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## SOME RESULTS CONCERNING MASTROIANNI OPERATORS BY POWER SERIES METHOD

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**ABSTRACT.** In this paper, we consider power series method which is also member of the class of all continuous summability methods. We study a Korovkin type approximation theorem for the Mastroianni operators with the use of power series method which includes Abel and Borel methods. We also give some estimates in terms of the modulus of continuity and the second modulus of smoothness.

### 1. INTRODUCTION

In the development of the theory of approximation by positive linear operators, the Korovkin theory has big importance. The classical Korovkin type theorems provide conditions for whether a given sequence of positive linear operators converges to the identity operator in the space of continuous functions on a compact interval [3], [11]. This theory has closely connections with real analysis, functional analysis and summability theory. In approximation theory, in order to correct the lack of convergence summability methods are used since it is well known that they provide a nonconvergent sequence to converge [4], [7], [10]. Also Holhos [9] has given a characterization of the functions which is uniformly approximated by Bernstein-Stancu operators.

In this paper, using power series method we give an approximation theorem and quantitative estimates by the Mastroianni operators [13] which contain many well known operators, such as Bernstein polynomials, Baskakov operators and Szász-Favard-Mirakjan operators.

First of all we recall some basic definitions and notations used in the paper.

Let  $(p_j)$  be real sequence with  $p_1 > 0$  and  $p_2, p_3, p_4, \dots \geq 0$ , and such that the corresponding power series  $p(t) := \sum_{j=1}^{\infty} p_j t^{j-1}$  has radius of convergence  $R$  with

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$0 < R \leq \infty$ . If, for all  $t \in (0, R)$ ,

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=1}^{\infty} x_j p_j t^{j-1} = L$$

then we say that  $x = (x_j)$  is convergent in the sense of power series method [12], [16]. Note that the power series method is regular if and only if

$$\lim_{t \rightarrow R^-} \frac{p_j t^{j-1}}{p(t)} = 0, \text{ for each } j \in \mathbb{N} \quad (1.1)$$

hold [5]. Throughout the paper we assume that power series method is regular.

Let for each  $j \in \mathbb{N}$ ,  $\phi_j : \mathbb{R}_+ \rightarrow \mathbb{R}$  be an infinitely differentiable function on  $\mathbb{R}^+ := [0, \infty)$  for which the following conditions hold:

- (i)  $\phi_j(0) = 1$
- (ii)  $(-1)^m \phi_j^{(m)}(x) \geq 0$  for every  $x \in \mathbb{R}_+$  and  $m \in \mathbb{N}_0$
- (iii) for every  $m \in \mathbb{N}_0$ , there exists a positive integer  $q(j, m) \in \mathbb{N}$  and a function  $\alpha_{j,m} : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that

$$\alpha_{j,m}(0) = O(j^m) \text{ as } j \rightarrow \infty$$

$$\phi_j^{(i+m)}(x) = (-1)^m \phi_{q(j,m)}^{(i)}(x) \alpha_{j,m}(x) \quad (1.2)$$

for every  $x \in \mathbb{R}_+$ ,  $i \in \mathbb{N}_0$  and also

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=1}^{\infty} \frac{p_j t^{j-1} j}{q(j, m)} = \lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,m}(0)}{j^m} = 1. \quad (1.3)$$

In [8] the following facts have been obtained for each  $m \in \mathbb{N}_0$  and  $x \in \mathbb{R}_+$

$$(a) \alpha_{j,m}(x) \geq 0 \text{ for every } j \in \mathbb{N}, (b) \phi_j^{(m)}(0) = O(j^m) \text{ as } j \rightarrow \infty. \quad (1.4)$$

**Lemma 1.** *The following statements are satisfied for each  $m \in \mathbb{N}_0$ ,*

- (i)  $\lim_{t \rightarrow R^-} \tau_t^{[m]} = 1$ ,
- (ii) for each  $v \in \mathbb{N}$ ,

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=1}^{\infty} (-1)^m \frac{p_j t^{j-1} \phi_j^{(m)}(0)}{j^{m+v}} = \lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,m}(0)}{j^{m+v}} = 0,$$

where  $\tau_t^{[m]} := \frac{1}{p(t)} \sum_{j=1}^{\infty} \frac{p_j t^{j-1} (-1)^m \phi_j^{(m)}(0)}{j^m}$ .

