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COMMUTATORS OF CLASSICAL OPERATORS IN A NEW VANISHING ORLICZ-MORREY SPACE

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Abstract. We study mapping properties of commutators of classical operators of harmonic analysis – commutators of maximal, singular and fractional operators in a new vanishing subspace of Orlicz-Morrey spaces. We show that the vanishing property defining that subspace is preserved under the action of those operators.

1. Introduction

The commutator of a sublinear operator T with a function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$[b,T]f := bT(f) - T(bf).$$

It is well known that commutators of classical operators of harmonic analysis play an important role in various topics of analysis and PDE. There are many papers in the literature dealing with commutators of classical operators in various function spaces.

Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ play an important role in the study of local behaviour and regularity properties of solutions to PDE. It is well known that the Morrey spaces are non-separable if $\lambda > 0$. The lack of approximation tools for the entire Morrey space has motivated the introduction of appropriate subspaces like vanishing spaces. The definition of the vanishing Morrey spaces involves several vanishing conditions. Each condition generate a closed subspace of $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$. We use the notation of [1] and show these conditions as (V_0), (V_∞) and (V^*).

The space $V_0\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$, often called in the literature just by vanishing Morrey space, was already introduced in [3, 22, 23] motivated by regularity results of elliptic equations. The vanishing generalized Morrey spaces were introduced and studied in [21], see also a study of commutators of Hardy operators in such spaces in [20]. The subspaces $V_\infty\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ and $V^{(*)}\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ were introduced in [1] to study the delicate problem in the approximation of Morrey functions by nice functions.

A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces

$$\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$$

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where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one.

We refer to [5, 6, 7, 15, 16, 18] for the preservation of the vanishing property (V_0) of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ by some classical operators and their commutators.

The authors recently introduced the vanishing Orlicz-Morrey space $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and showed that vanishing property (V_∞) is preserved under the action of maximal, singular and fractional operators in [9]. In this paper, we focus on the commutators of those classical operators for same problem.

We use the following notation: $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. The (Lebesgue) measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and χ_E denotes its characteristic function. We use C as a generic positive constant, i.e., a constant whose value may change with each appearance. The expression $A \lesssim B$ means that $A \leq CB$ for some independent constant $C > 0$, and $A \approx B$ means $A \lesssim B \lesssim A$.

2. Preliminaries

We recall the definition of Young function.

Definition 2.1. A function $\Phi : [0,\infty] \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \Phi(\infty) = \infty$.

A Young function Φ is said to satisfy the Δ_2-condition, denoted also by $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \qquad r > 0$$

for some $C > 0$.

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \qquad r \geq 0$$

for some $C > 1$.

Next we recall the generalized inverse of Young function Φ. For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

Definition 2.2 (Orlicz Space)**.** For a Young function Φ, the Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined by:

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}.$$

The space $L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all measurable functions f such that $f\chi_B \in L^\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$. We refer to [2, 17, 19] for Orlicz spaces in some other settings.

$L^\Phi(\mathbb{R}^n)$ is a Banach space under the Luxemburg-Nakano norm

$$\|f\|_{L^\Phi} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \leq 1 \right\}.$$

For $\Omega \subset \mathbb{R}^n$, let

$$\|f\|_{L^\Phi(\Omega)} := \|f\chi_\Omega\|_{L^\Phi}.$$

A tacit understanding is that f is defined to be zero outside Ω.

In [4], the Orlicz–Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz spaces and generalized Morrey spaces. The definition of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is as follows:

Definition 2.3. Let φ be a positive measurable function on $(0,\infty)$ and Φ any Young function. The Orlicz-Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is the space of functions $f \in L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{\Phi,\varphi}(f;x,r) < \infty,$$

where $\mathfrak{A}_{\Phi,\varphi}(f;x,r) := \frac{\|f\|_{L^{\Phi}(B(x,r))}}{\varphi(r)}$.

For a Young function Φ, we denote by $\mathcal{G}_{\Phi}$ the set of all almost increasing $\varphi : (0,\infty) \to (0,\infty)$ functions such that $t \in (0,\infty) \mapsto \varphi(t)\Phi^{-1}(t^{-n})$ is almost decreasing.

