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Some Characterizations of Constant Ratio Curves According to Type-2 Bishop Frame in Euclidean 3-space E^3

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ABSTRACT

In this paper, we study a twisted curve in the 3-dimensional Euclidean space E^3 as a curve whose position vector can be determined as linear combination of its type-2 Bishop frame. We research these curves according to their curvature functions. Moreover we obtain some results of T-constant and N-constant type curves in the 3-dimensional Euclidean space E^3.

Keywords: *Position vector; type-2 Bishop frame; constant ratio curves.*
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1. Introduction

The theory of curves has an important role in differential geometry. One of these curves is twisted curve, a curve $x : I \subset \mathbb{R} \to E^3$ which has non-zero Frenet curvatures $k_1(s)$ and $k_2(s)$ is called twisted curve. For a regular curve $x(s)$, the position vector of x can be decompose into its tangential and normal components at each point:

$$x = x^T + x^N. \tag{1.1}$$

A curve $x(s)$ with $k_1(s) > 0$ is called constant ratio if the ratio $\|x^T\| : \|x^N\|$ is constant on $x(I)$. Here $\|x^T\|$ and $\|x^N\|$ denote the length of x^T and x^N, respectively [4]. A curve in E^n is said to be T-constant (resp. N-constant) if the tangential component x^T (resp. the normal component x^N) of its position vector x is of constant length [4]. In recent years constant ratio curves are studied in Euclidean and Minkowski space [7, 3, 8, 2].

On the other hand, L.R. Bishop defined Bishop frame, which is known alternative or parallel frame of the curves with the help of parallel vector fields [1]. Then, S. Yilmaz and M. Turgut introduced a new version of the Bishop frame which is called type-2 Bishop frame [10]. Thereafter, E. Ozyilmaz studied classical differential geometry of curves according to type-2 Bishop trihedra [9].

In this study we researched a twisted curve in the 3-dimensional Euclidean space E^3 as a curve whose position vector satisfies the following parametric equation

$$x(s) = \lambda(s) N_1(s) + \mu(s) N_2(s) + \gamma(s) B(s) \tag{1.2}$$

where λ, μ, γ are differentiable functions and $\{N_1, N_2, B\}$ is its type-2 Bishop frame. We characterize these curves according to their curvature functions. Moreover we obtain some results of T-constant and N-constant type curves in the 3-dimensional Euclidean space E^3.

2. Preliminaries

The standard flat metric of 3-dimensional Euclidean space E^3 is given by

$$\langle\, ,\, \rangle : dx_1^2 + dx_2^2 + dx_3^2 \tag{2.1}$$

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where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. For an arbitrary vector x in E^3, the norm of this vector is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. α is called a unit speed curve, if $\langle \alpha', \alpha' \rangle = 1$. Suppose that $\{t, n, b\}$ is the moving Frenet-Serret frame along the curve α in E^3. For a unit speed curve α, the Frenet-Serret formulae can be given as

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t + \tau b \\ b' &= -\tau n \end{aligned} \tag{2.2}$$

where

$$\langle t, t\rangle = \langle n, n\rangle = \langle b, b\rangle = 1,$$
$$\langle t, n\rangle = \langle t, b\rangle = \langle n, b\rangle = 0.$$

and here, $\kappa = \kappa(s) = \|t'(s)\|$ and $\tau = \tau(s) = -\langle n, b'\rangle$. Furthermore, the torsion of the curve α can be given

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{\kappa^2}.$$

Along the paper, we assume that $\kappa \neq 0$ and $\tau \neq 0$.

Bishop frame is an alternative approachment to define a moving frame. Assume that $\alpha(s)$ is a unit speed regular curve in E^3. The type-2 Bishop frame of the $\alpha(s)$ is expressed as [10]

$$\begin{aligned} N_1' &= -k_1 B, \\ N_2' &= -k_2 B, \\ B' &= k_1 N_1 + k_2 N_2. \end{aligned} \tag{2.3}$$

The relation matrix may be expressed as

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}. \tag{2.4}$$

where $\theta(s) = \int_0^s \kappa(s)\, ds$. Then, type-2 Bishop curvatures can be defined in the following

$$k_1(s) = -\tau(s)\cos\theta(s),$$
$$k_2(s) = -\tau(s)\sin\theta(s).$$

On the other hand,

$$\theta' = \kappa = \frac{\left(\frac{k_2}{k_1}\right)'}{1 + \left(\frac{k_2}{k_1}\right)^2}.$$

The frame $\{N_1, N_2, B\}$ is properly oriented, τ and $\theta(s) = \int_0^s \kappa(s)\, ds$ are polar coordinates for the curve α . Then, $\{N_1, N_2, B\}$ is called type-2 Bishop trihedra and k_1, k_2 are called type-2 Bishop curvatures.

