+t)_n}{n!} \int_{\mathbb{B}_n} K_s(z,w)(1-|w|^2)^{s+t} D_s^t f(w) d\nu(w). \tag{2}$$

Let $f(z) := f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{(1+s+t)_n}{n!} \int_{\Omega_\epsilon^{\alpha,t}(f)} K_s(z,w)(1-|w|^2)^{s+t} D_s^t f(w) d\nu(w)$$

and

$$f_2(z) = \frac{(1+s+t)_n}{n!} \int_{\mathbb{B}_n \setminus \Omega_\epsilon^{\alpha,t}(f)} K_s(z,w)(1-|w|^2)^{s+t} D_s^t f(w) d\nu(w).$$

The proof will be done once we show that $\|f_2\|_{\mathcal{B}_\alpha} \lesssim \epsilon$ and $f_1 \in B_q^p$. Under the condition $\alpha + t > 0$, the norms on $\mathcal{B}_\alpha$ are equivalent. So we can take s, t for which (2) holds. Then, since $D_s^t K_s(z,w) = K_{s+t}(z,w)$, we obtain

$$\begin{aligned} |D_s^t f_2(z)| &= \left| \frac{(1+s+t)_n}{n!} \int_{\mathbb{B}_n \setminus \Omega_\epsilon^{\alpha,t}(f)} D_s^t K_s(z,w)(1-|w|^2)^{s+t} D_s^t f(w) d\nu(w) \right| \\ &= \left| \frac{(1+s+t)_n}{n!} \int_{\mathbb{B}_n \setminus \Omega_\epsilon^{\alpha,t}(f)} K_{s+t}(z,w)(1-|w|^2)^{s+t} D_s^t f(w) d\nu(w) \right| \\ &\leq \epsilon \frac{(1+s+t)_n}{n!} \int_{\mathbb{B}_n} |K_{s+t}(z,w)|(1-|w|^2)^{s-\alpha} d\nu(w). \end{aligned}$$

By our choice of s, t, notice that $s + t > \alpha + t - 1 > -1 > -(1+n)$. Hence by the definition of the reproducing kernel,

$$K_{s+t}(z,w) = \frac{1}{(1-\langle z,w\rangle)^{1+n+s+t}}.$$

Therefore,

$$\begin{aligned} |D_s^t f_2(z)| &\leq \epsilon \frac{(1+s+t)_n}{n!} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{s-\alpha}}{|1-\langle z,w\rangle|^{1+n+s+t}} d\nu(w) \\ &\lesssim \epsilon \frac{(1+s+t)_n}{n!} \frac{1}{(1-|z|^2)^{\alpha+t}}, \end{aligned}$$

where the second inequality follows from Lemma 3.1 with $s - \alpha > -1$ and $\alpha + t > 0$. This implies that $f_2(z) \in \mathcal{B}_\alpha$ and $\|f - f_1\|_{\mathcal{B}_\alpha} = \|f_2\|_{\mathcal{B}_\alpha} \lesssim \epsilon \frac{(1+s+t)_n}{n!}$. This also implies that $f_1 \in \mathcal{B}_\alpha$.

Now we are going to show that $f_1(z) \in B_q^p$. Let $t_1 \in \mathbb{R}$ be such that $q + pt_1 > -1$. Since s does not appear in this condition we can take $s \in \mathbb{R}$ for which (2) holds.

$$\begin{aligned} D_s^{t_1} f_1(z) &= \frac{(1+s+t)_n}{n!} \int_{\Omega_\epsilon^{\alpha,t}(f)} D_s^{t_1} K_s(z,w)(1-|w|^2)^{s+t} D_s^t f(w) d\nu(w) \\ &= \frac{(1+s+t)_n}{n!} \int_{\Omega_\epsilon^{\alpha,t}(f)} K_{s+t_1}(z,w)(1-|w|^2)^{s+t} D_s^t f(w) d\nu(w) \end{aligned}$$