It will be assumed that the functions φ are of the class $\mathcal{G}_{\Phi}$ in the sequel. We refer to [8, Section 5] for more information about these spaces.

We consider the following subspace of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$:

Definition 2.4. The vanishing Orlicz-Morrey space $V_{\infty}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(f;x,r) = 0.$$

The vanishing subspace $V_{\infty}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is nontrivial if $\varphi \in \mathcal{G}_{\Phi}$ satisfies the additional condition

$$\lim_{r\to\infty} \frac{1}{\varphi(r)} = 0,$$

since then it contains bounded functions with compact support.

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L^1_{\rm loc}(\mathbb{R}^n) : \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $b_{B(x,r)} = \frac{1}{B(x,r)} \int_{B(x,r)} b(y)dy$.

Lastly, we define operators investigated in this paper.

Definition 2.5. Let Φ be a Young function. For a function b, the sublinear commutator operator T_b will be called Φ-admissible commutator of a singular type operator T, if:

1) T_b satisfies the size condition of the form

$$\chi_{B(x,r)}(z)\left|T_b\Big(f\chi_{\mathbb{R}^n\setminus B(x,2r)}\Big)(z)\right| \le C\chi_{B(x,r)}(z)\int_{\mathbb{R}^n\setminus B(x,2r)} \frac{|b(y)-b(z)||f(y)|}{|y-z|^n}dy$$

for $x \in \mathbb{R}^n$, a.e. $z \in \mathbb{R}^n$ and $r > 0$;

2) T_b is bounded in $L^{\Phi}(\mathbb{R}^n)$.

The following classical operators are examples of Φ-admissible commutators of singular type operators: the maximal commutator

$$M_b f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x)-b(y)||f(y)|dy$$

and the commutators of Calderón-Zygmund operators $[b,S]$

$$[b,S]f := bS(f) - S(bf),$$

where S is Calderón-Zygmund operator with "standard kernel" (cf. [11, p. 99]), that is, a continuous function defined on $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\neq y\}$ and satisfying the estimates

$$|k(x,y)| \le C|x-y|^{-n} \text{ for all } x\neq y,$$
$$|k(x,y)-k(x,z)| \le C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma>0, \text{ if } |x-y|>2|y-z|,$$
$$|k(x,y)-k(\xi,y)| \le C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma>0, \text{ if } |x-y|>2|x-\xi|.$$

The known boundedness statement for the commutator operators $[b,S]$ and M_b on Orlicz spaces runs as follows.

Theorem 2.1. [12] *Let Φ be a Young function with $\Phi\in\Delta_2\cap\nabla_2$, $b\in BMO(\mathbb{R}^n)$. Then the operators $[b,S]$ and M_b are bounded on $L^{\Phi}(\mathbb{R}^n)$.*

Let $0<\alpha<n$. In this paper we also consider the fractional maximal commutator

$$M_{b,\alpha}f(x) := \sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}}\int_{B(x,r)}|b(x)-b(y)||f(y)|dy$$

and the commutator of the Riesz potential

$$[b,I_\alpha]f(x) := \int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}f(y)dy.$$

The operator $|b,I_\alpha|$ is defined by

$$|b,I_\alpha|f(x) := \int_{\mathbb{R}^n}\frac{|b(x)-b(y)|}{|x-y|^{n-\alpha}}f(y)dy.$$

Recall that, for $0<\alpha<n$,

$$M_{b,\alpha}(f)(x) \le C|b,I_\alpha|(|f|)(x). \tag{2.1}$$

3. Auxiliary Estimates

The following Guliyev-type local estimates play an essential role in the proof of our results.