3. Constant Ratio Curves According to type-2 Bishop Frame

Let $x(s)$ be a twisted curve whose position vector can be determined as linear combination of its type-2 Bishop frame, then its position vector can be written as

$$x(s) = \lambda(s) N_1(s) + \mu(s) N_2(s) + \gamma(s) B(s) \tag{3.1}$$

where λ, μ, γ are differentiable functions and $\{N_1, N_2, B\}$ is its type-2 Bishop frame. Differentiating the equation (3.1) and using equation (2.3) we get

$$\begin{aligned} x'(s) &= (\lambda' + \gamma k_1) N_1(s) + (\mu' + \gamma k_2) N_2(s) \\ &+ (\gamma' - \lambda k_1 - \mu k_2) B(s) \end{aligned}$$

where $k_1(s)$ and $k_2(s)$ are Bishop curvatures. On the other hand if N_1 is taken instead of tangent vector, and considering above equation we have the following

$$\begin{aligned} \lambda' + \gamma k_1 - 1 &= 0 \\ \mu' + \gamma k_2 &= 0 \\ \gamma' - \lambda k_1 - \mu k_2 &= 0. \end{aligned} \tag{3.2}$$

Definition 3.1. Let $x : I \subset \mathbb{R} \to E^n$ be a unit speed curve in E^n. Then the position vector of x can be decompose into its tangential and normal components at each point as

$$x = x^T + x^N$$

if the ratio $\|x^T\| : \|x^N\|$ is constant on $x(I)$ then x is said to be constant ratio [4].

For a unit speed curve x in E^n the gradient of the distance function $\rho = \|x(s)\|$ is given by

$$grad\rho = \frac{d\rho}{ds} x'(s) = \frac{\langle x(s), x'(s)\rangle}{\|x(s)\|} x'(s) \tag{3.3}$$

where T is the tangent vector of x. The following results can be given for constant ratio curves.

Theorem 3.1. *[5] Let $x : I \subset \mathbb{R} \to E^n$ be a unit speed regular curve in E^n. Then $\|grad\rho\| = c$ holds for a constant c if and only if the following three cases occurs:*

(i) $x(I)$ is contained in a hypersphere centered at the origin.

(ii) $x(I)$ is an open portion of a line through the origin.

(iii) $x(s) = csy(s), c \in (0,1)$, where $y = y(u)$ is a unit curve on the unit sphere of E^n centered at the origin and $u = \frac{\sqrt{1-c^2}}{c} \ln s$.

Corollary 3.1. *[5] Let $x : I \subset \mathbb{R} \to E^n$ be a unit speed regular curve in E^n. Then up to a translation of the arc length function s, we have*

(i) $\|grad\rho\| = 0 \iff x(I)$ is contained in a hypersphere centered at the origin.

(ii) $\|grad\rho\| = 1 \iff x(I)$ is an open portion of a line through the origin.

(iii) $\|grad\rho\| = c \iff \rho = \|x(s)\| = cs$ for $c \in (0,1)$.

(iv) If $n = 2$ and $\|grad\rho\| = c$ for $c \in (0,1)$, then the curvature of x satisfies

$$\kappa^2 = \frac{1-c^2}{c^2\sqrt{s^2+b}},$$

for some real constant b.

For twisted curves according to type-2 Bishop frame in E^3 we get the following results.

Proposition 3.1. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in E^3. If x is a curve of constant ratio then its position vector can be written as*

$$\begin{aligned} x(s) &= \left(c^2 s\right) N_1(s) - \left[\frac{c^2 s k_1}{k_2} + \frac{\left(1-c^2\right) k_1'}{k_1^2 k_2}\right] N_1(s) \\ &+ \left(\frac{1-c^2}{k_1}\right) B(s) \end{aligned} \tag{3.4}$$

for some differential functions, $c \in (0,1)$.

Proof. Let x be a curve of constant ratio, then from corollary 3.1. the distance function of x can be written as $\rho = \|x(s)\| = cs$ for some real constant c. Moreover considering (3.3) we have

$$\|grad\rho\| = \frac{\langle x(s), x'(s)\rangle}{\|x(s)\|}$$

Because of x is a twisted curve of E^3, the equation (3.1) is satisfied. Then we get $\lambda = c^2 s$. Therefore substituting $\lambda = c^2 s$ in the equation (3.2) we obtain

$$\begin{aligned} \mu &= -\frac{c^2 s k_1}{k_2} - \frac{(1-c^2)\, k_1'}{k_1^2 k_2}, \\ \gamma &= \frac{1-c^2}{k_1}. \end{aligned}$$

If we consider the above value of λ, μ, γ and substituting these value in equation (3.1) we obtain the equation (3.4) which complete the proof. □

4. T-Constant Curves

Definition 4.1. Let $x : I \subset \mathbb{R} \to E^n$ be a unit speed curve in E^n. If $\|x^T\|$ is constant then x is called a T-constant curve. For a T-constant curve x either $\|x^T\| = 0$ or $\|x^T\| = \eta$ for some non-zero smooth function η. Moreover, a T-constant curve x is called first kind if $\|x^T\| = 0$, otherwise second kind [6].

As a result of the equation (3.2), we obtain the following expression.

Theorem 4.1. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed twisted curve in E^3 that satisfies the equation (3.1). Then x is a T-constant curve of first kind if and only if*

$$\frac{k_2}{k_1} - \left(\frac{k_1'}{k_1^2 k_2}\right)' = 0 \tag{4.1}$$

where k_1, k_2 are Bishop curvatures.