*Proof.* If we choose  $i = 0$ ,  $x = 0$  in (1.2) and use (i), then we get

$$\frac{\phi_j^{(m)}(0)}{j^m} = (-1)^m \phi_{q(j,m)}(0) \frac{\alpha_{j,m}(0)}{j^m} = (-1)^m \frac{\alpha_{j,m}(0)}{j^m}.$$

Hence we obtain

$$\frac{1}{p(t)} \sum_{j=1}^{\infty} (-1)^m \frac{p_j t^{j-1} \phi_j^{(m)}(0)}{j^m} = \frac{1}{p(t)} \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,m}(0)}{j^m}. \tag{1.5}$$

Using (1.3) and (1.5), we obtain (i).

Since, for each  $v \in \mathbb{N}$ , the sequence  $(\frac{1}{j^v})$  is null sequence, for a given  $\varepsilon > 0$ , there exists a positive integer  $j_0 = j_0(\varepsilon, v)$  such that

$$\begin{aligned} 0 &\leq \frac{1}{p(t)} \sum_{j=1}^{\infty} \frac{p_j t^{j-1} (-1)^m \phi_j^{(m)}(0)}{j^{m+v}} \leq \frac{1}{p(t)} \sum_{j=1}^{j_0} \frac{p_j t^{j-1} (-1)^m \phi_j^{(m)}(0)}{j^m} \\ &\quad + \varepsilon \frac{1}{p(t)} \sum_{j=j_0+1}^{\infty} \frac{p_j t^{j-1} (-1)^m \phi_j^{(m)}(0)}{j^m}. \end{aligned}$$

Using the last inequality and also Lemma 1-(i) and (1.4)-(b), the next inequality is obtained for some  $M > 0$

$$0 \leq \lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=1}^{\infty} \frac{p_j t^{j-1} (-1)^m \phi_j^{(m)}(0)}{j^{m+v}} \leq M \lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=1}^{j_0} p_j t^{j-1} + \varepsilon$$

which implies (ii) by (1.1). □

Now let  $q \in \mathbb{N}_0$  and consider the following space

$$E_q := \{f \in C(\mathbb{R}_+) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^q} \text{ exists}\}.$$

This space is endowed with the norm  $\|\cdot\|_*$ ,  $\|f\|_* := \sup_{x \geq 0} \frac{f(x)}{1+x^q}$  is a Banach space. The classical Mastroianni operators are given as follows

$$M_j(f, x) = \sum_{m=0}^{\infty} (-1)^m f\left(\frac{m}{j}\right) x^m \frac{\phi_j^{(m)}(x)}{m!}$$

and map  $E_q(\mathbb{R}_+)$  into  $C(\mathbb{R}_+)$ . In case of  $\phi_j(x) = e^{-jx}$ ,  $q(j, m) = j$  and  $\alpha_{j,m} = j^m$ , we obtain Szász-Favard-Mirakjan operators also if  $\phi_j(x) = (1+x)^{-j}$ ,  $q(j, m) = j+m$  and  $\alpha_{j,m} = j(j+1)\dots(j+m-1)(1+x)^{-m}$  then we obtain Baskakov operators ([1], [14]).

In this paper we investigate the following operators

$$\mathcal{M}_t(f, x) = \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} M_j(f, x).$$

These operators can be written as

$$\mathcal{M}_t(f, x) = \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} \sum_{m=0}^{\infty} (-1)^m f\left(\frac{m}{j}\right) x^m \frac{\phi_j^{(m)}(x)}{m!}.$$

For the classical Mastroianni operators, we know that

$$\begin{aligned} M_j(1, x) &= 1, \\ M_j(y, x) &= -\frac{\phi_j'(0)}{j}x, \\ M_j(y^2, x) &= \frac{\phi_j''(0)}{j^2}x^2 - \frac{\phi_j'(0)}{j^2}x, \\ M_j((y-x)^2, x) &= \left(\frac{\phi_j''(0)}{j^2} + 2\frac{\phi_j'(0)}{j} + 1\right)x^2 - \frac{\phi_j'(0)}{j^2}x. \end{aligned}$$