Lemma 3.1. [15] *Let Φ be a Young function with $\Phi\in\Delta_2\cap\nabla_2$ and $b\in BMO(\mathbb{R}^n)$. Then for the Φ-admissible commutator operator T_b the following inequality is valid*

$$\|T_bf\|_{L^{\Phi}(B(x,r))} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L^{\Phi}(B(x,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \tag{3.1}$$

for any ball $B(x,r)$ and for all $f\in L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 3.2. [5] *Let Φ be a Young function with $\Phi\in\Delta_2\cap\nabla_2$ and $b\in BMO(\mathbb{R}^n)$, then the inequality*

$$\|M_bf\|_{L^{\Phi}(B(x,r))} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)}\sup_{t>r}\left(1+ln\frac{t}{r}\right)\Phi^{-1}(t^{-n})\|f\|_{L^{\Phi}(B(x,t))} \tag{3.2}$$

holds for any ball $B(x,r)$ and for all $f\in L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 3.3. [13] *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Then the inequality*

$$\|[b, I_\alpha]f\|_{L^\Psi(B(x,r))} \lesssim \frac{\|b\|_*}{\Psi^{-1}(r^{-n})} \int_r^\infty \left(1 + ln\frac{t}{r}\right) \Psi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x,t))} \frac{dt}{t} \quad (3.3)$$

holds for any ball $B(x,r)$ and for all $f \in L^\Phi_{\text{loc}}(\mathbb{R}^n)$.

Lemma 3.4. [14] *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Then the inequality*

$$\|M_{b,\alpha} f\|_{L^\Psi(B(x,r))} \lesssim \frac{\|b\|_*}{\Psi^{-1}(r^{-n})} \sup_{t>r} \left(1 + ln\frac{t}{r}\right) \Psi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x,t))} \quad (3.4)$$

holds for any ball $B(x,r)$ and for all $f \in L^\Phi_{\text{loc}}(\mathbb{R}^n)$.

4. Main Results

In this section, we show that the subspace $V_\infty \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is invariant with respect to Φ-admissible commutators of singular type operators. Moreover, we show that the vanishing property (V_∞) is preserved under the action of commutators of fractional operators $[b, I_\alpha]$ and $M_{b,\alpha}$.

Theorem 4.1. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $b \in BMO(\mathbb{R}^n)$ and $\varphi \in \mathcal{G}_\Phi$ satisfy the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \varphi(t) \Phi^{-1}(t^{-n}) \frac{dt}{t} \le C\varphi(r)\Phi^{-1}(r^{-n}) \quad (4.1)$$

where C does not depend on r. Then Φ-admissible commutator operator T_b is bounded on $V_\infty \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Proof. Since the Orlicz-Morrey norm inequalities are already known [15, Theorem 4.14], it remains to show that $V_\infty \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is invariant with respect to T_b:

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(f; x, r) = 0 \quad \Longrightarrow \quad \lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(T_b f; x, r) = 0.$$

If $f \in V_\infty \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ then for any $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(f; x, t) < \epsilon \qquad \text{for every } t \ge R.$$

Using inequality (3.1) and the condition (4.1), we get

$$\mathfrak{A}_{\Phi,\varphi}(T_b f; x, r) \lesssim \frac{\|b\|_*}{\varphi(r)\Phi^{-1}(r^{-n})} \int_r^\infty \left(1 + \ln\frac{t}{r}\right) \Phi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x,t))} \frac{dt}{t} \lesssim \epsilon$$

for any $x \in \mathbb{R}^n$ and every $r \ge R$ (with the implicit constants independent of x and r). Therefore

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(T_b f; x, r) = 0$$

and hence $T_b f \in V_\infty \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$. □

Corollary 4.1. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $\varphi \in \mathcal{G}_\Phi$ satisfy the condition (4.1). Then operators M_b and $[b, S]$ are bounded on $V_\infty \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.*

In view of (3.2) we can give a better result for maximal commutator. More precisely, we have the following result.

Theorem 4.2. *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $\varphi \in \mathcal{G}_\Phi$ satisfy the condition*

$$\sup_{t>r}\left(1+ln\frac{t}{r}\right)\varphi(t)\Phi^{-1}(t^{-n}) \le C\varphi(r)\Phi^{-1}(r^{-n}) \tag{4.2}$$

where C does not depend on r. Then the operator M_b is bounded on $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Proof. Since M_b is bounded in $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ (cf. [5, Theorem 5.6]) we only have to show that it preserves the vanishing property (V_∞). This can be done as in proof of Theorem 4.1, but now using the estimate (3.2). $\square$