Proof. Suppose that x is a T-constant curve of first kind. Then using the first and third equation of (3.2) we get

$$\begin{aligned} \gamma &= \frac{1}{k_1}, \\ \mu &= -\frac{k_1'}{k_1^2 k_2}, \end{aligned}$$

where k_1, k_2 are Bishop curvatures. Substituting above equation into the second equation of (3.2) we obtain the desired result. □

Theorem 4.2. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed twisted curve in E^3 that satisfies the equation (3.1). If x is a T-constant curve of second kind then the position vector of the curve is given by*

$$x = \lambda N_1(s) - \left(\frac{\lambda k_1}{k_2} + \frac{k_1'}{k_1^2 k_2}\right) N_2(s) + \frac{1}{k_1} B(s) \tag{4.2}$$

where λ is a constant function.

Proof. Suppose that x is a T-constant curve of second kind. Then using the equation (3.2) we have

$$\gamma = \frac{1}{k_1}$$

and considering the value of γ in the third equation of (3.2) we obtain

$$\mu = -\frac{k_1'}{k_1^2 k_2} - \frac{\lambda k_1}{k_2}$$

where λ is a constant function. So, substituting the value of μ, γ into the equation (3.1) we obtain the result. □

Corollary 4.1. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed twisted curve in E^3. If x is a T-constant curve of second kind then the functions λ, μ, γ satisfied the following equation*

$$\gamma^2 + \mu^2 = 2\lambda s + c \tag{4.3}$$

Proof. Suppose that x is a T-constant curve of second kind. Then using the equation (3.2) we get

$$\begin{aligned} k_1 &= \frac{1}{\gamma}, \\ k_2 &= -\frac{\mu'}{\gamma}. \end{aligned}$$

Then substiuting this values into the third equation of (3.2) we have the following differential equation

$$\gamma\gamma' + \mu\mu' = \lambda$$

which is the solution of (4.3). □

5. N-Constant Curves

Definition 5.1. Let $x : I \subset \mathbb{R} \to E^n$ be a unit speed curve in E^n. If $\|x^N\|$ is constant then x is called a N-constant curve. For a N-constant curve x either $\|x^N\| = 0$ or $\|x^N\| = \nu$ for some non-zero smooth function ν. Moreover, a N-constant curve x is called first kind if $\|x^N\| = 0$, otherwise second kind [6].

For a N-constant curve x the following equation satisfied

$$\|x^N(s)\|^2 = \mu^2(s) + \gamma^2(s) = \omega \tag{5.1}$$

where ω is a constant function.

Considering the equation (3.1), (3.2) and (5.1) we obtain some results as follows.

Lemma 5.1. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in E^3. Then x is a N-constant curve if and only if*

$$\begin{aligned} \lambda' &= 1 - \gamma k_1 \\ \mu' &= -\gamma k_2 \\ \gamma' &= \lambda k_1 + \mu k_2 \\ 0 &= \gamma\gamma' + \mu\mu' \end{aligned} \tag{5.2}$$

the above equation hold, where $\lambda(s), \mu(s), \gamma(s)$ are differentiable functions.

Proposition 5.1. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in E^3. Then x is a N-constant curve of first kind if and only if $x(I)$ is an open portion of a straight line [3].*

Proof. Let x is a N-constant curve in E^3, so the equation (5.1) holds. Moreover if x is a N-constant curve of first kind then using (5.1) we have $\mu = \gamma = 0$ which implies that $k_1 = k_2 = 0$. So x becomes a part of straight line. □

Theorem 5.1. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed twisted curve in E^3. If x is a N-constant curve of second kind then the curve has the following parametrization*

$$x(s) = \left(-\frac{k_1'}{k_1^2 k_2}\right) N_2(s) + \frac{1}{k_1} B(s) \tag{5.3}$$

or

$$x(s) = (s + a) N_1(s) + c N_2(s) \tag{5.4}$$

where a and c are real constants.

Proof. Suppose that x is a N-constant curve of second kind then substituting the second and third equation of (5.2) into the last equation of (5.2) we have

$$\begin{aligned} \mu\left(-\gamma k_2\right)+\gamma\left(\lambda k_1+\mu k_2\right) &= 0 \\ \gamma\lambda k_1 &= 0 \end{aligned}$$

Since $k_1 \neq 0$ we have two possibilities that $\lambda = 0$ or $\gamma = 0$. If $\lambda = 0$ then x is a T-constant curve and from the first and third equation of (5.2) x has the following parametrization

$$x\left(s\right)=\left(-\frac{k_1'}{k_1^2 k_2}\right)N_2\left(s\right)+\frac{1}{k_1}B\left(s\right). \tag{5.5}$$

If $\gamma = 0$ then using the equation (5.2) we obtain

$$\begin{aligned} \lambda' &= 1 \\ \mu' &= 0 \end{aligned}$$

Then x satisfied the equation (5.4) which complete the proof. □

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