We claim that  $\mathcal{M}_t(e_q, x)$  is well defined for the functions  $e_q(y) = y^q, q = 0, 1, 2, \dots$ . It is known from Lemma 3 of [15] that

$$M_j(e_q, x) = \frac{1}{j^q} \sum_{v=1}^q \alpha_{j,v}(0) \sigma(q, v) x^v,$$

where  $\alpha_{j,v}(0)$  is the same as in (iii) and  $\sigma(q, v), v = 1, 2, \dots, q$  are Stirling numbers of the second kind. Then we get

$$\begin{aligned} \mathcal{M}_t(e_q, x) &= \frac{1}{p(t)} \sum_{v=1}^q \sigma(q, v) x^v \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,v}(0)}{j^q} \\ &= \frac{1}{p(t)} x^q \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,q}(0)}{j^q} + \frac{1}{p(t)} \sum_{v=1}^{q-1} \sigma(q, v) x^v \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,v}(0)}{j^q}. \end{aligned}$$

The last inequality implies that

$$0 \leq \mathcal{M}_t(e_q, x) \leq \frac{1}{p(t)} x^q \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,q}(0)}{j^q} + \frac{1}{p(t)} \sum_{v=1}^{q-1} |\sigma(q, v)| x^v \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,v}(0)}{j^q}$$

and this proves our claim. So for all  $f \in E_q$ ,  $\mathcal{M}_t(f, x)$  are well defined.

**Theorem 1.** *Let  $q$  be a positive integer such that  $q \geq 2$ . For every  $f \in E_q, x \in \mathbb{R}_+$ , we have*

$$\lim_{t \rightarrow R^-} \mathcal{M}_t(f, x) = f(x) \tag{1.6}$$

*uniformly on every compact subsets of  $\mathbb{R}_+$  or equivalently  $\{M_j(f, x)\}_{j \in \mathbb{N}}$  is power series summable to  $f(x)$  uniformly on every compact subsets of  $\mathbb{R}_+$ .*

*Proof.* We follow the similar procedure in the Korovkin type approximation theory. We use the approximation theorems by Altomare-Campiti [2]. Thus, it is enough to prove (1.6) holds for three test functions  $e_0, e_1, e_q$ . It is easy to observe that

$$\mathcal{M}_t(e_0, x) = \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} \{M_j(e_0, x)\} = \tau_t^{[0]}.$$

Hence from the regularity of the power series method we see that

$$\lim_{t \rightarrow R^-} \mathcal{M}_t(e_0, x) = e_0(x).$$

We also see that

$$\mathcal{M}_t(e_1, x) = \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} \{M_j(e_1, x)\} = x \tau_t^{[1]}.$$

Hence, by Lemma 1-(i), we obtain that

$$\lim_{t \rightarrow R^-} \mathcal{M}_t(e_1, x) = e_1(x) = x.$$

Finally we can write

$$\mathcal{M}_t(e_q, x) = \frac{1}{p(t)} x^q \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,q}(0)}{j^q} + \frac{1}{p(t)} \sum_{v=1}^{q-1} \sigma(q, v) x^v \sum_{j=1}^{\infty} \frac{p_j t^{j-1} \alpha_{j,v}(0)}{j^q}$$

and by (1.3) and Lemma 1-(ii) we have

$$\lim_{t \rightarrow R^-} \mathcal{M}_t(e_q, x) = x^q = e_q(x).$$

The pointwise approximation in (1.6) with respect to  $x$  becomes uniform on every compact subsets of  $\mathbb{R}_+ [1]$ . This completes the proof.  $\square$

The next result concerns with the pointwise order of approximation in Theorem 1.