So,

$$\int_{\mathbb{B}_n} \left|(1-|z|^2)^{t_1} D_s^{t_1} f_1(z)\right|^p (1-|z|^2)^q d\nu(z) = \int_{\mathbb{B}_n} \left|D_s^{t_1} f_1(z)\right|^p (1-|z|^2)^{q+pt_1} d\nu(z)$$

$$\leq \left[\frac{(1+s+t)_n}{n!}\right]^p \left[\int_{\Omega_\epsilon^{\alpha,t}(f)} \left(\int_{\mathbb{B}_n} |K_{s+t_1}(z,w)|^p (1-|w|^2)^{p(s+t)} |D_s^t f(w)|^p (1-|z|^2)^{q+pt_1} d\nu(z)\right)^{1/p} d\nu(w)\right]^p$$

$$\leq \left[\frac{(1+s+t)_n}{n!}\right]^p \|f\|_{\mathcal{B}_\alpha}^p \left[\int_{\Omega_\epsilon^{\alpha,t}(f)} \left(\int_{\mathbb{B}_n} |K_{s+t_1}(z,w)|^p (1-|w|^2)^{p(s-\alpha)} (1-|z|^2)^{q+pt_1} d\nu(z)\right)^{1/p} d\nu(w)\right]^p$$

$$= \left[\frac{(1+s+t)_n}{n!}\right]^p \|f\|_{\mathcal{B}_\alpha}^p \left[\int_{\Omega_\epsilon^{\alpha,t}(f)} (1-|w|^2)^{s-\alpha} \left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{q+pt_1}}{|1-\langle z,w\rangle|^{p(1+n+s+t_1)}} d\nu(z)\right)^{1/p} d\nu(w)\right]^p,$$

where the first inequality follows from Lemma 3.2 and the second one follows from the fact that $f \in \mathcal{B}_\alpha$. Above the last equation follows from the definition of the reproducing kernel since $t_1 > \frac{n}{p} - \alpha$ implies that $s + t_1 > \alpha - 1 + \frac{n}{p} - \alpha = -1 + \frac{n}{p} > -(n+1)$. In order to apply Lemma 3.1 to the integral

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{q+pt_1}}{|1-\langle z,w\rangle|^{p(1+n+s+t_1)}} d\nu(z),$$

we write $p(1+n+s+t_1) = 1+n+q+pt_1+p(1+n+s)-1-n-q$. Now recall that $q = \alpha p - (1+n)$. Hence $p(1+n+s+t_1) = 1+n+q+pt_1+p(1+n+s-\alpha)$. Since $1+n+s-\alpha > n \geq 1$, we can apply Lemma 3.1 with $c = p(1+n+s-\alpha)$. Thus

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{q+pt_1}}{|1-\langle z,w\rangle|^{p(1+n+s+t_1)}} d\nu(z) \lesssim \frac{1}{(1-|w|^2)^{p(1+n+s-\alpha)}}.$$

Therefore,

$$\int_{\mathbb{B}_n} \left|(1-|z|^2)^{t_1} D_s^{t_1} f_1(z)\right|^p (1-|z|^2)^q d\nu(z)$$

$$\lesssim \left[\frac{(1+s+t)_n}{n!}\right]^p \|f\|_{\mathcal{B}_\alpha}^p \left[\int_{\Omega_\epsilon^{\alpha,t}(f)} \frac{(1-|w|^2)^{s-\alpha}}{(1-|w|^2)^{1+n+s-\alpha}} d\nu(w)\right]^p$$

$$= \left[\frac{(1+s+t)_n}{n!}\right]^p \|f\|_{\mathcal{B}_\alpha}^p \left[\int_{\Omega_\epsilon^{\alpha,t}(f)} (1-|w|^2)^{-(n+1)} d\nu(w)\right]^p$$

which is finite by (1). So $f_1 \in B_q^p$. This finishes the proof.

□

We obtain the following corollaries from Theorem 5.1.

Corollary 5.2 *Let $1 \le p < \infty$, $\alpha \in \mathbb{R}$, $q \le \alpha p - (n+1)$ and choose $t \in \mathbb{R}$ such that $0 < \alpha + t$. If $f \in \mathcal{B}_\alpha$, then the following quantities are equivalent in the sense of $\approx$:*

(i) $dist_{\mathcal{B}_\alpha}(f, \mathcal{B}_{\alpha 0})$,
(ii) $dist_{\mathcal{B}_\alpha}(f, B_q^p)$,
(iii) $inf\{\epsilon : \frac{\chi_{\Omega_\epsilon^{\alpha,t}(f)}(z)}{(1-|z|^2)^{n+1}} d\nu(z)$ *is a finite measure*$\}$.