Remark 4.1. We find it important to underline the result for the M_b maximal commutator is obtained under weaker assumption than derived from Theorem 4.1. More precisely, the supremal condition (4.2) is weaker than the corresponding integral condition (4.1). To show that we first note that $\Phi^{-1}(\tau)/\tau$ is decreasing, since $\Phi^{-1}(0) = 0$ and Φ^{-1} concave. By this fact, we have

$$\Phi^{-1}(s^{-n}) \approx \Phi^{-1}(s^{-n})s^n \int_s^\infty \frac{dt}{t^{n+1}} \lesssim \int_s^\infty \Phi^{-1}(t^{-n})\frac{dt}{t}.$$

It follows from this inequality

$$\begin{aligned}
\varphi(r)\Phi^{-1}(r^{-n}) &\gtrsim \int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi(t)\Phi^{-1}(t^{-n})\frac{dt}{t}\\
&\gtrsim \left(1+\ln\frac{s}{r}\right)\varphi(s)\int_s^\infty \Phi^{-1}(t^{-n})\frac{dt}{t}\\
&\gtrsim \left(1+\ln\frac{s}{r}\right)\varphi(s)\Phi^{-1}(s^{-n}),
\end{aligned}$$

where we took $s \in (r,\infty)$ arbitrarily, so that

$$\sup_{s>r}\left(1+\ln\frac{s}{r}\right)\varphi(s)\Phi^{-1}(s^{-n}) \lesssim \varphi(r)\Phi^{-1}(r^{-n}).$$

Theorem 4.3. *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Suppose that $\varphi_1 \in \mathcal{G}_\Phi$ and $\varphi_2 \in \mathcal{G}_\Psi$ satisfy the condition*

$$\int_r^\infty \left(1+ln\frac{t}{r}\right)\Psi^{-1}(t^{-n})\varphi_1(t)\frac{dt}{t} \le C\varphi_2(r)\Psi^{-1}(r^{-n}), \tag{4.3}$$

where $C > 0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha]$ from $V_\infty\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V_\infty\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. The ($\mathcal{M}^{\Phi,\varphi_1} \to \mathcal{M}^{\Psi,\varphi_2}$) boundedness of $[b, I_\alpha]$ follows from [13, Theorem 34]. So, we only need to check the action of the $[b, I_\alpha]$. Hence, it remains to show that

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi_1}(f;x,r) = 0 \quad \Longrightarrow \quad \lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Psi,\varphi_2}([b, I_\alpha]f;x,r) = 0.$$

This can be done as in proof of Theorem 4.1, but now using the estimate (3.3) and condition (4.3). □

Remark 4.2. From the proof of [13, Theorem 34], we know that the boundedness result is still true if one has $|b, I_\alpha|$ instead of $[b, I_\alpha]$, see, for example, [10, Remark 3]. Although $M_{b,\alpha}$ is pointwise dominated by $|b, I_\alpha|$ (see (2.1)), and consequently, the results for the former could be derived from the results for the latter, we consider them separately, because we are able to study the boundedness of $M_{b,\alpha}$ under weaker assumptions than it derived from the results for the operator $|b, I_\alpha|$. More precisely, the supremal condition (4.4) is weaker than the corresponding integral condition (4.3) (cf. Remark 4.1).

Theorem 4.4. *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Suppose that $\varphi_1 \in \mathcal{G}_\Phi$ and $\varphi_2 \in \mathcal{G}_\Psi$ satisfy the condition*

$$\sup_{t>r} \left(1 + ln\frac{t}{r}\right)\varphi_1(t)\Psi^{-1}(t^{-n}) \le C\varphi_2(r)\Psi^{-1}(r^{-n}), \tag{4.4}$$

where C does not depend on r. Then the operator $M_{b,\alpha}$ is bounded from $V_\infty\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V_\infty\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. The $(\mathcal{M}^{\Phi,\varphi_1} \to \mathcal{M}^{\Psi,\varphi_2})$ boundedness of $M_{b,\alpha}$ follows from [14, Theorem 5.13]. So, we only need to check the action of the $M_{b,\alpha}$. Hence, it remains to show that

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi_1}(f; x, r) = 0 \quad \Longrightarrow \quad \lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Psi,\varphi_2}(M_{b,\alpha}f; x, r) = 0.$$

This can be done as in proof of Theorem 4.1, but now using the estimate (3.4) and condition (4.4). □

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