**Lemma 2.** *For every  $x \in \mathbb{R}_+$ , we have*

$$\mathcal{M}_t(\Psi_x^2, x) \leq \delta_t^2(x)$$

where  $\Psi_x(y) := y - x$  and

$$\delta_t(x) := \sqrt{(x^2 + x) \max\{\tau_t^{[0]} - 2\tau_t^{[1]} + \tau_t^{[2]}, \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} \frac{[-\phi_j'(0)]}{j^2}\}}.$$

*Proof.* By simple calculation, we obtain

$$\mathcal{M}_t(\Psi_x^2, x) = (\tau_t^{[0]} - 2\tau_t^{[1]} + \tau_t^{[2]})x^2 + x \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} \frac{[-\phi_j'(0)]}{j^2}$$

which completes the proof.  $\square$

By  $w(f, \delta) = \sup_{|y-x| \leq \delta} |f(y) - f(x)|$ ;  $\delta > 0$ ;  $x, y \in \mathbb{R}_+$  we denote the usual modulus of continuity of a function  $f \in C_b(\mathbb{R}_+)$ , the space of all continuous and bounded functions on  $\mathbb{R}_+$ . Now we will estimate the rate of convergence in terms of modulus of continuity.

**Theorem 2.** *Let  $f \in C_b(\mathbb{R}_+)$ ,  $x \in \mathbb{R}_+$ . Then we have*

$$|\mathcal{M}_t(f, x) - f(x)| \leq |f(x)| |\tau_t^{[0]} - 1| + w(f, \delta_t(x)) (\tau_t^{[0]} + \sqrt{\tau_t^{[0]}}).$$

*Proof.* Following Theorem 5.1.2 of [2], we obtain for any  $\delta > 0$ , that

$$\begin{aligned} |\mathcal{M}_t(f, x) - f(x)| &\leq |f(x)| |\mathcal{M}_t(e_0, x) - 1| + w(f, \delta) (\mathcal{M}_t(e_0, x)) \\ &\quad + \frac{w(f, \delta)}{\delta} \sqrt{\mathcal{M}_t(e_0, x)} \sqrt{\mathcal{M}_t(\Psi_x^2, x)}. \end{aligned}$$

Then from Lemma 2, taking  $\delta := \delta_t(x)$ , we get

$$|\mathcal{M}_t(f, x) - f(x)| \leq |f(x)| |\tau_t^{[0]} - 1| + w(f, \delta_t(x)) \tau_t^{[0]} + \frac{w(f, \delta)}{\delta} \sqrt{\tau_t^{[0]} \delta_t^2(x)}.$$

□

The following result gives an estimation on the space of differentiable functions.

**Theorem 3.** *Let  $f \in E_q(\mathbb{R}_+)$ , ( $q \in \mathbb{N}_0$ ) for which  $f$  is differentiable on  $\mathbb{R}_+$  and  $f' \in C_b(\mathbb{R}_+)$  and for every  $x \in \mathbb{R}_+$ , we have*

$$|\mathcal{M}_t(f, x) - f(x)| \leq |f(x)| |\tau_t^{[0]} - 1| + |xf'(x)| |\tau_t^{[1]} - \tau_t^{[0]}| + w(f', \delta_t(x)) \delta_t(x) (\sqrt{\tau_t^{[0]}} + 1).$$

*Proof.* Following Theorem 5.1.2 of [2], we obtain for any  $\delta > 0$ , that

$$\begin{aligned} |\mathcal{M}_t(f, x) - f(x)| &\leq |f(x)| |\mathcal{M}_t(e_0, x) - 1| + w(f', \delta) \sqrt{\mathcal{M}_t(\Psi_x^2, x)} \left( \sqrt{\mathcal{M}_t(e_0, x)} \right. \\ &\quad \left. + \frac{1}{\delta} \sqrt{\mathcal{M}_t(\Psi_x^2, x)} \right) + |f'(x)| |\mathcal{M}_t(\Psi_x, x)|. \end{aligned}$$

By using Lemma 2, we obtain that

$$\begin{aligned} |\mathcal{M}_t(f, x) - f(x)| &\leq |f(x)| |\tau_t^{[0]} - 1| + |xf'(x)| |\tau_t^{[1]} - \tau_t^{[0]}| \\ &\quad + w(f', \delta_t(x)) \sqrt{\delta_t^2(x)} \left( \sqrt{\tau_t^{[0]}} + \frac{1}{\delta} \sqrt{\delta_t^2(x)} \right). \end{aligned}$$