Corollary 5.3 *Let $1 \le p_1 < p_2 < \infty$. Then*

$$dist_{\mathcal{B}_\alpha}(f, B^{p_1}_{\alpha p_1 - (n+1)}) = dist_{\mathcal{B}_\alpha}(f, B^{p_2}_{\alpha p_2 - (n+1)}).$$

Next we prove Theorem 1.3. Since the proof of this theorem is similar to the proof of Theorem 5.1, we will only give the skecth of the proof. The novelty here is that one needs sufficiently higher order derivatives.

Theorem 5.4 *Let $1 \le p < \infty$, $\alpha \in \mathbb{R}$, $\alpha p - (n+1) < q \le \alpha p - 1$ and choose $t \in \mathbb{R}$ such that $\alpha + t > 0$. If $f \in \mathcal{B}_\alpha$, then the following conditions are equivalent:*

(i) $f \in C_{\mathcal{B}_\alpha}(B_q^p \cap \mathcal{B}_\alpha)$.
(ii) There exists $t_0 \ge t$ with $\alpha + t_0 > n$ and $q + pt_0 > -1$ such that for every $\epsilon > 0$,

$$\int_{\Omega_\epsilon^{\alpha,t_0}(f)} (1-|z|^2)^{q-\alpha p} d\nu(z) < \infty.$$

Proof To show first direction, suppose $f \in C_{\mathcal{B}_\alpha}(B_q^p \cap \mathcal{B}_\alpha)$ and $\epsilon > 0$. Then there exists a function $g \in B_q^p \cap \mathcal{B}_\alpha$ such that $\|f - g\|_{\mathcal{B}_\alpha} \le \frac{\epsilon}{2}$ for some $s, t_0 \in \mathbb{R}$ such that $\alpha + t_0 > 0$. In particular, we can take $t_0 \ge t + \frac{n}{p}$. Then for such t_0, as before one can show that $\Omega_\epsilon^{\alpha,t_0}(f) \subseteq \Omega_{\frac{\epsilon}{2}}^{\alpha,t_0}(g)$. Note that $q + pt_0 > \alpha p - (n+1) + pt + n > -1$. Now, similar to the proof of Theorem 5.1, since $g \in B_q^p$ one can also show that

$$\int_{\Omega_\epsilon^{\alpha,t_0}(f)} (1-|z|^2)^{q-\alpha p} d\nu(z) < \infty.$$

To prove other direction, fix $\epsilon > 0$ and let $f \in \mathcal{B}_\alpha$ satisfy

$$\int_{\Omega_\epsilon^{\alpha,t_0}(f)} (1-|z|^2)^{q-\alpha p} d\nu(z) < \infty,$$

for some $t_0 \ge t$, where $\alpha + t_0 > n$ and $q + pt_0 > -1$ hold. Since $\alpha + t_0 > 0$, choose $s \in \mathbb{R}$ such that $\alpha < s + 1$. Then f can be written as in (2). Following the procedure of Theorem 5.1, we can write $f(z) = f_1(z) + f_2(z)$. Now it is easy to see that $\|f_2\|_{\mathcal{B}_\alpha} \lesssim \epsilon$ which also imples that $f_1 \in \mathcal{B}_\alpha$. Proof will be done once we show that $f_1 \in B_q^p$, i.e. we need to show that

$$\int_{\mathbb{B}_n} \left| I_s^{t_0} f_1(z) \right|^p (1-|z|^2)^q d\nu(z) < \infty.$$

One can show this by using the facts f, $f_1 \in \mathcal{B}_\alpha$, and Fubini Theorem, and Lemma 3.1. □

Corollary 5.5 *If* $1 \le p < \infty$, $\alpha \in \mathbb{R}$ *and* $\alpha p - (n+1) < q \le \alpha p - 1$, *then* $\mathcal{B}_{\alpha 0} \subsetneqq C_{\mathcal{B}_\alpha}(B_q^p \cap \mathcal{B}_\alpha)$.

Proof There exists a function $f_\alpha \in B_q^p \cap \mathcal{B}_\alpha \backslash \mathcal{B}_{\alpha 0}$ (cf. [10, Example 3.2]). □

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Declarations

Conflicts of interest Not applicable.

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