□

Now, we consider the approximation property. It is said to be that a function  $f \in E_q(\mathbb{R}_+)$ , ( $q \in \mathbb{N}_0$ ), satisfies the locally Lipschitz condition on a subset  $U$  of  $\mathbb{R}_+$  provided that

$$|f(x) - f(y)| \leq c_f |x - y|^\alpha, (x, y) \in (\mathbb{R}_+, U), \quad (1.7)$$

holds for some positive constant  $c_f$  depending on  $\alpha$  and  $f$ , where  $\alpha \in (0, 1]$ . The next result gives an estimation on the class of functions satisfying locally Lipschitz condition.

**Theorem 4.** *For every  $x \in \mathbb{R}_+$  the following estimate*

$$|\mathcal{M}_t(f, x) - f(x)| \leq c_f \{ (\tau_t^{[0]})^{\frac{(2-\alpha)}{2}} \delta_t^\alpha(x) + 2\tau_t^{[0]} d^\alpha(x, U) \} + |f(x)| |\tau_t^{[0]} - 1|$$

holds for any function  $f \in E_q(\mathbb{R}_+)$ , ( $q \in \mathbb{N}_0$ ) satisfying the locally Lipschitz condition on a subset  $U$  of  $\mathbb{R}_+$  as in (1.7), where  $\tau_t^{[0]}$  and  $\delta_t(x)$  are given before and  $d(x, U)$  denotes the distance  $x$  from  $U$ , i.e.,

$$d(x, U) := \inf\{|x - y| : y \in U\}.$$

*Proof.* Let  $x \in \mathbb{R}_+$ . By the definition of  $d(x, U)$ , there exists a point  $x_0 \in \overline{U}$ , such that  $d(x, U) = |x - x_0|$ . Then using the fact that  $|f(x) - f(y)| \leq |f(y) - f(x_0)| + |f(x) - f(x_0)|$  for any  $y \in \mathbb{R}_+$ , it follows from the positivity and linearity of the operators that

$$\begin{aligned} |\mathcal{M}_t(f, x) - f(x)| &\leq \mathcal{M}_t(|f(y) - f(x)|, x) + |f(x)| |\mathcal{M}_t(e_0, x) - 1| \\ &\leq \mathcal{M}_t(|f(y) - f(x_0)|, x) + |f(x) - f(x_0)| \mathcal{M}_t(e_0, x) \\ &\quad + |f(x)| |\mathcal{M}_t(e_0, x) - 1|. \end{aligned}$$

By (1.7), we can write that

$$\begin{aligned} |\mathcal{M}_t(f, x) - f(x)| &\leq c_f \mathcal{M}_t(|y - x_0|^\alpha, x) + c_f |x - x_0|_t^\alpha \mathcal{M}_t(e_0, x) + |f(x)| |\mathcal{M}_t(e_0, x) - 1| \\ &\leq c_f \mathcal{M}_t(|y - x|^\alpha, x) + 2c_f |x - x_0|_t^\alpha \mathcal{M}_t(e_0, x) + |f(x)| |\mathcal{M}_t(e_0, x) - 1| \\ &= c_f \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} \sum_{m=0}^{\infty} (-1)^m \left| \frac{m}{j} - x \right|^\alpha \frac{x^m \phi_j^{(m)}(x)}{m!} \\ &\quad + 2c_f |x - x_0|_t^\alpha \mathcal{M}_t(e_0, x) + |f(x)| |\mathcal{M}_t(e_0, x) - 1|. \end{aligned}$$

Now, using the Hölder's inequality with the Hölder conjugates  $\frac{2}{\alpha}$  and  $\frac{2}{2-\alpha}$ , one can see that

$$\begin{aligned} |\mathcal{M}_t(f, x) - f(x)| &\leq c_f (\tau_t^{[0]})^{\frac{(2-\alpha)}{2}} \left\{ \frac{1}{p(t)} \sum_{j=1}^{\infty} p_j t^{j-1} \left( \sum_{m=0}^{\infty} (-1)^m \left( \frac{m}{j} - x \right)^2 \frac{x^m \phi_j^{(m)}(x)}{m!} \right) \right\}^{\frac{\alpha}{2}} \\ &\quad + 2c_f |x - x_0|_t^\alpha \mathcal{M}_t(e_0, x) + |f(x)| |\mathcal{M}_t(e_0, x) - 1| \\ &= c_f \left\{ (\tau_t^{[0]})_t^{\frac{(2-\alpha)}{2}} \mathcal{M}_t^{\frac{\alpha}{2}}(\Psi_x^2, x) + 2\tau_t^{[0]} |x - x_0|^\alpha \right\} + |f(x)| |\tau_t^{[0]} - 1| \\ &\leq c_f \left\{ (\tau_t^{[0]})^{\frac{(2-\alpha)}{2}} \delta_t^\alpha(x) + 2\tau_t^{[0]} |x - x_0|^\alpha \right\} + |f(x)| |\tau_t^{[0]} - 1|. \end{aligned}$$

Therefore, the proof is completed.  $\square$

Let  $w_2(f, \delta)$ ,  $\delta > 0$  denote the second modulus of smoothness of a function  $f \in C_b(\mathbb{R}_+)$ . Then we get the following theorem.

**Theorem 5.** For every  $f \in C_b(\mathbb{R}_+)$  and  $x \in \mathbb{R}_+$  we have

$$|\mathcal{M}_t(f, x) - f(x)| \leq C(\tau_t^{[0]} + 1) \{w_2(f, \sqrt{\Omega_t(x)}) + \Omega_t(x) \|f\|\},$$

where  $C$  is a positive constant and

$$\Omega_t(x) := \max\{|\tau_t^{[0]} - 1|, |x(\tau_t^{[1]} - \tau_t^{[0]})|, \frac{\delta_t^2(x)}{2}\}.$$

*Proof.* Let  $g \in C_b^2(\mathbb{R}_+)$ . Then, by the Taylor's formula, one can write that

$$g(y) = g(x) + (y - x)g'(x) + \frac{1}{2}g''(\xi), \quad y \in \mathbb{R}_+$$

where  $\xi$  lies between  $y$  and  $x$ . So,

$$\mathcal{M}_t(g, x) = g(x)\mathcal{M}_t(e_0, x) + g'(x)\mathcal{M}_t(\Psi_x, x) + \frac{1}{2}\mathcal{M}_t(\Psi_x^2 g''(\xi), x)$$

which implies that

$$\begin{aligned} |\mathcal{M}_t(g, x) - g(x)| &\leq \|g\| |\tau_t^{[0]} - 1| + \|g'\| |x(\tau_t^{[1]} - \tau_t^{[0]})| + \frac{\|g''\|}{2} \mathcal{M}_t(\Psi_x^2, x) \\ &\leq \|g\| |\tau_t^{[0]} - 1| + \|g'\| |x(\tau_t^{[1]} - \tau_t^{[0]})| + \frac{\|g''\|}{2} \delta_t^2(x) \\ &\leq \Omega_t(x) (\|g\| + \|g'\| + \|g''\|) \\ &= \Omega_t(x) \|g\|_{C_b^2(\mathbb{R}_+)} \end{aligned}$$

where  $\|g\|_{C_b^2(\mathbb{R}_+)} = (\|g\| + \|g'\| + \|g''\|)$ . Then it is easy to see that

$$|\mathcal{M}_t(f, x) - f(x)| \leq |\mathcal{M}_t(f - g, x)| + |\mathcal{M}_t(g, x) - g(x)| + |f(x) - g(x)|.$$

By the definition of the operators we can write that

$$\begin{aligned} |\mathcal{M}_t(f, x) - f(x)| &\leq \|f - g\| (\tau_t^{[0]} + 1) + \Omega_t(x) \|g\|_{C_b^2(\mathbb{R}_+)} \\ &\leq (\tau_t^{[0]} + 1) (\|f - g\| + \|g\|_{C_b^2(\mathbb{R}_+)}) \Omega_t(x) \end{aligned}$$

and also by taking infimum over  $g \in C_b^2(\mathbb{R}_+)$  we obtain that

$$|\mathcal{M}_t(f, x) - f(x)| \leq (\tau_t^{[0]} + 1) K(f, \Omega_t(x))$$

where

$$K(f, \delta) := \inf_{g \in C_b^2(\mathbb{R}_+)} \{ \|f - g\| + \|g\|_{C_b^2(\mathbb{R}_+)} \delta \}$$

known as the Peetre's K-functional. Now, using the fact that

$$K(f, \delta) \leq C(w_2(f, \sqrt{\delta}) + \|f\| \min\{1, \delta\})$$

for some positive constant  $C$  independent of  $\delta, f$  [6], we get

$$|\mathcal{M}_t(f, x) - f(x)| \leq C(\tau_t^{[0]} + 1) \{w_2(f, \sqrt{\Omega_t(x)}) + \Omega_t(x) \|f\|\}$$

which completes the proof.  $\square$



## 2. CONCLUDING REMARKS

- In the case of  $R = 1$ ,  $p(t) = \frac{1}{1-t}$  and for  $j \geq 1$ ,  $p_j = 1$  the power series method coincides with Abel method which is a sequence-to-function transformation.
- In the case of  $R = \infty$ ,  $p(t) = e^t$  and for  $j \geq 1$ ,  $p_j = \frac{1}{(j-1)!}$  the power series method coincides with Borel method.

We can therefore give all of the theorems of this paper for Abel and Borel convergences.

## REFERENCES

- [1] O. Agratini and B. Della Vecchia, Mastroianni operators revisited, *Facta Univ. Ser. Math. Inform* 19 (2004), 53-63.
- [2] F. Altomare and M. Campiti, Korovkin-type approximation theory and its applications, *De Gruyter Studies in Mathematics*, 17, Walter de Gruyter Co., Berlin (1994).
- [3] F. Altomare, Korovkin-type theorems and approximation by positive linear operators, *Surv. Approx. Theory*, 5 (2010), 92-164.
- [4] O. G. Atlihan and C. Orhan, Matrix summability and positive linear operators, *Positivity* 11 (2007), 387-398.
- [5] J. Boos, *Classical and Modern Methods in Summability*, Oxford University Press (2000).
- [6] P. L. Butzer and H. Berens, *Semi-groups of operators and approximation*, *Die Grundlehren der Mathematischen Wissenschaften*, 145, Springer, New York, (1967).
- [7] O. Duman, M. K. Khan and C. Orhan, A statistical convergence of approximating operators, *Math. Inequal. Appl.* 6 (2003), 689-699.
- [8] O. Duman, Summability process by Mastroianni operators and their generalizations, *Mediterr. J. Math.* 12 (2015), 21-35.
- [9] A. Holhos, Uniform approximation of functions by Bernstein-Stancu operators, *Carpathian J. Math.* 31 (2015), 205-212.
- [10] A. D. Gadjević and C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* 32 (2002), 129-138.
- [11] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Co., Delhi, (1960).
- [12] W. Kratz and U. Stadtmüller, Tauberian theorems for  $J_p$ -summability, *J. Math. Anal. Appl.* 139 (1989), 362-371.
- [13] G. Mastroianni, On a linear positive operator, *Rend. Acad. Sci. Fis. Mat. Napoli* 46 (1979), 161-176.
- [14] A. J. López-Moreno and J. M. Latorre-Palacios, Localization results for generalized Baskakov/Mastroianni and composite operators, *J. Math. Anal. Appl.* 380 (2011), 425-439.
- [15] A. J. López-Moreno, J. Martínez-Moreno and F. Muñoz-Delgado and J. M. Quesada, Some properties of linear positive operators defined in terms of finite differences. *Multivariate approximation and interpolation with applications (Almuñécar, 2001)* 87-96, *Monogr. Real Acad. Ci. Exact. Fís.-Quím. Nat. Zaragoza*, 20, Acad. Cienc. Exact. Fís. Quím. Nat. Zaragoza, Zaragoza (2002).
- [16] U. Stadtmüller and A. Tali, On certain families of generalized Nörlund methods and power series methods, *J. Math. Anal. Appl.* 238 (1999), 44-66.

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